Multivariate Continuous Time Models through Copula

Shang Chan Chiou

1 Abstract

This paper proposes a new class of multivariate continuous time model. Different from the commonly-used multivariate normal model, our model splits the marginal behaviors from the dependence relation. Hence, it provides a very flexible structure in modeling multivariate financial assets. In particular, the marginal processes are characterized by a stochastic jump-diffusion model, and the dependence relation is modeled by a ”mixed” copula function. The innovation of this paper is threefold. First, it generalizes the basic stochastic volatility model to allow time-varying leverage effect and double jumps. Secondly, instead of a familiar parametric copula, we develop a mixed copula which best describes the dependence among multiple processes. Third, the dependence is further extended to a dynamic structure, instead of a fixed one.

2 Introduction

As the development of the financial markets, investors and institutions are inevitably facing multifarious financial instruments. Hence, there is a soaring thirsty for correct assessment of the complexities among multiple assets. This is extremely important to
banks and insurance companies since their portfolios usually include a variety of assets. For the long time, the study of multivariate time series has been dominated by few models, say, multivariate normal or t distributions. However, these models are seldom verified by empirical financial data. In particular, they are incapable of fully capturing the stylized features, say, extreme events appear in cluster. Furthermore, they restrict the association between financial variables to be linear. A new alternative is copula-based multivariate models, which go beyond the linear correlation to more advanced associations, including linear, nonlinear and tail dependence. Recently, a number of authors investigated the dependence structure among variables using copula, including Mendes and Souza (2004), Embrechts, Lindskog and McNeil (2001), Cherubini, Luciano and Vecchiato (2004) and Joe (1997), among others. The distinguishing property of copula is allowing the separation of marginal processes and the dependence relation. This separation, like the divide-and-conquer strategy, proves to be very helpful not only in modeling but also in estimation. One contribution of this paper is to construct the optimal copula which best describes the dependence among financial asset returns.

Linked by copula, the marginal elements also plays a important role in modeling. It is well known that financial time series often exhibit heavy tails, non-symmetry and random volatility. Many models have been developed to capture these features, say ARCH, GARCH, diffusion-jumps and regime-switching. Another contribution of this paper is to develop a comprehensive model for the marginal process, not only generating time-varying leverage effects but also including jumps in both returns and volatility.

This paper is organized as follows. In section 3, we set up the model and introduce some key concepts of copula which will be heavily used in this paper. In section 4,
we estimate the model by firstly estimating the marginal processes and then the copula function. The data for the marginal processes are S&P500 and DJIA, both ranging from January 3, 2000 to May 31, 2005. Section 5 concludes.

3 The model: Multivariate jump-diffusion through Copula

We are going to set each of the marginal processes as a jump-diffusion model and then link them by a copula. In particular, each margin is a 2-dimensional diffusion process satisfying the following stochastic differential equation.

$$dX_t = \mu(X_t; \theta)dt + \sigma(X_t; \theta)dW_t + \xi_t dN_t$$

where $X_t \equiv (Y_t, V_t)'$, $W_t = (W_t^y, W_t^v)'$ and $N_t = (N_t^y, N_t^v)$. $\mu$ and $\sigma$ are mean function and volatility function, respectively.

The association among the margins are represented by a copula. For example, if we have two marginal processes, the joint distribution function are:

$$P(Y_t^{(1)} \leq y_t^{(1)}, Y_t^{(2)} \leq y_t^{(2)}) = C[P(Y_t^{(1)} \leq y_t^{(1)}), P(Y_t^{(2)} \leq y_t^{(2)})]$$

Where $C$ is the copula function. The validity of this equation will be verified when we introduce Sklar’s theorem later on. To get an idea about the copula, we briefly review some important concepts of copulas.

3.1 Brief Introduction of Copula

The name ”Copula” was chosen to emphasize how a copula ”couples” a joint distribution to its marginal distributions. In what follows, we will introduce some important concepts
of copulas. Readers are referred to Nelsen (1999) for details.

Definition: A two-dimensional copula C is a real function defined on $[0, 1] \times [0, 1]$, with range $[0, 1]$. Furthermore, for every element $(u, v)$ in the domain, $C(u, 0) = C(0, v) = 0$, $C(u, 1) = u$ and $C(1, v) = v$. For every rectangle $[u_1, u_2] \times [v_1, v_2]$ in the domain such that $u_1 \leq u_2$ and $v_1 \leq v_2$, $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$.

Theorem (Sklar’s theorem) Let $F(x, y)$ be a joint cumulative distribution function with marginal cumulative distribution $F_1$ and $F_2$. There exists a copula $C$ such that for all real $(x, y)$,

$$F(x, y) = C(F_1(x), F_2(y))$$

If both $F_1$ and $F_2$ are continuous, then the copula is unique; otherwise, $C$ is uniquely determined on $(\text{range of } F_1) \times (\text{range of } F_2)$. Conversely, if $C$ is a copula and $F_1$ and $F_2$ are cumulative distribution functions, then $F(x, y)$ defined above is a joint cumulative distribution function with $F_1$ and $F_2$ as margins.

