Uninsured Idiosyncratic Investment Risk and Aggregate Saving

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Abstract

This paper augments the neoclassical growth model to study the macroeconomic effects of uninsured idiosyncratic investment risk. The general equilibrium is solved in closed form under standard assumptions for preferences and technologies. As compared to complete markets, the steady state is characterized by both a lower interest rate and a lower capital stock when the elasticity of intertemporal substitution is sufficiently high relative to the contribution of private equity in total wealth. For empirically plausible parametrizations, the reduction in aggregate saving and income is significant. These findings contrast with Bewley-type models, where labor-income risk leads to higher aggregate saving. Finally, idiosyncratic investment risk is also shown to introduce two potential sources of amplification in the transitional dynamics.

JEL codes: D52, E13, E32, G11, O16, O41.

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1 Introduction

The macroeconomic effects of uninsured idiosyncratic labor-income risk and precautionary saving have been the subject of a voluminous literature. Although disagreement remains in the profession about quantitative significance, no ambiguity exists about qualitative effects: labor-income risk necessarily increases aggregate saving and thereby capital and output in a neoclassical economy.

In contrast, the macroeconomic effects of uninsured idiosyncratic investment risk are far less understood. Clearly, such risks are paramount in less developed economies, where the bulk of production and investment takes place in privately-owned businesses, and where risk-sharing opportunities are rather limited. But even in the United States private businesses account for almost half of the aggregate capital stock. Indeed, the “representative” investor in the US economy has a very poorly diversified portfolio: the median of the richest 20 percent—who account for almost 95 percent of the entire US non-housing wealth—holds more than one half of her non-housing wealth in private equity.\textsuperscript{1} It thus seems difficult to assess the macroeconomic effects of incomplete markets without first understanding the general-equilibrium effects of idiosyncratic investment risk.

What is more, the issue here is not just quantitative; even the sign of the effects is ambiguous. On the one hand, like labor-income risk, investment risk increases the precautionary supply for savings; in equilibrium this leads to a reduction in the risk-free rate, which in turn stimulates capital accumulation. On the other hand, unlike labor-income risk, investment risk directly discourages capital accumulation by introducing a (private) risk premium on the return to capital. Can we then tell which case—higher or lower aggregate saving and income—is more relevant empirically? Can we quantify the effects? And, how do these effects play over the business cycle?

This paper attempts a first pass at these questions with a minimal modification of the neoclassical growth model. The main contribution is theoretical: to develop a tractable analytical framework for studying the macroeconomic effects of uninsured idiosyncratic investment risk. Some quantitative lessons, however, also emerge: on the one hand, large negative effects on aggregate saving and output are found for plausible parametrizations, leading the way for richer quantitative exercises in future work; on the other hand, the effects are shown to depend critically on parameters which are hard to pin down empirically, uncovering a potential challenge for future quantitative exercises.

The benchmark model is as follows. Households supply labor in a competitive labor market and invest capital in privately-held firms. Firms operate a neoclassical technology, with constants returns to scale (CRS) in capital and labor. Firm-specific productivity shocks translate to idiosyncratic capital-income risk. Households have homothetic preferences (CRRA/CEIS) and can freely borrow and lend in a riskless bond. If they could also diversify their idiosyncratic capital-income risk, the model would reduce to the standard neoclassical growth model; but they cannot.

\textsuperscript{1}For more details on these facts see Quadrini (2000), Gentry and Hubbard (2000), Carroll (2001), and Moskowitz and Vissing-Jørgensen (2002).
A key property of the neoclassical growth model is nevertheless unaffected: capital accumulation exhibits diminishing returns at the aggregate level, but linear returns at the individual level. For a given sequence of prices, the households’ decision problem is homothetic and the optimal decision rules are therefore linear in individual wealth. As a result, the aggregate dynamics do not depend on the wealth distribution. This avoids the “curse of dimensionality” and permits closed-form recursive solution of the general-equilibrium dynamics.

Idiosyncratic investment risk introduces a risk premium on the return to private investment, which reduces the demand for capital at any given interest rate. This effect would unambiguously lead to a lower capital stock if the interest rate were exogenously fixed. However, an Aiyagari-like effect works in the opposite direction: an increase in precautionary savings pushes the interest rate down, below the discount rate, thus stimulating investment. As a result, the general-equilibrium effect on capital accumulation is ambiguous in general.

Yet, thanks to the tractability of the model, a simple necessary and sufficient condition for the risk-premium effect to dominate the Aiyagari-like effect is identified for the case of small risks: incomplete markets lead to a lower capital stock if and only if $\theta > \phi$, where $\theta$ stands for the elasticity of intertemporal substitution and $\phi$ for the ratio of private equity to total (financial plus human) wealth. Numerical simulations then suggest that the same condition, $\theta > \phi$, remains sufficient for a negative effect on capital also in the case of large risks.

Is $\theta > \phi$ the empirically relevant case? In the United States, private equity is about one half of financial (non-human) wealth; translating this fact into the model implies that 0.5 is an upper bound for $\phi$, while plausible parametrizations of the model give a value for $\phi$ around 0.3. The elasticity of intertemporal substitution, on the other hand, is harder to pin down. Hall (1981) estimates it to be below 0.1 using macro data, but this kind of estimates suffer from aggregation bias and identification problems. Micro estimates typically give much higher estimates, frequently above 1, especially if one focuses on households that participate in the stock market (e.g., Vissing-Jørgensen and Attanasio, 2003). Estimates using variation in capital taxation to identify the elasticity of intertemporal substitution suggest an even higher value, as high as 2 (Gruber, 2005).

Hence, on the basis of the predictions of the model and the available evidence for the EIS, the empirically most relevant case appears to be that idiosyncratic investment risk reduces aggregate savings and income. What is more, for plausible numerical simulations this reduction is found to be quantitatively large. However, the main quantitative lesson is rather a message of caution: any quantitative assessment of the impact of idiosyncratic investment risk, whether conducted in the stylized model of this paper or in a richer model, has to confront the significant empirical uncertainty we face regarding the elasticity of intertemporal substitution.

In the benchmark model the entire capital stock is held in private firms. I next extend the model so that a fraction of aggregate savings is in “public equity,” where idiosyncratic risks are fully insurable. Because the low risk-free rate stimulates investment in public equity, the negative impact
of incomplete markets on aggregate savings is significantly mitigated. Nevertheless, incomplete markets now also reduce aggregate total factor productivity by shifting resources away from the more risky but also more productive private equity. As a result, the impact on aggregate output remains relatively large. Indeed, aggregate output can fall even if aggregate saving increases.

Once again, uncertainty about the elasticity of intertemporal substitution and the magnitude of idiosyncratic investment risk precludes precise quantitative analysis. I nevertheless show that quantitatively large effects on savings and income are consistent with modest idiosyncratic risks and low excess returns in private equity. When I calibrate the model so that private equity accounts for half the capital stock (as in US data) and the associated risk premium is as low as 1% or 2%, the saving rate is about 2 or 3 percentage points lower than under complete markets and aggregate income is about 10% lower. Further examining the quantitative effects of idiosyncratic investment risks thus appears a promising, albeit challenging, line for future research.

The model abstracts from aggregate uncertainty, but business-cycle properties can be extrapolated from the structure of transitional dynamics. This leads me to identify two potential sources of amplification. First, cyclical variation in the magnitude of uninsured investment risk generates cyclical variation in private risk premia, which reinforces the cyclicality in private investment. Second, the general-equilibrium interaction of wealth and risk taking introduces a novel macroeconomic complementarity: for given interest rates and given expected returns, the anticipation of higher income in the future increases the willingness to take risk, and hence investment, in the present; but higher investment now means higher capital and higher income in the future, which feeds back to higher risk taking and higher investment today, and so on. A sort of “Keynesian accelerator” thus emerges in a very neoclassical economy.

Both effects highlight the differential role played by uninsured idiosyncratic investment risk along the business cycle. Indeed, whereas counter-cyclical investment risk tends to amplify the transitional dynamics by reducing the demand for investment during a recession, counter-cyclical labor-income risk can have a mitigating effect in a neoclassical economy by contributing to higher savings during a recession. Moreover, the complementarity discussed above emerges only because uninsurable risk impacts investment returns: if markets were complete or if risk was only in labor income, investment would be pinned down by the equality of the marginal product of capital with the interest rate and would be independent of future income.

In plausible numerical simulations, however, the complementarity alone turns out to have a rather modest effect. This is because a “neoclassical” price effect is also at work: the endogenous reaction of interest rates counteracts the amplification effect of the complementarity. In contrast, cyclical risks are found to have strong effects: a flight to quality during recessions generates endogenous variation in the Solow residual, thus amplifying the transitional dynamics.

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2 The business-cycle implications of precautionary saving may differ in a Keynesian economy, where more saving during a recession means less consumption and hence less aggregate demand.
The rest of the introduction discusses the contribution of the paper relative to the pertinent literature. Section 2 introduces the basic model. Sections 3 and 4 analyze the general equilibrium and the steady state. Section 5 revisits the steady-state effects after adding public equity. Section 6 examines the transitional dynamics. Section 7 concludes. All proofs are in the Appendix.

**Relation to the literature.** The paper is most closely related to Bewley-type models such as Aiyagari (1994), Huggett (1997) and Krusell and Smith (1998, 2004). These models also introduce uninsurable idiosyncratic risk in the neoclassical growth model, but focus on labor-income rather than capital-income risk. Given the very different general-equilibrium effects of these two types of risk, exercises like the one conducted here seem necessary if one wishes a more complete understanding of the macroeconomic effects of idiosyncratic risk.

The paper also makes a methodological contribution by presenting an incomplete-markets economy that admits exact aggregation. This result is instrumental for the clean characterization of the steady state and the transitional dynamics; but it is also interesting on its own for it clarifies that non-linearity of individual policy rules—equivalently, the dependence of aggregates on wealth inequality—is not essential for idiosyncratic risk to have significant macroeconomic effects.

In complementary earlier work, Angeletos and Calvet (2000, 2003) also obtain exact aggregation for an economy with both endowment and investment risks; but they do so by assuming constant absolute risk aversion (CARA), thus killing altogether the effect of wealth on precautionary savings, risk taking, and investment. Here instead I assume standard CRRA/CEIS preferences, along with a competitive labor market and a public-equity sector, thus improving both the qualitative and the quantitative content of the analysis.

The role of investment, or rate-of-return, risk has been studied before in AK models (e.g., Levhari and Srinivasan, 1969; Sadmo, 1970; Obstfeld, 1994; Devereux and Smith, 1994; Jones, Manuelli, and Stacchetti, 2000; Krebs, 2003). The AK framework shares the prediction that the effect of risk on savings depends on the elasticity of intertemporal substitution, but by ignoring labor income it underestimates the potential for a negative effect on savings. Moreover, the AK framework obtains tractability only by eliminating transitional dynamics. In contrast, this paper obtains novel insights about the business-cycle implications of idiosyncratic investment risk; and by embedding the analysis in the standard neoclassical growth framework, it makes a step closer to a quantitative assessment of the effects at interest.

Still, the methodological approach taken here is different from, but also complementary to, the one in many recent quantitative papers on incomplete markets (e.g., Castañeda, Diaz-Giménez, and Ríos-Rull, 2003; Cagetti and De Nardi, 2005). These papers use rich models that better capture the phenomenon at interest—heterogeneity and inequality—but inevitably constrain the analysis to a limited set of numerical exercises. Here instead I focus on a single friction—uninsured idiosyncratic investment risk—but offer a more comprehensive analysis of this particular friction.
At the same time, I provide some guidance for future quantitative work on this topic by uncovering the sensitivity of the results to parameters that are hard to pin down empirically.

Various authors have introduced borrowing constraints and occupational choice in the neoclassical growth paradigm to examine the impact of wealth inequality on entrepreneurial activity and capital accumulation. Some focus on the intensive margin as this paper (e.g., Bernanke and Gertler, 1989, 1990; Aghion, and Bolton, 1997; Kiyotaki and Moore, 1997; Kocherlakota, 2000), others on the extensive (e.g., Banerjee and Newman, 1993; Quadrini, 2000; Cagetti and De Nardi, 2005; Buera, 2005). Borrowing constraints imply a wedge between the interest rate and the marginal product of capital. A similar wedge is featured here: the private risk premium on investment. However, the origins and the implications of this wedge are quite different. First, the wedge reflects a reduction in willingness, not ability, to invest; its sensitivity to wealth is due to diminishing absolute risk aversion, not collateral constraints. Second, because risk taking increases with wealth, the impact of this wedge on savings and investment, unlike that of borrowing constraints, need not vanish as agents get wealthier. Finally, although wealth matters for investment at the individual level, wealth inequality does not matter for aggregates.

The paper also makes a contribution to the literature on macroeconomic complementarities (e.g., Matsuyama, 1991; Benhabib and Farmer, 1994; Cooper, 1999). The complementarity identified here is reminiscent of the one in Kiyotaki and Moore (1997): in both papers current investment depends, not only on expected future returns, but also on expected future income. But whereas there channel is the effect of future income on asset prices and thereby on the value of collateral today, here the channel is the effect of future income on current risk taking.

Finally, as in Aiyagari (1994), Krusell and Smith (1998, 2004) and most other Bewley-type models, the lack of insurance in this paper is exogenously imposed. Meh and Quadrini (2005) argue that optimal risk-sharing arrangements under a certain specification of the underlying informational frictions can take the economy much closer to complete insurance than the market structure assumed here. How the extent of risk sharing depends on different assumptions about the underlying informational (or other) frictions, is an important issue beyond the limits of this paper.

2 The Model

Time is discrete, indexed by \( t \in \{0, 1, \ldots, \infty\} \). There is a continuum of infinitely-lived households, indexed by \( i \) and distributed uniformly over \([0, 1]\). Each household owns a single private firm, so that firm \( i \) is identified as the firm owned by household \( i \). Firms employ labor in a competitive labor market but use the capital stock accumulated by their respective household-owner. Households, on the other hand, are each endowed with one unit of labor, which they supply inelastically in a competitive labor market; they can invest capital in the firm they own, but in no other firm; and they can freely trade a riskless bond, but not other financial asset.
Preferences. I assume an Epstein-Zin specification with constant elasticity of intertemporal substitution (CEIS) and constant relative risk aversion (CRRA). A stochastic consumption stream \( \{c_t^i\}_{t=0}^\infty \) generates a stochastic utility stream \( \{u_t^i\}_{t=0}^\infty \) according to the recursion

\[
u_t^i = U(c_t^i) + \beta \cdot U(\mathbb{E}_t[U^{-1}(u_{t+1}^i)])
\]

(1)

where \( \mathbb{E}_t(u_{t+1}^i) = \mathcal{Y}^{-1}[\mathbb{E}_t[\mathcal{Y}(u_{t+1}^i)]] \) represents the certainty equivalent of \( u_{t+1} \) conditional on period-\( t \) information. The utility functions \( U \) and \( \mathcal{Y} \) aggregate consumption across dates and states, respectively, and are given by

\[
U(c) = \frac{c^{1-1/\theta}}{1-1/\theta} \quad \text{and} \quad \mathcal{Y}(c) = \frac{c^{1-\gamma}}{1-\gamma},
\]

(2)

where \( \theta > 0 \) is the elasticity of intertemporal substitution and \( \gamma > 0 \) the coefficient of relative risk aversion.

None of the results of the paper relies on the Epstein-Zin preference specification. Standard expected utility is nested by letting \( \theta = 1/\gamma \), in which case (1) reduces to \( u_t^i = \mathbb{E}_t \sum_{s=0}^\infty \beta^s U(c^i_{t+s}) \).

Technology and idiosyncratic risk. The capital income of household \( i \) in period \( t \) is given by

\[
c_t^i + k_{t+1}^i + b_{t+1}^i = \pi_t^i + R_t b_t^i + \omega_t,
\]

(3)

where \( c_t^i \) denotes consumption, \( k_{t+1}^i \) investment in physical capital, \( b_{t+1}^i \) savings in the risk-free bond, and \( \pi_t^i \) capital income (or the value of firm \( i \), to be specified below).³ Naturally, consumption and physical capital can not be negative: \( c_t^i \geq 0 \) and \( k_{t+1}^i \geq 0 \). Finally, households can freely borrow in the riskless bond up to the “natural” solvency constraint that debt is low enough to be paid out even under the worst realization of idiosyncratic uncertainty.

Budgets. Let \( \omega_t \) denote the wage rate in period \( t \) and \( R_t \) the gross risk-free rate between periods \( t-1 \) and \( t \). The budget constraint of household \( i \) in period \( t \) is given by

\[
c_t^i + k_{t+1}^i + b_{t+1}^i = \pi_t^i + R_t b_t^i + \omega_t,
\]

(4)

where \( n_t^i \) denotes the amount of labor firm \( i \) hires in period \( t \) and \( y_t^i \) the gross output it produces in the same period. Output in turn is given by

\[
y_t^i = F(k_t^i, n_t^i, A_t^i),
\]

where \( A_t^i \) is a shock specific to firm \( i \) and \( F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+ \) a neoclassical production technology.⁴

³The budget constraint in (3) is expressed in terms of stock variables: \( R_t \) equals 1 plus the net risk free rate and \( \pi_t^i \) includes the value of the beginning-of-period non-depreciated capital stock installed in firm \( i \).

⁴That is, \( F \) exhibits constant returns to scale with respect to \( K \) and \( L \), has positive and strictly diminishing marginal products, and satisfies the familiar Inada conditions.
The shock $A_i^t$ is realized in the beginning of period $t$, after capital $k_i^t$ is installed but before employment $n_i^t$ is chosen. It is independently and identically distributed across $i$ and $t$, with continuous p.d.f. $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. In order to interpret a higher $A_i^t$ as higher productivity (or higher profitability), I impose $F_A > 0$, $F_{KA} > 0$, and $F_{LA} > 0$. I finally let $F(K, L, 0) = 0$, meaning that the worst idiosyncratic event leads to zero output, and normalize $\bar{A} \equiv \int A\psi (A) \, dA = 1$.

**Equilibrium.**  The initial position of the economy is given by the distribution of $(k_0^i, b_0^i)$ across households. Households choose plans \(\{c_t^i, n_t^i, k_t^i, b_t^i\}_{t=0}^\infty\) contingent on the history of their idiosyncratic shocks so as to maximize their life-time utility. Idiosyncratic uncertainty, however, washes out at the aggregate. I thus define an equilibrium as a deterministic sequence of prices \(\{\omega_t, R_t\}_{t=0}^\infty\), a deterministic macroeconomic path \(\{C_t, K_t, Y_t\}_{t=0}^\infty\), and a collection of contingent plans \(\{c_t^i, n_t^i, y_t^i, k_t^i, b_t^i\}_{t=0}^\infty\), $i \in [0, 1]$, such that the following conditions hold:  
\[\text{(i) Optimality: } \{c_t^i, n_t^i, y_t^i, k_t^i, b_t^i\}_{t=0}^\infty \text{ maximizes } u_0^i \text{ for every } i.\]
\[\text{(ii) Labor-market clearing: } \int_i n_t^i = 1 \text{ in all } t.\]
\[\text{(iii) Bond-market clearing: } \int_i b_t^i = 0 \text{ in all } t.\]
\[\text{(iv) Aggregation: } C_t = \int_i c_t^i, \ Y_t = \int_i y_t^i, \ K_t = \int_i k_t^i \text{ in all } t.\]

3  **Equilibrium Characterization**

3.1  **Individual behavior**

The idiosyncratic state of agent $i$ in period $t$ is summarized by $(k_i^t, b_i^t, A_i^t)$. For a given price sequence, the value function can therefore be denoted by $V(k, b, A; t)$. Since, by the assumption $F(K, L, 0) = 0$, the worst possible realization of capital income is zero, the natural solvency constraint reduces to $b_{t+1}^i \geq -h_t$, where
\[
h_t \equiv \sum_{j=1}^\infty \frac{\omega_{t+j}}{R_{t+1} \ldots R_{t+j}}
\]
denotes the present value of future labor income (a.k.a. "human wealth"). The household’s problem can thus be represented by the following dynamic program:
\[
V(k, b, A; t) = \max_{c,n,k,b} U(c) + \beta \cdot U\Theta^{-1}(\int \Theta U^{-1}V(k', b', A'; t+1)\psi(A')\,dA')
\begin{align*}
s.t. \quad & c + k' + b' = \pi + Rb + \omega \\
& \pi = F(k, n, A) - \omega n \\
& c \geq 0 \quad k' \geq 0 \quad b' \geq -h_t
\end{align*}
\]
This reduces to the more familiar $V(\cdot; t) = \max\{U(\cdot) + \beta \mathbb{E}_t V(\cdot; t+1)\}$ in the case of expected utility ($\theta = 1/\gamma$). 

\footnote{With some abuse of notation, whenever I write $\int_i x_t^i$ for some variable $x \in \{c, n, y, k, b\}$ I mean the cross-sectional expectation of $x$ in period $t$. I also suppress the dependence of individual variables on the history of shocks.}
Lemma 1 Given \((\omega_t, A^i_t, k^i_t)\), labor demand and capital income are linear in \(k^i_t\), decreasing in \(\omega_t\), and increasing in \(A^i_t\):

\[
n^i_t = n(A^i_t, \omega_t)k^i_t \quad \text{and} \quad \pi^i_t = r(A^i_t, \omega_t)k^i_t,
\]

where \(r(A, \omega) \equiv \max_L [F(1, L, A) - \omega L]\) and \(n(A, \omega) \equiv \arg\max_L [F(1, L, A) - \omega L]\).

Let \(w^i_t \equiv \pi^i_t + R_t b^i_t + \omega_t\) denote the financial wealth of household \(i\) in period \(t\). The budget constraint reduces to \(c^i_t + k^i_{t+1} + b^i_{t+1} = w^i_t\) and, by Lemma 1,

\[
w^i_t = r(A^i_t, \omega_t)k^i_t + R_t b^i_t + \omega_t.
\]

Note then that conditioning on \((k^i_t, b^i_t, A^i_t)\) is useful only for evaluating the optimal \(n^i_t\) and the associated \(w^i_t\) in (7). It follows that the household’s savings problem reduces to

\[
V(w; t) = \max_{(c', b', A') \in \mathbb{R}_+^2 \times [-h_t, \infty)} U(c) + \beta \cdot U^{-1}(\int \mathcal{Y}U^{-1}(V(w'; t + 1)|A')dA')
\]

\[
s.t. \quad c + k' + b' = w, \quad w' = r(A', \omega_{t+1})k' + R_{t+1}b' + \omega_{t+1}.
\]

(With slight abuse of notation, \(V\) now denotes the value function in terms of financial wealth.)

This problem is formally similar to the classic portfolio problem studied by Samuelson (1969) and Merton (1969): preferences are homothetic (by assumption) and wealth is linear in assets (by Lemma 1). That the risky asset is a privately-held business rather than a publicly-traded financial security, that the payoff of this asset depends on the wage rate and thereby on the aggregate capital stock, or that the risk is idiosyncratic, are important for the general-equilibrium properties of the economy, but do not affect the mathematical structure of the individual decision problem.

Lemma 2 Given prices, optimal consumption, investment and bond holdings are linear in wealth:

\[
c^i_t = (1 - s_t)(w^i_t + h_t)
\]

\[
k^i_{t+1} = s_t \phi_t(w^i_t + h_t)
\]

\[
b^i_{t+1} = s_t (1 - \phi_t)(w^i_t + h_t) - h_t
\]
where \( w^i_t \) and \( h_t \) are given by (7) and (5), and where

\[

t = \frac{1 + \left[ \sum_{\gamma=t}^{\infty} \prod_{j=\gamma}^{t} \beta^\rho_{\gamma-1} \right]^{-1}}{1 - \beta^\rho} \quad (12)
\]

\[
\rho_t = \rho \omega_{t+1},R_{t+1} = \max_{\varphi \in [0,1]} CEt\left[ \varphi r(A_{t+1}, \omega_{t+1}) + (1 - \varphi)R_{t+1} \right] \quad (13)
\]

\[
\phi_t = \phi (\omega_{t+1},R_{t+1}) = \arg \max_{\varphi \in [0,1]} CEt\left[ \varphi r(A_{t+1}, \omega_{t+1}) + (1 - \varphi)R_{t+1} \right] \quad (14)
\]

To interpret the above conditions, note that the sum \( w^i_t + h_t \) represents the “effective” wealth of household \( i \), \( s_t \) the saving rate out of effective wealth, \( \phi_t \) the fraction of savings allocated to capital, and \( \rho_t \) the risk-adjusted return to savings (a.k.a. the certainty equivalent of the overall portfolio return). Condition (12) follows from the Euler condition and gives the saving rate as a function of current and future risk-adjusted returns.

Conditions (13) and (14), on the other hand, mean that the allocation of savings between private equity and bonds maximizes the risk-adjusted return to savings. Note that

\[
\phi_t \approx (\ln \bar{r}_{t+1} - \ln R_{t+1})/\left(\gamma \sigma^2_{t+1}\right) \quad \text{and} \quad \rho_t \approx R_{t+1} \exp \left\{ (\ln \bar{r}_{t+1} - \ln R_{t+1})^2 / (2\gamma \sigma^2_{t+1}) \right\}, \quad (15)
\]

where \( \bar{r}_{t+1} \equiv E_t [r(A_{t+1}, \omega_{t+1})] \) in the mean and \( \sigma_{t+1} \equiv \text{Var}_t [\ln r(A_{t+1}, \omega_{t+1})] \) the volatility of private returns (see Appendix for the derivation). Assuming that \( \sigma_{t+1} \) and \( \omega_{t+1} \) are unrelated, \( \phi_t \) and \( \rho_t \) are decreasing in both \( \sigma_{t+1} \) and \( \omega_{t+1} \). The first effect is due to risk aversion; the second is because a higher wage reduces capital income for every realization of the productivity shock.

### 3.2 General equilibrium

By Lemma 1 and the fact that there is a continuum of agents and the shocks are i.i.d. across them, aggregate employment and capital income are given by \( N_t = \bar{n}(\omega_t)K_t \) and \( \Pi_t = \bar{r}(\omega_t)K_t \), where \( \bar{n}(\omega) \equiv \int n(A, \omega)\psi(A)dA \) and \( \bar{r}(\omega) \equiv \int r(A, \omega)\psi(A)dA \). It follows that the labor market clears in period \( t \) if and only if \( \omega_t = \omega(K_t) \), where \( \omega(K) \equiv \bar{n}^{-1}(1/K) \). Similarly, aggregate gross output (including the value of non-depreciated capital) is given by \( Y_t = \Pi_t + \omega_t = f(K_t) \), where \( f(K) \equiv \bar{r}(\omega(K))K + \omega(K) \). By Lemma 2, in turn, consumption, bond holdings, and private investment are linear in individual wealth and therefore the corresponding aggregates are not affected by wealth inequality. Using these properties and aggregating across agents, we arrive at the following closed-form recursive characterization of the general equilibrium.

**Proposition 1 (General Equilibrium)** In equilibrium, the aggregate dynamics satisfy

\[
C_t + K_{t+1} = Y_t = f(K_t) \quad (16)
\]

\[
C_t = (1 - s_t) \left[ f(K_t) + H_t \right] \quad (17)
\]

\[
(1 - s_t)^{-1} = 1 + \beta^\rho \rho_{t+1}^{-1} (1 - s_{t+1})^{-1} \quad (18)
\]

\[
K_{t+1} = \phi_t s_t \left[ f(K_t) + H_t \right] \quad (19)
\]
\[ \bar{n}(\omega_t)K_t = 1 \]  
\[ H_t = \frac{\omega_{t+1} + H_{t+1}}{R_{t+1}} \]

where \( \phi_t = \phi(\omega_{t+1}, R_{t+1}) \) and \( \rho_t = \rho(\omega_{t+1}, R_{t+1}) \).

The interpretation of these conditions is straightforward: (16) is the resource constraint; (17) gives aggregate consumption and (18) the associated Euler condition; (19) gives the aggregate capital stock and (21) the clearing condition for the labor market; and (??) is the present value of aggregate labor income. Indeed, the same system characterizes the complete-markets case, with one modification: the optimality condition for investment, \( \phi_t = \phi(\omega_{t+1}, R_{t+1}) \equiv \arg \max_{\phi} CE_t \{ \varphi_r(A_{t+1}, \omega_{t+1}) + (1 - \varphi)R_{t+1} \} \), reduces to the arbitrage condition \( R_{t+1} = f'(K_{t+1}) \).

Note that the equilibrium system is recursive in \((K_t, H_t, s_t)\). To see this, use (16), (21) and (??) to eliminate \( C_t, \omega_t, \) and \( R_{t+1} \). The equilibrium dynamics then reduce to a three-dimension, first-order, difference-equation system in \((K_t, H_t, s_t)\). If \( \theta = 1 \) (logarithmic utility), then \( s_t = \beta \) for all \( t \) and the dimensionality further reduces to two. This means a dramatic gain in tractability as compared to other incomplete-market models where the entire wealth distribution—an infinitely-dimensional object—is a relevant state variable. This gain is reflected both in the next section, where I analyze the steady state, and in Section 6.2, where I examine the transitional dynamics.

The exact-aggregation result presented above is interesting also for the following reason. With diminishing absolute risk aversion, the precautionary motive associated with labor-income risk leads to strict concavity in the consumption function. Such non-linearity in individual policy rules is important because it breaks the representative-agent framework and makes wealth inequality matter for aggregates (see, e.g., the discussion in Carroll, 2000). However, whether an economy admits exact aggregation—as in the model of this paper—or approximate aggregation—as, for example, in Krusell and Smith (1997)—is not necessarily related to how significant the macroeconomic impact of idiosyncratic risk is.

4 Steady State

4.1 Characterization

A steady state is a fixed point of the dynamic system (16)-(21).\(^6\) Since the general equilibrium was characterized in closed form for any kind of idiosyncratic risk, so does the steady state as well.

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\(^6\)Note that the log of individual wealth follows a random walk in the steady state. Hence, although the steady state is well defined in terms of aggregates, there is no stationary distribution for the wealth distribution. This could easily be fixed, without affecting the results, by having a fraction of the population being replaced each period with new households that are endowed with the mean wealth of the exiting households—but this exercise would be of little economic interest. A more natural mean-reverting force in individual wealth is the one introduced by labor-income risk, borrowing constraints, or diminishing returns at the individual level—but this is beyond the scope of this paper.
For expository purposes, however, it is useful to consider the case that the productivity shock is augmented to capital and lognormally distributed. Thus, I henceforth impose the following.

**Assumption A1.** \( F(K, L, A) = F(AK, L, 1) \) and \( \ln A \sim \mathcal{N}(-\sigma^2/2, \sigma^2) \).

The standard deviation \( \sigma \) then parsimoniously parameterizes the amount of uninsured idiosyncratic risk in private production and investment.\(^7\)

**Proposition 2 (Steady State)** In steady state, the capital stock \( K \) and the interest rate \( R \) solve

\[
\beta^\theta \rho^\theta - 1 \left[ \phi f'(K) + (1 - \phi)R \right] = 1 \tag{22}
\]

\[
\frac{f(K) - f'(K)K}{(R - 1)K} = \frac{1 - \phi}{\phi} \tag{23}
\]

where \( \phi = \phi(\omega(K), R) \) and \( \rho = \rho(\omega(K), R) \).

Condition (22) follows from combining the resource constraint with the Euler condition and has a simple interpretation. The first term in the left-hand side of (22), \( \beta^\theta \rho^\theta - 1 = s \), is the steady-state value of the saving rate; this is increasing (respectively, decreasing) in the risk-adjusted return \( \rho \) if and only if \( \theta > 1 \) (\( \theta < 1 \)) and reduces to \( s = \beta \) when \( \theta = 1 \). The second term, \( \phi f'(K) + (1 - \phi)R \), represents the aggregate return to savings; this is a weighted average of the marginal product of capital and the risk-free rate. The product of these two terms gives the growth rate of aggregate effective wealth. In steady state, aggregate wealth must be constant, which gives (22). Condition (23), on the other hand, follows from clearing the bond market and requires that the ratio of the present value of labor income to the capital stock is consistent with the individuals’ optimal allocation of savings between private equity and the riskless bond.

When markets are complete, the optimality condition for \( \phi \) reduces to the familiar arbitrage condition \( f'(K) = R \), while (22) reduces to \( R = 1/\beta \) and (23) pins down \( \phi \). When instead markets are incomplete, (22) together with \( \phi = \phi(\omega(K), R) \) gives \( K \) as a function of \( R \), and (23) can then be solved for \( R \).

Clearly, it must be that \( R < 1/\beta \), or otherwise aggregate consumption would explode to infinity and a steady state would not exist. This is similar to Aiyagari (1994). But, whereas in Aiyagari (1994) there is no investment risk and therefore \( f'(K) = R \), here it must be that \( f'(K) > R \), or otherwise agents would hold no capital in equilibrium and a steady state would again fail to exist. In other words, there is a tension between the precautionary motive for savings and the risk premium on investment. This leaves open the possibility that \( f'(K) > 1/\beta \). That is, unlike Aiyagari (1994), the impact of incomplete markets on the capital stock is ambiguous in general.

\(^7\)The first part of Assumption A1 implies that \( f(K) = F(K, 1, 1) \) and \( \bar{\sigma}(\omega(K)) = F_K(K, 1, 1) = f'(K) \), for every \( \sigma \) and \( K \); an increase in \( \sigma \) is hence equivalent to a mean-preserving spread in individual returns. Note also that, under A1, one can also interpret \( A_{t+1} \) as random capital depreciation.
Can we find a condition that resolves this ambiguity? The following result provides the answer: the steady-state effect on savings and income is negative if and only if the elasticity of intertemporal substitution is sufficiently high.

**Proposition 3** (i) For any economy \( E = (\sigma; \beta, \gamma, \theta, F) \) with \( \sigma > 0 \), there exists \( \theta = \theta (E) < 1 \) such that the steady-state levels of capital, output, and consumption are lower under incomplete markets if and only if \( \theta > \theta \).

(ii) For \( \sigma \approx 0 \),

\[
\theta \approx \phi \quad \text{and} \quad \phi \approx \frac{\alpha - \chi}{1 - \chi} \leq \alpha, \tag{24}
\]

where \( \alpha \) is the income share of capital and \( \chi \) the investment-to-GDP ratio.\(^8\)

The first part of condition (24) states that \( \theta \), the critical value for the elasticity of intertemporal substitution (EIS), is approximately equal to \( \phi \), the contribution of private equity to effective wealth. The basic intuition behind this result is quite simple. An increase in \( \sigma \) implies a reduction in the (risk-adjusted) rate of return to savings. This in turn has opposing income and substitution effects: the income effect contributes to higher savings, the substitution effect to lower savings. The strength of the income effect depends on \( \phi \): the smaller the contribution of private equity to total effective wealth, the weaker the income effect. The strength of the substitution effect is governed by \( \theta \): the higher the elasticity of intertemporal substitution, the stronger the substitution effect. The overall effect of risk on aggregate saving thus depends on the relation between \( \theta \) and \( \phi \).\(^9\)

Clearly, this intuition is closely related to the one about the role of rate-of-return risk in simple decision-theoretic or \( AK \) models (e.g., Levhari and Srinivasan, 1969; Sadmo, 1970; Obstfeld, 1994; Devereux and Smith, 1994). An important difference, however, emerges in the equilibrium determination of \( \phi \), and this is reflected in the second part of (24). In \( AK \) models risky capital accounts for the entire wealth of a household, so that \( \phi = 1 \). In the neoclassical economy of this paper, instead, risky capital is only a fraction of effective wealth, so that \( \phi < 1 \). In particular, since the ratio of stock of capital to the present value of labor income is directly related to the income share of capital, we have \( \phi \leq \alpha \). In other words, it is the existence of labor income—or other income beyond private equity, such as income from land properties and public equity—that explains why the critical threshold for the EIS is lower here than in \( AK \) models.

Another important difference is that here there is a novel general-equilibrium feedback. As risk makes all agents cut back in their investments, wages falls in equilibrium. Because of diminishing absolute risk aversion, the reduction in the present value of wages discourages risk taking and

\[^8\] That is, \( \alpha \equiv \frac{\dot{f} (K) K}{\dot{f} (K)} \) and \( \chi = \frac{I}{\dot{f} (K)} \), where \( \dot{f} (K) = f (K) - (1 - \delta) K \) is GDP and \( I = \dot{f} (K) - C = \delta K \) is gross saving, both evaluated at steady state. Note that \( \chi \neq s \) and that \( \chi > (\leq)0 \) if \( \delta > (=)0 \).

\[^9\] This intuition and the formulas in (24) are exact only in the limit as \( \sigma \to 0 \). Simulations suggest that \( \theta \) falls with \( \sigma \), and that \( \theta > \phi \) remains sufficient for the steady-state level of capital to be lower under incomplete markets. However, I have not been able to prove this formally.
triggers a further reduction in investment. This in turn causes a further reduction in wages, and so on. This interaction between wealth and risk taking introduces a complementarity—a “multiplier”—that increases the overall impact of market incompleteness. I will revisit the role of this complementarity in transitional dynamics.

Finally, note that the formulas in (24) are exact only in the limit as $\sigma \to 0$. In the simulations that I consider later, $\theta \approx \phi$ remains a good approximation. More generally, simulations suggest that $\theta$ falls with $\sigma$, and that $\theta > \phi$ remains sufficient for the steady-state level of capital to be lower under incomplete markets. However, I have not been able to prove this formally.

4.2 Empirically relevant case?

Assessing the quantitative impact of labor-income risk and precautionary savings has been the subject of a long debate (e.g., Zeldes, 1989; Caballero, 1990; Aiyagari, 1994; Krusell and Smith, 1998; Carroll, 2000; Gourinchas and Parker, 2001), but at least the qualitative effect is unambiguous: labor-income risk necessarily increases savings as compared to complete markets. With investment risk the case is more delicate: before arguing about magnitudes, one has to settle about signs. In this section I thus try to assess whether a negative or positive effect on capital seems more likely empirically (from the perspective, of course, of the model). For this purpose, I need an estimate for $\theta$ and an estimate for $\theta$ (or, equivalently, for $\phi$).

Consider first $\theta$. Using (24) together with $\chi = 20\%$ and $\alpha = 36\%$ (approximately US data) gives $\theta \approx 0.2$, which is dramatically lower than the value, $\theta = 1$, predicted by an AK model.

However, at this point it is important to generalize the formula for $\theta$ in a way that takes the model a step closer to capturing an important fact: the significant heterogeneity in wealth and investment choices across the population. One way to do this without complicating the analysis is to separate the population in two groups: “investors”, who are like the households modeled so far; and “hand-to-mouth workers” as in Campbell and Mankiw (1989). These new agents do not hold any assets, consume their entire labor income every period, and serve a single purpose in the model: they absorb a fraction $\xi \in (0, 1)$ of aggregate human wealth. Then, the equality between $\theta$, the critical threshold for the EIS, and $\phi$ continues to hold provided we reinterpret the latter as the ratio of private equity to the effective wealth held by the class of investors rather than that held by the entire population. Accordingly, condition (24) becomes

$$\theta \approx \phi \approx \left(1 - \xi\right) \left(\frac{\alpha - \chi}{1 - \chi}\right)^{-1} \left(\frac{1}{1 + \xi}\right)^{-1},$$ (25)

where $\alpha$ and $\chi$ are again the income share of capital and the investment-to-GDP ratio.

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10 This presumes $\theta > \theta$; the multiplier works in the opposite direction, amplifying the increase of savings, if $\theta < \theta$. Also note that this multiplier effect is clearest when $R$ is fixed—as in the case of an open economy—for then a reduction the steady-state wage translates one-to-one to a reduction in the steady-state $H$. The endogenous adjustment of $R$ in the closed economy of this paper tends to mitigate this effect.
Not surprisingly, $\theta$ increases with $\xi$, meaning that under-accumulation becomes less likely the higher the fraction of non-proprietary income absorbed by hand-to-mouth (non-investing) agents. At the one extreme, $\xi = 0$ nests the benchmark case (condition (24)); at the other, $\xi = 1$ nests the AK case in the sense that the threshold for the EIS becomes 1 when investors have no source of income other than their private equity. For plausible values of $\xi$, however, $\theta$ remains quite below 1. For example, let $\chi = 20\%$ and $\alpha = 36\%$ (approximately US data) and calibrate $\xi$ so that hand-to-mouth workers account for 50% of aggregate consumption, which is the fraction estimated by Campbell and Mankiw (1989). Then, the threshold for the EIS raises from $\theta = 0.2$ to $\theta = 0.4$.11

Some information about the contribution of private equity to the wealth of the typical US investor, and hence about the threshold $\theta$, is also contained in the data. Carroll (2001) and Moskowitz and Vissing-Jørgensen (2002) document that private equity accounts about one half of financial wealth, whether one looks at the entire US economy, or at the portfolio of the rich 2%-10% (who account for the bulk of US wealth). Adjusting this for human wealth implies that 0.5 is an upper bound for $\phi$ and therefore for $\theta$ as well.12

The empirical estimates of the EIS vary a lot. Hall (1988) and Campbell and Mankiw (1989) obtain very low estimates using US macro data on aggregate consumption growth and T-bill rates. However, these kind of estimates suffer from aggregation bias and ignore heterogeneity. For example, using British data, Attanasio and Weber (1993) show that correcting for aggregation bias raises the estimated EIS from about 0.3 to about 0.7. Moreover, the estimated EIS tends to increase with wealth and/or asset holdings (Blundell, Browning and Meghir, 1994; Attanasio and Browning, 1995; Vissing-Jørgensen, 2002; Attanasio, Banks and Tanner, 2002; and Guvenen 2005b). The exact estimates vary a lot across studies and different specifications, but in most cases they are above 0.5, and frequently around or above 1, especially if one concentrates on the top layers of wealth or asset holdings. For example, using data from the US Consumer Expenditure Survey (CEX) and an Epstein-Zin specification as in this paper, Vissing-Jørgensen and Attanasio (2003) report baseline estimates between 1 and 1.4 for stockholders.

Although my model cannot accommodate heterogeneity in the EIS, what seems most relevant for the exercises conducted here is the EIS of rich investors, for they hold almost the entire capital stock in the economy. Indeed, if the poorer households are well approximated by the hand-to-mouth workers of the exercise conducted above, then their EIS is irrelevant. Consistent with this intuition are Guvenen’s (2005a) results. He augments a standard RBC model for heterogeneity in stock-

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11To see this, let $\lambda$ denote the fraction of aggregate consumption that goes to hand-to-mouth agents (the Campbell-Mankiw fraction) and $Y$ the GDP in steady state. Then, the total consumption of hand-to-mouth agents is $\xi(1-\alpha)Y = \lambda(1-\chi)Y$. Solving for $\xi$ gives $\xi = \lambda(1-\chi)/(1-\alpha)$. Using then this together with (25), $\lambda = .5$, $\chi = .2$, and $\alpha = .35$, gives $\theta = .4$.

12In the model analyzed so far, private equity accounts for the entire financial wealth. This however can be relaxed either by letting the safe asset to be in exogenously fixed positive supply, or by introduce a public-equity sector; see Section 5.
market participation and in intertemporal substitution preferences. In particular, he assume that the EIS is 1 for (wealthy) stockholders and 0.1 for (poor) non-stockholders. He then shows that the poor are important for the business-cycle properties of aggregate consumption but virtually irrelevant for those of aggregate investment and output: a Hall-type regression based on aggregate consumption data generated by the model estimates an EIS around 0.25, but aggregate investment and output behave as in a representative-agent economy with an EIS equal to 1.

This discussion already suggests setting \( \theta \) close to 1; the case for a high \( \theta \) is further supported by the following observations. The aforementioned empirical studies— with either macro or micro data— typically attempt to identify the EIS by instrumenting current rates of return with lagged rates of return. This identification strategy, however, is invalid if persistent macroeconomic shocks impose a correlation between interest rates and the precautionary motive (the residual in the linearized Euler regressions estimated in these studies). A better identification strategy is probably to instrument real returns with predictable variation in tax rates on capital income. This strategy, which is employed by Mulligan (2002) for macro data and by Gruber (2005) for micro data, delivers consistently high point estimates: Mulligan estimates the EIS to be above 1; Gruber estimates it to be close to 2. What is more, unlike many other studies, Gruber’s point estimates are quite robust to different specifications and different choices on whether one measures the rate of return with T-bill rates or stock-market returns. However, the standard errors are typically too large to rule out the case that the EIS is lower than 0.5. For example, Gruber’s baseline point estimate is 2.032 with a standard error of 0.796 (see Table 2, “Tax IV” raw, “T-bill rate” column, in his paper).

To recap, on the basis of the results of this paper and the more recent empirical estimates of the EIS, the most likely scenario appears to be that \( \theta > \theta_0 \), and therefore that idiosyncratic investment risk leads to lower aggregate savings and income. However, the opposite scenario can not be ruled out given the large uncertainty we face about the EIS.

### 4.3 Some numerical results

The lack of estimates about the magnitude of idiosyncratic investment risk together with the uncertainty surrounding the EIS precludes any precise quantification of the steady-state effects; the purpose of the numerical exercises conducted here and later in the paper is to give a sense of the order of magnitudes.\(^{13}\)

Thanks to the tractability of the model, numerical solution of the steady state is trivial: substituting \( \phi = \phi (\omega (K) , R) \) and \( \rho = \rho (\omega (K) , R) \) into (22)-(23) gives a simple system of two equations in two unknowns, the steady-state levels of \( K \) and \( R \).

\(^{13}\)Levine and Zamme (2002) have shown that the effect of market incompleteness in Bewley economies vanishes as the discount rate converges to zero. An analogue of this limit result appears to hold here: as \( \beta \to 1 \), (22) and (23) can hold only if \( R \to 1 \) and \( \phi \to 0 \), which in turn implies that the risk premium vanishes. Nevertheless, the simulations presented below deliver large quantitative effects with low discount rates and modest i.i.d. shocks.
With a Cobb-Douglas technology and a lognormal productivity, the economy is fully parameterized by \((\beta, \gamma, \theta, \alpha, \delta, \sigma)\). For my benchmark calibration, I let the time period be one year, the discount rate \(1 - \beta^{-1} = 5\%\), the coefficient of relative risk aversion \(\gamma = 2\), the income share of capital \(\alpha = 40\%\), and the depreciation rate \(\delta = 5\%\). These values are standard in the literature. Following the discussion in the previous section regarding the elasticity of intertemporal substitution, I set \(\theta = 1\) for my benchmark parametrization and latter vary \(\theta\) between .5 and 2.

What remains is \(\sigma\), the standard deviation of the individual return in private equity.\(^{14}\) There are various indications that idiosyncratic investment risks are both large and significant for the typical investor in the US economy. The probability that a privately-held firm survives over the first 5 years of its life is less than 40 percent. The variation of returns in the cross-section of private investors is very large even conditional on survival. The estimated value of private equity in the United States is about as high as the value of public equity today; and it was about twice as large in the 70’s and 80’s. More than 75% of aggregate private equity is owned by households for whom private equity constitutes at least half of their total net worth. The median rich (top 5% or so) household holds almost half of his total wealth, or almost 60% of his non-housing wealth, in private equity; and more than 70% of this is invested in a single company in which the household has an active management interest.\(^{15}\)

Unfortunately, however, there are no precise estimates of the level of idiosyncratic investment risk. For example, Moskowitz and Vissing-Jørgensen (2002) and Bitler, Moskowitz, and Vissing-Jørgensen (2005) explore the cross-section of private-equity investors in the Survey of Consumer Finances, but are unable to provide a reliable measure for the risk faced by individual investors due to lack of enough time-series variation in the data. For their numerical exercises, they instead proxy \(\sigma\) with the standard deviation of the annual return to an individual publicly-traded stock.

Campbell et al. (2001) report that the standard deviation of the annual return to a publicly-traded stock is around 50%. One possibility is that privately-held firms, being on average younger and smaller than publicly-held firms, face even higher risks. Perhaps more likely, however, is that publicly-held firms they are willing to engage in more risky projects than privately-held firms, as well as there is “noise” is stock-market returns. Hence, I would guess that 50% is an upper bound for \(\sigma\). On the other hand, I would expect that \(\sigma\) is no lower than the aggregate stock-market volatility, which pools all idiosyncratic risk and is about 17%. I thus choose \(\sigma = 20\%\) as my baseline value, but also consider \(\sigma = 40\%\) and \(\sigma = 10\%\) for comparison.

The results are presented in Table 1. The first raw corresponds to the baseline parametrization; the rest check different values for \(\sigma, \gamma, \theta, \beta, \alpha,\) and \(\delta.\)^{16}

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\(^{14}\)Note that, under Assumption A1, \(\text{Var}[\ln r(A_{t+1}, \omega_{t+1})] = \sigma^2\).

\(^{15}\)For further details on these facts, see Carroll (2001) and Moskowitz and Vissing-Jørgensen (2002).

\(^{16}\)For the quantitative results, income is measured by \(\text{GDP}_t \equiv \hat{f}(K_t) \equiv f(K_t) - (1 - \delta)K_t = K_t^\alpha L^{1-\alpha};\) the risk-free rate and the mean excess return on private equity by \(R_t - 1\) and \(\bar{r}_t - R_t\), respectively; and the saving rate by \(I_t/\text{GDP}_t\), where \(I_t = K_{t+1} - (1 - \delta)K_t\).
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Table 1: Steady-state effects in the benchmark model. $\Delta(Saving \ Rate)$ is the change in the aggregate saving rate between complete and incomplete markets; $\Delta(GDP)$ is the corresponding percentage change in the aggregate level of income; interest rate is the rate of return on the riskless bond under incomplete markets; private premium is the mean excess return on private equity.

Under the baseline parametrization, the steady-state saving rate falls by 2.4 percentage points, from 20% under complete markets to 17.6% under incomplete markets. Consequently, aggregate capital falls by 19% and aggregate income by 8% as compared to complete markets. The risk-free rate, on the other hand, falls from 5% to 4.6%. Finally, the associated risk premium on private equity is 1.7%.

Not surprisingly, the effects are much stronger if $\sigma = 40\%$ and more modest if $\sigma = 10\%$. Raising risk aversion $\gamma$ from 2 to 4 (which is within the range of empirically plausible values) raises the reduction in saving rates from 2.4 to 4.1 percentage points and the reduction in income from 8% to 14%. Raising $\alpha$ to 60% also increases the reduction in output; to the extent that a higher $\alpha$ is interpreted as a broader definition of capital, this is indicative of the potential importance of idiosyncratic risk in human-capital accumulation.17 As for the discount and depreciation rates, their impact on the magnitudes of interest are rather modest.

Given the earlier theoretical result about the critical role played by the EIS, a more interesting exercise is to see how sensitive the quantitative findings here are to variation in $\theta$. The baseline parametrization features $\theta \approx 0.22$. If we reduce $\theta$ below 0.22, the reported output and capital losses turn to gains. However, letting $\theta$ be as low as 0.5 (lower bound for most recent empirical studies

17 For this issue, see also Krebs (2003).
on the EIS) reduces the magnitudes of capital and output losses only to a modest degree: the reduction in the saving rate is 2.1 percentage points when $\theta = 0.5$ as compared to 2.4 percentage points when $\theta = 1$. Conversely, raising $\theta$ to 2 (Gruber’s estimate) increases the capital and output losses, but again only to a modest degree. Hence, the quantitative impact of the EIS appears to be quite non-linear, and the capital and output losses remain large for a wide range of empirically plausible values for the EIS.

5 Two Sectors: Private and Public Equity

The analysis so far has assumed that all investment is subject to idiosyncratic risk. This is not necessarily a bad benchmark for less developed economies, in which production is dominated by privately-held firms. Nevertheless, it is important to understand the robustness of the results to the availability of a safe asset that is in positive net supply.

A partial way to address this issue is to allow for the riskless bond—the only safe asset in the model analyzed so far—to be in exogenously fixed positive supply.\(^{18}\) The economy then remains unchanged except for one modification: clearing the bond market now requires $\int_t b_t = \bar{B}$, where $\bar{B} > 0$ is the fixed supply of the safe asset. Proposition 1 thus directly extends if we simply $f(K_t) + H_t$ in conditions (17) and (19) with $f(K_t) + H_t + \bar{B}$. Similarly, Propositions 2 and 3 also extend if we adjust the equilibrium value of $\phi$ for the fact that effective wealth now includes $\bar{B}$. In particular, condition (24) becomes

$$\theta \approx \phi \approx \left( (\frac{\alpha - \chi}{1 - \chi})^{-1} + \left( \frac{\kappa}{1 - \kappa} \right)^{-1} \right)^{-1} \leq \min \left\{ \frac{\alpha - \chi}{1 - \chi}, \kappa \right\},$$

where $\kappa \equiv K / (\bar{B} + K)$ denotes the ratio of private equity to financial wealth (both evaluated at the steady state). Hence, the critical value for the EIS is bounded from above by $\kappa$, which we can directly measure in the data.\(^{19}\) Moreover, the critical value for the EIS is now lower than in the benchmark model, which is intuitive since the safe asset reduces the contribution of private equity to total effective wealth, making it easier for the substitution effect to dominate the wealth effect. For example, letting $\kappa \approx 50\%$, $\chi \approx 20\%$, and $\alpha \approx 36\%$ (approximately US data) gives $\theta \approx 0.17$, as compared to $\theta \approx 0.20$ in the benchmark model, while introducing hand-to-mouth workers that absorb one half of aggregate consumption gives $\theta \approx 0.34$, as compared to $\theta \approx 0.40$ in the benchmark.

This approach, however, leaves unmodeled what the safe asset is and where its supply comes from—and this is an important omission if the size of this asset depends on the level of idiosyncratic risk faced by the agents, which is exactly the case in Aiyagari (1994). In what follows, I thus endogenize the supply of this asset by introducing a second sector of production, to be identified as

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\(^{18}\)Note that agents also have an *implicit* safe asset in the form of the present value of their labor income.

\(^{19}\)The property that $\theta \leq \kappa$ is robust to adding hand-to-mouth workers as in the previous section.
public equity, in which ownership of capital is freely traded across agents and therefore idiosyncratic risks are fully diversified. This also introduces an additional margin in the analysis: the allocation of resources between privately-held and publicly-traded firms.

5.1 General equilibrium

Let $X_t$ and $L_t$ denote the total capital and labor allocated to the public-equity sector in period $t$. Total output for this sector is given by $G(X_t, L_t)$, where $G$ is a neoclassical production function. Since public equity is risk-free, simple arbitrage implies that its return must equal the return to the riskless bond. Moreover, by profit maximization, $\omega_t = G_L(X_t, L_t)$ and $R_t = G_X(X_t, L_t)$. The rest of the equilibrium characterization is like in the benchmark model. Lemma 1 remains unaffected, whereas Lemma 2 extends with a minor modification, namely replacing bond holdings with the sum of bond and public-equity holdings. We thus obtain the following variant of Proposition 1.

**Proposition 4 (General Equilibrium)** In an equilibrium in which both sectors are active, the aggregate dynamics satisfy

\begin{align*}
C_t + K_{t+1} + X_{t+1} &= Y_t = F(K_t, N_t, 1) + G(X_t, L_t) \quad (26) \\
C_t &= (1 - s_t)(Y_t + H_t) \quad (27) \\
(1 - s_t)^{-1} &= 1 + \beta^\theta \rho_t (1 - s_{t+1})^{-1} \quad (28) \\
R_t &= G_X(X_t, L_t) \quad \omega_t = G_L(X_t, L_t) \quad (29) \\
K_{t+1} &= \phi_t s_t (Y_t + H_t) \quad N_t = \bar{n}(\omega_t)K_t \quad (30) \\
N_t + L_t &= 1 \quad (31) \\
H_t &= (\omega_{t+1} + H_{t+1}) / R_{t+1} \quad (32)
\end{align*}

where $\rho_t = \rho(\omega_{t+1}, R_{t+1})$ and $\phi_t = \phi(\omega_{t+1}, R_{t+1})$.

The system is again recursive, as in the benchmark model, and the interpretation of the conditions is equally simple. (26) is the resource constraint of the economy. (27) and (28) give the equilibrium consumption and the Euler condition. (29) gives the familiar conditions characterizing the equilibrium capital and employment in public equity, whereas (30) gives the analogues for private equity. Finally, (31) is the clearing condition for the labor market and (32) the present value of aggregate labor income in recursive form.

5.2 Steady state

A steady state in which both sectors are active is a fixed point of the dynamic system (26)-(32). To simplify the analysis, it is useful to assume that the capital intensity of the technology used by

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20 The difference is that the state vector is now $(K_t, X_t, H_t, s_t)$, as compared to $(K_t, X_t, s_t)$ in the benchmark; see also Lemma 3 in the next section.
public-equity firms is identical to the one in privately-held firms, in which case the productivity difference between the two sectors is parameterized by a single scalar.

Assumption A2. \( G(X, L) = F(X, L, 1/\mu) \) for some \( \mu > 1 \).

The restriction \( \mu > 1 \) simply means that (risky) private equity has a higher mean return than (riskless) public equity, which is necessary for a positive amount of private equity to be held in equilibrium. On the other hand, if \( \mu \) is too high relative to \( \sigma \), the equilibrium may involve zero resources allocated to the public-equity sector, in which case the model reduces to the one-sector benchmark examined before. In what follows, I thus focus on the case where both sectors are active in equilibrium.

Letting \( R(\omega) \equiv \max_l [G(1, l) - \omega l] \) and \( l(\omega) \equiv \arg \max_l [G(1, l) - \omega l] \), we can restate (29) as \( R_t = R(\omega_t) \) and \( L_t = l(\omega_t) X_t \). Moreover, the combination of A1 and A2 gives \( \bar{r}(\omega) = \mu R(\omega) \), so that \( \mu \) pins down the risk premium on private equity. It follows that \( \phi(\omega, R) = \varphi \) and \( \rho(\omega, R) = \rho R \), where \( \varphi \) and \( \rho \) are determined by the exogenous parameters \( \mu, \sigma \) and \( \gamma \) alone:

\[
\varphi \equiv \arg \max_{\varphi \in [0, 1]} \mathbb{E}[\phi \mu A_{t+1} + 1 - \phi] \quad \text{and} \quad \rho \equiv \max_{\varphi \in [0, 1]} \mathbb{E}[\phi \mu A_{t+1} + 1 - \phi]. \quad (33)
\]

These properties greatly simplify the characterization of the steady state.

Proposition 5 (Steady State) In a steady state in which both sectors are active:

(i) The interest rate is

\[
R = \beta^{-1} \rho^{1/\theta - 1} (\varphi \mu + 1 - \varphi)^{-1/\theta} < 1/\beta \quad (34)
\]

where \( \varphi \) and \( \rho \) are given by (33); the wage rate is then given by \( R(\omega) = R \) and the capital stocks by

\[
K = \frac{1/l(\omega)}{\mu} + \frac{\omega/(R-1)}{1/\phi - 1} \quad \text{and} \quad X = 1/l(\omega) - \mu K. \quad (35)
\]

(ii) There exists \( \theta = \theta(\sigma, \gamma, \mu) < 1 \) such that \( \theta > \theta \) suffices for a local increase in \( \sigma \) to raise the interest rate and reduce private equity, TFP, aggregate output, and consumption.

As compared to the benchmark model, public equity has three novel implications. First, an Aiyagari-like effect applies to public equity: since the risk-free rate is necessarily lower than the discount rate, the capital-labor ratio in publicly-traded firms is unambiguously higher than what under complete markets. As a result, a higher \( \sigma \) can possibly lead to higher aggregate savings even when it leads to less investment in private equity.

Second, an increase in idiosyncratic risk triggers a reallocation of resources (both capital and labor) from the more risky but more productive sector (private equity) to the less risky but less productive one (public equity), thus causing a reduction in aggregate total factor productivity. As a result, aggregate output can fall with \( \sigma \) even when aggregate capital does not.
Third, even though the risk-free rate is always below the discount rate, an increase in $\sigma$ may locally increase the risk-free rate when both private and public equity are held. This is unlike either the Bewley class of models or the one-sector model of the previous section. The reason for this new effect is that the technology in the public equity sector imposes a negative relation between the wage rate and the interest rate, namely the relation implied by the equation of the input price ratio with the marginal rate of technical substitution. When an increase in $\sigma$ causes a reallocation of resources from private to public equity, thus reducing aggregate productivity and wages, the reduction in wages is necessarily associated with an increase in interest rates. In the Bewley class of models, the same negative relation between wages and interest rates is present, since all capital is public, but it works the other way round: higher labor-income risk leads to a lower interest rate and thereby to a higher capital-labor ratio and a higher wage rate. In the one-sector model of the previous sections, on the other hand, the negative relation between wages and interest rates was broken, because the interest rate is not equated to the marginal product of capital.

5.3 Some numerical results

Although a precise quantification is again precluded by the uncertainty we face about $\theta$ and $\sigma$, it is useful to get a sense of the quantitative importance of introducing public equity in the model.

The new parameter that needs to be calibrated is $\mu$. Other things equal, $\mu$ determines the allocation of resources between the two sectors. As mentioned earlier, private and public equity each claim roughly one half of aggregate wealth in the United States. For any given set of values for $(\sigma, \beta, \gamma, \theta, \alpha, \delta)$, I thus calibrate $\mu$ so that the implied steady-state shares of private and public equity in the aggregate capital stock are 50% each. The results are reported in Table 2.

In the baseline parametrization (first raw of Table 2), the reduction in the saving rate is 1.1 percentage point with public equity as compared to 2.4 percentage points in the benchmark model. Similarly, the reduction in the capital stock is now 12% as compared to 19% before. On the other hand, the reduction in the steady-state level of income is 7% as compared to 9% before. Hence, the impact of incomplete markets on aggregate savings is significantly mitigated by the introduction of public equity, but the effect on aggregate income remains strong. The reason is that incomplete markets now distort also aggregate total factor productivity.

This finding is quite robust to different parameter specifications (rest of Table 2). For example, the reduction in steady-state income is 12% when $\gamma = 4$ as compared to 14% in the benchmark model. The premium on private equity, on the other hand, is considerably lower. In the baseline parametrization, for example, the private premium is as low as 0.9% (compared to 1.7% without public equity). Even when $\gamma = 4$ or $\sigma = 40\%$, the premium is no more than 3%. This is simply because the excess return required for holding a risky asset (here private equity) is lower the lower the fraction of savings invested in this asset. Large effects on aggregate savings and income are thus consistent with modest risks and low private premia.
Table 2: Steady-state effects in the model with public equity.

### 6 Transitional Dynamics and Amplification

Since the model abstracts from aggregate uncertainty, I can not examine literally the business-cycle properties of the model. Nevertheless, as in the standard neoclassical growth model, some insight into the business-cycle properties is provided by studying the transitional dynamics.

For this purpose, I modify the model as follows. Let $Z_t$ denote the aggregate labor productivity in period $t$; with a Cobb-Douglas production function, variation in $Z_t$ is equivalent to variation in total factor productivity for both sectors. I assume that $Z_t$ follows the deterministic analogue of an AR(1) process:

$$\ln Z_{t+1} = \rho \ln Z_t, \quad (36)$$

where $\rho \in [0, 1)$ measures the persistence of productivity. I also allow for the level of idiosyncratic risk to vary with the state of the economy:

$$\sigma_t \equiv \left[ \text{Var}_t (\ln A^i_{t+1}) \right]^{1/2} = \sigma [1 - \eta \ln Z_t], \quad (37)$$

where $\sigma \geq 0$ and $\eta \geq 0$ parameterize, respectively, the steady-state level and the cyclical elasticity of idiosyncratic risk. I can then mimic a recession with a reduction in $Z_0$ starting from steady state.

Thanks to the simple closed-form recursive structure of the general equilibrium, it is easy to compute the response of the economy to such a shock (or, more generally, the transitional dynamics...
from any given initial conditions). Letting \( \theta = 1 \) (baseline value for the EIS) further simplifies the equilibrium recursion by ensuring that \( s_t = \beta \) for all \( t \).\(^{21}\)

**Lemma 3** Suppose \( \theta = 1 \) and \((Z_t, \sigma_t)\) satisfy (36)-(37). There is a unique mapping \( \Omega : \mathbb{R}^4 \to \mathbb{R}^8 \) such that, for all \( t \geq 0 \),

\[
(Z_{t+1}, K_{t+1}, X_{t+1}, H_t; C_t, Y_t, \omega_t, R_t) = \Omega (Z_t, K_t, X_t, H_{t-1}).
\]

For any given \((Z_0, K_0, X_0, H_{-1})\), the whole path is computed simply by iterating on \( \Omega \). Since \((Z_0, K_0, X_0)\) are historically given, one only needs to find the equilibrium value for \( H_{-1} \). Starting with an arbitrary guess for \( H_{-1} \), iterating on \( \Omega \) to compute the implied \( \{\omega_t, R_t\}_{t=0}^T \) for \( T \) large enough, and letting \( H'_{-1} = \sum_{t=1}^T \omega_t / (R_1...R_{t-1}) \), gives a mapping from \( H_{-1} \) to \( H'_{-1} \). Iterating on this mapping till \( H'_{-1} \approx H_{-1} \) gives the equilibrium.\(^{22}\)

### 6.1 Two potential sources of amplification

The model features two potential sources of amplification: the cyclical variation in the level of uninsurable idiosyncratic risk and the equilibrium interaction of wealth and risk taking.

Consider first the role of (exogenous) cyclical variation in \( \sigma_t \). As the economy enters a recession (that is, after a negative shock in \( Z_0 \)), the level \( \sigma_t \) of uninsurable investment risk increases, implying a reduction in the willingness to invest in private equity. That is, the demand for investment is low during a recession, not only because interest rates are high—the standard reason in the complete-markets neoclassical model—but also because private risk premia are high. Moreover, as resources are diverted away from private equity towards public equity, an endogenous reduction in aggregate productivity takes place—the Solow residual itself is amplified.

Consider next the interaction of wealth and risk taking. To gain some insight, ignore for a moment the presence of public equity and the equilibrium variation in \( R, s, \) or \( \phi \). Conditions (18) and (19) then reduce to the following:

\[
K_{t+1} = s\phi [f(K_t) + H_t], \tag{38}
\]

\[
H_t = \sum_{j=1}^{\infty} R^{-j} \omega(K_{t+j}). \tag{39}
\]

On the one hand, (38) implies that, other things equal, \( K_{t+1} \) increases with either \( K_t \) or \( H_t \) and therefore the path of capital \( \{K_{t+1}, K_{t+2}, ...\} \) increases with the path of human wealth \( \{H_t, H_{t+1}, ...\} \). This effect reflects a *decision-theoretic* property: the dependence of individual risk taking on individual wealth. On the other hand, (39) implies that \( \{H_{t+1}, H_{t+2}, ...\} \) increases with \( \{K_{t+1}, K_{t+2}, ...\} \). This feedback reflects a *general-equilibrium* effect: the dependence of labor income on aggregate

\(^{21}\)If, instead, \( \theta \neq 1 \), the “state vector” \((Z_{t+1}, K_{t+1}, X_{t+1}, H_t)\) must be expanded to include \( s_t \).

\(^{22}\)This is not a contraction mapping, but I obtained a fixed point for the simulations reported below.
capital. The combination of these two effects gives rise to a dynamic macroeconomic complementarity: the anticipation of low income tomorrow leads every agent to invest less today, which implies lower aggregate capital and hence lower income tomorrow, which in turn feeds back to less investment today, and so on.

Few remarks are worth making about this complementarity. First, it introduces a short of “Keynesian accelerator” in an RBC economy: for given interest rate and given productivity, investment demand depends on anticipated income. Second, it derives from a pecuniary externality: it would be absent if wages and interest rates, and therefore $H_t$, were exogenously fixed. Third, it relies on investment being subject to uninsured idiosyncratic risk and risk taking being sensitive to future income.

To the best of my knowledge, the complementarity identified here is novel to the literature. Indeed, it is absent in Bewley models (e.g., Aiyagari, 1994), because there is no uninsured idiosyncratic investment risk. It is also absent in AK models (e.g., Obstfeld, 1994, Krebs, 2003) and in the CARA-normal economies of Angeletos and Calvet (2000, 2003), because in these frameworks there is no feedback from future income to today’s risk-taking. Finally, it is absent in credit-constraint models with risk-neutral entrepreneurs (e.g., Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997), simply because agents are risk neutral. Yet, there is an interesting similarity between the complementarity here and the one in Kiyotaki and Moore (1997): in both papers current investment depends, not only on expected future returns, but also on expected future income. The origins, however, are different: whereas the channel in Kiyotaki-Moore is the effect of future income on asset prices and thereby on the value of collateral today, here the channel is the effect of future income on current risk taking.\textsuperscript{23}

The two sources of amplification identified here—the cyclicality in risk and the complementarity in investment—are indicative of the very different business-cycle effects that idiosyncratic investment risk can have as compared to idiosyncratic labor-income risk. But they are only part of the story: there may be other counteracting effects in general equilibrium. In particular, the discussion assumed a constant interest rate. This would be fine if the economy were a small economy open to an international market for the riskless bond; but in the closed economy here it is a priori unclear how the equilibrium interest rates may interact with the forces discussed above. To obtain a more complete picture of the overall general-equilibrium effects, I next simulate the response of the economy to an unanticipated shock in $Z_0$.

\section{6.2 A numerical example}

There are various indications that idiosyncratic investment risks are highly cyclical—proxies such as bankruptcy rates, firm-exit rates, and firm-specific volatility in publicly traded firms vary a lot

\footnote{In Kiyotaki and Moore (2005), idiosyncratic liquidity risk translates to idiosyncratic rate-of-return risk; reinterpreting the investment risk in my paper as liquidity risk is an intriguing but unverified possibility.}
Figure 1: The response of the economy to a negative 1% shock in aggregate productivity. Solid lines for incomplete markets with cyclical idiosyncratic risk; dashed lines for complete markets. All variables normalized by their respective steady-state levels.
over the business cycle—but there is no hard evidence on which to base the calibration of their cyclical sensitivity ($\eta$). I thus make a plausible but random guess: I calibrate $\eta$ so that a 2% reduction in output below steady state is associated with a 5% increase in $\sigma_t$. For the rest of the parameters, I use the baseline values ($\sigma = 20\%$, $\gamma = 2$, $\theta = 1$, $1 - \beta^{-1} = 5\%$, $\alpha = 40\%$, $\delta = 5\%$) along with $\rho = 95\%$ and $\mu$ such that, again, 50% of capital is private equity.

Figure 1 illustrates the response of the economy to a negative 1% productivity shock starting from steady state (that is, $\ln Z_0 = -1%/ (1 - \alpha)$, $K_0 = K_\infty$, and $X_0 = X_\infty$). The solid lines correspond to incomplete markets, the dashed ones to complete markets. The amplification effect is quite strong: the impact of the exogenous shock on aggregate output, consumption, and investment is, respectively, 63%, 92%, and 53% higher than the impact of the shock under complete markets. Importantly, note how the amplification shows up in aggregate productivity: the reallocation of resources away from private equity contributes 42% of the overall reduction in the Solow residual.$^{24}$

These results do not distinguish whether the main source of amplification is the cyclical variation in risk or the macroeconomic complementarity discussed earlier. To isolate the role of the latter, Figure 2 repeats the same exercise as Figure 1 setting $\eta = 0$. Clearly, the amplification effect almost vanishes. The reason is precisely the offsetting effect of interest rates: the reduction in interest rates during the recession counteracts the reduction in expected future wages and thereby mitigates the reduction in the demand for private equity. In other words, the “Keynesian accelerator” is here offset by a “neoclassical” price effect.

To recap, the complementarity is of theoretical interest on its own—for it is likely to extend

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$^{24}$For each variable $X$, the amplification effect is measured by taking the maximal value of the ratio $\hat{X}_t^{\text{inco}}/\hat{X}_t^{\text{com}}$ over the first 4 periods, where $\hat{X}_t^{\text{inco}} = (X_t - X_\infty)/X_\infty$ denotes the period-$t$ percentage change relative to steady state under incomplete markets, and $\hat{X}_t^{\text{com}}$ the corresponding change under complete markets; the numbers reported in the text are $\hat{X}_t^{\text{inco}}/\hat{X}_t^{\text{com}} - 1$. 

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to a larger class of models where agents face idiosyncratic investment risk—but it fails to generate strong amplification in the closed-economy model of this paper. In contrast, cyclical variation in idiosyncratic risk appears to have a more significant quantitative potential.

7 Concluding Remarks

This paper augmented the standard neoclassical growth model in order to study the macroeconomic effects of uninsured idiosyncratic investment risk. The main contribution of the paper was theoretical; the quantitative results were at best preliminary. Yet, the picture that emerged is that large negative effects on aggregate savings and income can be consistent with modest investment risks and low private premia.

The merit but also the limitation of the analysis is that it modified the neoclassical growth model in only one dimension, that of idiosyncratic investment risk. This was not an accident: the purpose of the exercise was to understand the role of incomplete risk sharing in isolation from that of other frictions, such as borrowing constraints, or other aspects of entrepreneurial activity and wealth heterogeneity, such as occupational choice. But extending the analysis in these dimensions is necessary for a better understanding of the macroeconomic effects of entrepreneurship.25

Further exploring the business-cycle implications of idiosyncratic investment risk is another interesting direction for future research. Although the results presented here regarding the structure of transitional dynamics were only suggestive, cyclicality in investment risk, or in the willingness to take such risk, may have important positive, as well as normative, implications. For example, Krusell and Smith (2004) examine the welfare and distributional effects of eliminating business cycles in an incomplete-markets economy. By removing cyclical variation in labor-income risk, their analysis predicts a reduction in aggregate savings and wages; the opposite prediction may hold if one considers removing cyclical variation in idiosyncratic investment risk.

A cornerstone result in the Ramsey paradigm of optimal taxation is that the optimal tax on capital income is zero under complete markets (Chamley, 1986; Judd, 1985; Atkeson, Chari and Kehoe, 1999). Introducing labor-income risk in this framework leads to a positive tax on capital (Aiyagari, 1995). The dynamic Mirrlees paradigm has also focused on labor-income risk; a central result of this literature is the optimality of a positive wedge (tax) on savings (Werning, 2001; Golosov, Kocherlakota and Tsyvinski, 2003; Albanesi and Sleet, 2006). How idiosyncratic investment risk affects the properties of optimal taxation in either paradigm is another intriguing open question.26

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25 Progress in this direction is made by Buera (2005) and Cagetti and De Nardi (2005).
26 Preliminary work in this direction indeed suggests that an investment subsidy can be optimal even in cases where there is over-saving relative to the first best.
Appendix: Proofs

Proof of Lemma 1. By the linear homogeneity of $F(K, L, A)$ in $(K, L)$,
\[
\frac{\pi_t^i}{k_t^i} = F(1, n_t^i, A_t^i) - \omega_t \frac{n_t^i}{k_t^i}.
\]
Since $k_t^i$ and $A_t^i$ are known when $n_t^i$ is chosen, the optimal $n_t^i/k_t^i$ maximizes (40) for any $A_t^i$, which gives (6). By definition of $n(\cdot)$ and $r(\cdot)$, $F_L(1, n(A, \omega), A) \equiv \omega$ and $r(A, \omega) \equiv F(1, n(A, \omega), A) - \omega n(A, \omega)$. Hence, $F(K, L, 0) = 0$ implies $n(0, \cdot) = r(0, \cdot) = 0$, whereas $n(A, \cdot) > 0$ and $r(A, \cdot) > 0$ for $A > 0$. Applying the implicit function theorem and using.

Proof of Lemma 2. For notational simplicity, I drop the superscript $i$ and use $r_{t+1}$ as a short-cut for $r(A_{t+1}, \omega_{t+1})$. I propose the following solution:
\[
V(w; t) = U(a_t + w + h_t), \quad c(w; t) = (1 - s_t)(w + h_t), \quad k(w; t) = \phi_t s_t(w + h_t),
\]
where $a_t$, $s_t$, and $\phi_t$ are coefficients (time-varying but non-stochastic) to be determined. From the budget constraint and (41), we then infer $b(w; t) = (1 - \phi_t)s_t(w + h_t) - h_t$. From (2) and (41), the certainty equivalent of the value of wealth is
\[
\mathbb{E}_t[U^{-1}V_{t+1}(w_{t+1})] = \gamma^{-1} \mathbb{E}_t[(\gamma U^{-1}V_{t+1}(w_{t+1}))] = a_{t+1} \mathbb{E}_t[w_{t+1} + h_{t+1}]^{1-\gamma}.
\]
Hence, the first-order conditions with respect to $k_{t+1}$ and $b_{t+1}$ give:
\[
\begin{align*}
(c_t)^{-1/\theta} &= \beta a_{t+1}^{1-\theta} \cdot (w_{t+1} + h_{t+1})^{1-\gamma} \cdot (1-\gamma)/(1-\gamma) \mathbb{E}_t[(w_{t+1} + h_{t+1})^{-\gamma}r_{t+1}] , \\
(c_t)^{-1/\theta} &= \beta a_{t+1}^{1-\theta} \cdot (w_{t+1} + h_{t+1})^{1-\gamma} \cdot (1-\gamma)/(1-\gamma) \mathbb{E}_t[(w_{t+1} + h_{t+1})^{-\gamma}R_{t+1}].
\end{align*}
\]
Combining the two conditions, and using
\[
w_{t+1} = r_{t+1}k_{t+1} + R_{t+1}b_{t+1} + \omega_{t+1} = [\phi_t r_{t+1} + (1 - \phi_t)R_{t+1}] s_t(w_t + h_t) - h_{t+1},
\]
we get $\mathbb{E}_t\{(R_{t+1} + \phi_t(r_{t+1} - R_{t+1}))^{-\gamma}(r_{t+1} - R_{t+1})\} = 0$, or equivalently $\phi_t = \phi(\omega_{t+1}, R_{t+1})$. Next, the envelope condition, $V'(w_t; t) = U''(c_t)$, or equivalently $a_t^{-1/\theta}(w_t + h_t)^{-1/\theta} = (c_t)^{-1/\theta}$, along with $c_t = (1 - s_t)(w_t + h_t)$ from (41), implies
\[
a_t^{-1/\theta} = (1 - s_t)^{-1/\theta}.
\]
Multiplying (42) and (43) with $\phi_t$ and $(1 - \phi_t)$, respectively, summing up, substituting $w_{t+1}$ in the resulting relation from (44), and rearranging, gives the saving rate in recursive form:
\[
(1 - s_t)^{-1} = 1 + \beta^\theta \rho_t^{\theta-1}(1 - s_{t+1})^{-1},
\]
(46)
where \( \rho_t = \rho(\omega_{t+1}, R_{t+1}) \). For any \( \{\omega_t, R_t\}_{t=0}^{\infty} \) that is part of an equilibrium, \( \sum_{s=t}^{\infty} \prod_{r=t}^{s-1} \beta^s \rho_r \) is finite. Forward iteration of (46) thus yields (12), with \( s_t \in (0,1) \). Using (41), we then verify that \( c_t > 0, k_{t+1} > 0 \), and \( b_{t+1} > -h_t \). Finally, we verify that (41) solves the Bellman equation. Substituting (41) into (8) gives

\[
U(a_t(w_t + h_t)) = U((1 - s_t)(w_t + h_t)) + \beta U(a_{t+1}[E_t(w_{t+1} + h_{t+1})^{1-\gamma}]^{1/(1-\gamma)}).
\]

Dividing both sides by \( U(w_t + h_t) \) and using (44) and (46), the above reduces to \( a_t^{1-1/\theta} = (1 - s_t)^{-1/\theta}(1 - s_t) + a_{t+1}^{1-1/\theta}(1 - s_{t+1})^{1/\theta} s_t \), which is satisfied by (45).

**Proof of Condition (15).** To simplify notation, let \( r_{t+1}^i = r(A_t^i, \omega_{t+1}) + \tau_{t+1} = \mathbb{E}_t r_{t+1}^i \), and \( \sigma^2_{t+1} = \text{Var}_t[\ln r_{t+1}^i] \). A second-order Taylor approximation for \( \ln \rho_t \) around \( \sigma_t = 0 \) gives

\[
\ln \rho_t \approx \phi_t \mathbb{E}_t[\ln r_{t+1}^i] + (1 - \phi_t) \ln R_{t+1} + \frac{1}{2} \phi_t(1 - \phi_t) \sigma^2_{t+1} + \frac{1}{2} \frac{\gamma}{\sigma^2_{t+1}}.
\]

Since \( \phi_t \) maximizes \( \rho_t \), the above also implies

\[
\phi_t \approx \frac{\mathbb{E}_t[\ln r_{t+1}^i] - \ln R_{t+1} + \sigma^2_{t+1}/2}{\gamma \sigma^2_{t+1}}.
\]

(These two equations are the analogues of (2.24) and (2.25) in Chapter 2 of Campbell and Viceira (2002); see there for a detailed derivation.) Combining the two conditions above and using \( \mathbb{E}_t[\ln r_{t+1}^i] \approx \ln \mathbb{E}_t r_{t+1}^i - \text{Var}_t[\ln r_{t+1}^i]/2 = \ln \tau_{t+1} - \sigma^2_{t+1}/2 \) gives (15).

**Proof of Proposition 1.** Note that \( \phi_t \) and \( s_t \) are identical across agents. Aggregating the conditions in Lemma 2 over all \( i \) and using the facts that \( A_t^i \) and \( k_t^i \) are independent and that \( \Pi_t + \omega_t = \bar{r}(\omega_t) K_t + \omega_t = f(K_t) = Y_t \), we infer

\[
W_t = \Pi_t + RB_t + \omega_t = f(K_t) + RB_t \tag{49}
\]

\[
C_t = (1 - s_t)(W_t + H_t) \tag{50}
\]

\[
K_{t+1} = s_t \phi_t(W_t + H_t) \tag{51}
\]

\[
B_{t+1} = s_t(1 - \phi_t)(W_t + H_t) - H_t \tag{52}
\]

where \( B_t = \int b_t^i di \). The bond market clears if and only if \( B_t = 0 \) and therefore \( W_t = f(K_t) = Y_t \). Along with (50) and (51), this immediately gives (17) and (18). Next, adding up (50)-(52) gives the resource constraint (16), whereas (19) follows directly from (5). Finally, the labor market clears if and only if \( 1 = \int n_t^i = \bar{n}(\omega_t) K_t \), which gives (21).

**Proof of Proposition 2.** Evaluating (49)-(52) [equivalently, (16)-(21)] in the steady state and combining, we get

\[
K + H = s(W + H) = s(\bar{r}(\omega) K + RH) = s[\phi \bar{r}(\omega) + (1 - \phi) R](K + H),
\]

or equivalently

\[
1 = s[\phi \bar{r}(\omega) + (1 - \phi) R],
\]
which is simply the stationarity condition for aggregate wealth. Substituting \( s = \beta^\theta \rho^{\theta-1} \) into the above gives condition (22) in the Proposition. Next, by (??), (51), (52), and the property that, under A1, \( \bar{r}(\omega) = F_K(K,1,1) = f'(K) \) and \( \omega = f(K) - f'(K)K \), we have

\[
\frac{H}{K} = \frac{1 - \phi}{\phi} \quad \text{and} \quad H = \frac{\omega}{R-1} = \frac{f(K) - f'(K)K}{R-1}.
\]

Combining gives condition (23) in the Proposition. ■

**Proof of Proposition 3.** By (23), \( \phi \in (0,1) \). For that to be true, it must be that \( f'(K) > R \), or otherwise the bond would dominate private equity. By risk aversion, then, \( R < \rho < \phi f'(K) + (1 - \phi)R \), which together with (22) gives also \( \rho < 1/\beta \). Combining, we have

\[
R < \rho < \phi f'(K) + (1 - \phi)R < f'(K) \quad \text{and} \quad R < \rho < 1/\beta.
\]  

Next, taking logarithms of (22) and rearranging gives

\[
\theta \log[\beta f'(K)] = -\log[\phi + (1 - \phi)R/f'(K)] - (\theta - 1)\log[\rho/f'(K)].
\]

It follows that \( \beta f'(K) > 1 \) if and only if \( \theta > \theta_0 \) where

\[
\theta_0 = 1 - \frac{\log[\phi + (1 - \phi)R/f'(K)]}{\log[\rho/f'(K)]}.
\]  

Note that \( \theta_0 \) above is expressed in terms of endogenous variables, but (53) ensures that \( \theta_0 < 1 \). Next, presuming continuity of the steady state in \( \sigma \), consider the limit as \( \sigma \to 0 \). Letting stars indicate the steady-state values under complete markets, we have that \( R, \rho, f'(K) \to R^* = \beta^{-1}, K \to K^* = f^{-1}(1/\beta) \), and \( \phi \to \phi^* \), where

\[
\frac{1 - \phi^*}{\phi^*} = \frac{f(K^*) - f'(K^*)K^*}{(R^* - 1)K^*} = \frac{f(K^*)/K^* - R^*}{R^* - 1} \in (0,1).
\]  

Thus, using L’Hopital’s rule in (54), we have that \( \theta_0 \to 1 - (1 - \phi^*) = \phi^* \) as \( \sigma \to 0 \). Finally, note that GDP is given by \( \hat{f}(K) = f(K) - (1 - \delta)K \) and let \( \chi = \delta K/\hat{f}(K) \) and \( \alpha = \hat{f}(K)/\hat{f}(K) \) denote, respectively, the saving rate out of income and the income share of capital, both evaluated at the complete-markets steady state. Then (55) gives

\[
\phi^* = \frac{R^* - 1}{f(K^*)/K^* - 1} = \frac{\alpha - \chi}{1 - \chi} \leq \alpha,
\]

(with equality if and only if \( \alpha = 1 \) or \( \chi = 0 \)), which completes the proof. ■

**Proof of Proposition 4.** The budget constraint of household \( i \) in period \( t \) reduces to

\[
c^i_t + k^i_{t+1} + (x^i_{t+1} + b^i_{t+1}) \leq w^i_t = r(A^i_t, \omega_t)k^i_t + R_t(x^i_t + b^i_t) + \omega_t.
\]
Hence, Lemma 2 continues to apply provided we replace $b$ with $x + b$; that is,

$$c_t^i = (1 - s_t)(w_t^i + h_t)$$
$$k_{t+1}^i = s_t\phi_t(w_t^i + h_t)$$
$$x_{t+1}^i + b_{t+1}^i = s_t(1 - \phi_t)(w_t^i + h_t) - h_t$$

where $\phi_t$, $\rho_t$, and $s_t$ are defined again as in Lemma 2. Conditions (26), (27), (30), and (32) then follow from aggregating across agents and using the bond market clearing condition, as in Proposition 1. Finally, (29) follows from profit maximization in the public-equity sector and (31) from labor market clearing.

**Proof of Proposition 5.** We first prove that $\rho(\omega, R) = \varrho R$ and $\phi(\omega, R) = \varphi$, where $\varrho$ and $\varphi$ are given by (33). Under $A_1$, $n(A, \omega) = A\bar{n}(\omega)$, $r(A, \omega) = A\bar{r}(\omega)$, and $\bar{r}(\omega) = F_K(1, \bar{n}(\omega), 1)$. It follows that

$$\phi(\omega, R) = \arg \max_{\varphi} \{ \int [\varphi AF_K(1, \bar{n}(\omega), 1) + (1 - \varphi)R]^{1-\gamma} \psi(A) dA \}^{1/(1-\gamma)}$$
$$\rho(\omega, R) = \max_{\varphi} \{ \int [\varphi AF_K(1, \bar{n}(\omega), 1) + (1 - \varphi)R]^{1-\gamma} \psi(A) dA \}^{1/(1-\gamma)}$$

Under Assumption $A_2$, on the other hand, $R = G_K(1, l(\omega)) = F_K(1, \bar{n}(\omega), 1) / \mu$. Combining gives the result. We now prove the proposition.

(a) As in the one-sector case, stationarity of aggregate savings requires $s [\varphi \bar{r}(\omega) + (1 - \varphi)R] = 1$, where $s = \beta^\varrho \beta - 1$. Using $R = R(\omega)$ and $\bar{r}(\omega) = \mu R(\omega)$, we have $\rho = \varrho R$ and $[\varphi \bar{r}(\omega) + (1 - \varphi)R] = (\varphi \mu + 1 - \varphi)R$, and therefore the stationarity condition reduces to (34). This together with $R = R(\omega)$ gives a unique $R$ and a unique $\omega$. Next, in steady state, $K = \varphi s[W + H]$ and $X + H = (1 - \varphi)[W + H]$, and therefore $(X + H) / K = (1 - \varphi) / \varphi$. On the other hand, the clearing condition for the labor market gives $\bar{n}(\omega)K + l(\omega)X = 1$. Using $\bar{n}(\omega) = \mu l(\omega)$, and solving the above two conditions for $K$ and $X$, we get

$$K = \varphi [1 + l(\omega)H] / (\varphi \mu + 1 - \varphi)l(\omega)$$

or equivalently (35). This completes the characterization of the steady state. Uniqueness is obvious. As for existence, note that any $\mu > 1$ implies $\varphi > 0$ and therefore $K > 0$ necessarily. On the other hand, $X > 0$ if and only if $\varphi$ is sufficiently small, which is the case as long as $\sigma$ is sufficiently large.

(b) Since $R'(\omega) < 0$, $\omega$ decreases with $\sigma$ if and only if $R$ increases with $\sigma$. From condition (34),

$$\frac{d \ln R}{d \sigma} = -\frac{1}{\theta} \left[ (\theta - 1) \frac{d \ln \varrho}{d \sigma} + \frac{\mu - 1}{\varphi \mu + 1 - \varphi} \frac{d \varphi}{d \sigma} \right].$$

It follows that $dR/d\sigma > 0$, and therefore $d\omega/d\sigma < 0$, if and only if $\theta > \theta(\mu, \sigma, \gamma)$, where

$$\theta(\mu, \sigma, \gamma) = 1 - \frac{\mu - 1}{\varphi \mu + 1 - \varphi} \frac{d \varphi}{d \sigma} \frac{d \ln \varrho}{d \sigma},$$

and therefore

$$\frac{d \ln R}{d \sigma} = -\frac{1}{\theta} \left[ (\theta - 1) \frac{d \ln \varrho}{d \sigma} + \frac{\mu - 1}{\varphi \mu + 1 - \varphi} \frac{d \varphi}{d \sigma} \right].$$

This completes the proof.
Clearly, \( d\varphi/d\sigma < 0, d\ln g/d\sigma < 0 \), and therefore \( \bar{\theta} < 1 \). Next, since \( l'(\omega) < 0 \) and \( R'(\omega) < 0 \), from (35) we infer that \( K \) is increasing in \( \omega \) and decreasing in \( \varphi \), and \( X \) is increasing in \( \varphi \) but (possibly) non-monotonic in \( \omega \). Since

\[
K + X = 1/l(\omega) - (\mu - 1)\frac{1/l(\omega) + \omega/(R - 1)}{\mu + 1/\varphi - 1} = 1/l(\omega) - (\mu - 1)K,
\]

total capital \( K + X \) is increasing in \( \varphi \) but non-monotonic in \( \omega \). Finally, aggregate output is \( Y = F(1, \bar{n}(\omega), \bar{A})K + G(1, l(\omega))X \). Using \( \bar{n}(\omega) = \mu l(\omega) \) and \( F(K/\mu, N, A) = F(K, N, A/\mu) = G(K, N) \), we have \( F(1, \bar{n}(\omega), 1) = F(1, \mu l(\omega), 1) = G(1, l(\omega))\mu \) and therefore \( Y = G(1, l(\omega))[\mu K + X] \), which together with (35) gives \( Y = G(1/l(\omega), 1) \). Output thus increases with \( \omega \), reflecting the fact that \( \omega \) increases if and only if resources are shifted from less productive public equity to more productive private equity. By implication, \( Y/(K + X) \) and \( C = Y - (K + X) \) also increase with \( \omega \) and decrease with \( \varphi \). Hence, \( \theta > \bar{\theta} \) suffices for a higher \( \sigma \) to raise \( R \) and reduce \( \omega, K, Y/(N + L) \), and \( Y/(K + X) \). ■

**Proof of Lemma 3.** Let \( \omega_t \) denote the wage rate per effective unit of labor and take a given \((Z_t, K_t, X_t, H_{t-1})\). The labor-market clearing condition, \( \bar{n}(\omega_t)K_t + l(\omega_t)X_t = Z_t \), gives a unique \( \omega_t \). Next, let \( R_t = R(\omega_t), H_t = R(\omega_t)H_{t-1} - \omega_t Z_t \), and \( Y_t = f(\omega_t)K_t + g(\omega_t)X_t \), where \( f(\omega_t) \equiv F(1, \bar{n}(\omega_t), 1) \) and \( g(\omega_t) \equiv G(1, l(\omega_t)) \). Next, denote with \( \psi_t(A) \) the p.d.f. for the lognormal distribution \( \ln A \sim \mathcal{N}(-\sigma_t^2/2, \sigma_t^2) \), where \( \sigma_t = \sigma [1 - \eta \ln Z_t] \), and let \( \phi_t = \arg \max_\varphi \{ \int (\varphi A^\mu + 1 - \mu)^{1-\gamma} \psi_t(A) dA \}^{1/(1-\gamma)} \approx \mu/\{\gamma \sigma_t^2\} \). Finally, let \( C_t = (1 - \beta)[Y_t + H_t], K_{t+1} = \phi_t \beta [Y_t + H_t], X_{t+1} = (1 - \phi_t) \beta [Y_t + H_t] - H_t, \) and \( Z_{t+1} = Z_{t+1}^0 \), which completes the construction of \((Z_{t+1}, K_{t+1}, X_{t+1}, H_t)\) and \((C_t, Y_t, \omega_t, R_t)\). ■

**References**