To describe the dependence structure of financial time series, we need some measures of dependence or association. Ideally, a dependence measure, denoted as $M$, should have the following desired properties:

1. $M(X, Y) = M(Y, X)$
2. $-1 \leq M(X, Y) \leq 1$
3. $M(X, Y) = 1$ iff $X, Y$ is comonotonic$^1$; $M(X, Y) = -1$ iff $X, Y$ is countermonotonic$^2$.

$^1$X, Y are comonotonic if $F(X, Y) = \min(F_1(X), F_2(Y))$, which is the Frechet upper bound.
$^2$X, Y are countercomonotonic if $F(X, Y) = \max(F_1(X) + F_2(Y) - 1, 0)$, which is the Frechet lower bound.
4. M is invariant under strictly increasing transformation. i.e. \( M(X,Y) = M(T_1(X), T_2(Y)) \),

where \( T_1 \) and \( T_2 \) are two (possibly different) strictly increasing transformation.

The widely used measure is Pearson coefficient (linear correlation coefficient, \( \rho \)), which measures the linear association between two variables. However, without the normality assumption, \( \rho \), may be problematic. In particular, linear correlation coefficient violates property 3 and 4. Indeed, as shown in Frechet (1957), the upper bound and lower bound of \( \rho \) may be different from \([-1, 1]\). Cherbini and Luciano (2002) also provide an example of log-normal marginal distributions. Furthermore, \( \rho \) is only invariant under "linear" strictly increasing transformation, but not under general strictly increasing transformation.

In addition, the variance of \( X \) and \( Y \) must be finite for \( \rho \) to be well-defined. This is not desired, especially when fat-tailed distributions are of interests. For example, bivariate t distribution with degree of freedom less than 3 does not have finite variance because of the fat tails. This is a serious problem because fat-tailed distributions are empirically supported by financial data.

Thus, \( \rho \) is not a suitable measure of dependency while nonlinear relationships are of interests. In the following, we quote two of the most commonly used non-parametric measures: the Kendall’s \( \tau \) and the Spearman’s \( \rho \). Given two random variable \( X \) and \( Y \), their Kendall’s \( \tau \) is defined as:

\[
\text{Kendall's } \tau = P\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - P\{(X_1 - X_2)(Y_1 - Y_2) < 0\}
\]

Where \((X_1, Y_1)\) and \((X_2, Y_2)\) are independent and identically distributed random vec-
tors, each with joint distribution as \((X, Y)\). Clearly, Kendall’s \(\tau\) measures the relative frequency with which a change in one of the variable is accompanied by a change in the same direction of the other variable.

In the other hand, Spearman’s \(\rho\) is defined as:

\[
\text{Spearman’s } \rho = \text{Pearson’s } \rho \text{ of } \{F_1(X), F_2(Y)\}
\]

Therefore, Spearman’s \(\rho\) attempts to remove the effect of the marginal distribution and measures the dependence of the transformed variables.

In the copula framework, we have

\[
\begin{align*}
\text{Kendall’s } \tau &= 4 \int \int_{[0,1] \times [0,1]} C(u, v) dC(u, v) - 1 \\
\text{Spearman’s } \rho &= 12 \int \int_{[0,1] \times [0,1]} C(u, v) dudv - 3
\end{align*}
\]

Where \(u = F_1(X)\) and \(v = F_2(Y)\).

It can be shown that Kendall’s \(\tau\) and Spearman’s \(\rho\) satisfy the aforementioned properties of a desired measure. We will use these two non-parametric measures to describe the association between variables.

In what follows, we introduce four widely-used copula. Their definition are summarized below:
1. Frank Copula

\[ C(u, v|\alpha) = \frac{1}{\alpha} \ln \left\{ 1 + \frac{(\exp(\alpha u) - 1)(\exp(\alpha v) - 1)}{\exp(\alpha) - 1} \right\}, \text{ where } \alpha \in (-\infty, \infty) \setminus \{0\} \]

Spearman’s \( \rho = 1 - \frac{12}{\alpha} [D_2(-\alpha) - D_1(-\alpha)] \)

Kendall’s \( \tau = 1 - \frac{4}{\alpha} [D_1(-\alpha) - 1] \)

where \( D_K \) is the so-called ”Debye” function, which is defined as \( D_K(y) \equiv \frac{K}{\alpha^2} \int_0^y \frac{t^K}{\exp(t) - 1} dt \)

2. Plackett Copula

\[ C(u, v|\alpha) = \begin{cases} \frac{1}{\pi(\alpha - 1)} \left\{ 1 + (\alpha - 1)(u + v) - \sqrt{[1 + (\alpha - 1)(u + v)]^2 - 4uv\alpha(\alpha - 1)} \right\}, & \text{if } \alpha \neq 1 \\ uv & \text{if } \alpha = 1 \end{cases} \]

Spearman’s \( \rho = \begin{cases} \frac{\alpha + 1}{\alpha - 1} - \frac{2\alpha}{(\alpha - 1)^2} \ln(\alpha), & \text{if } \alpha \neq 1 \\ 0 & \text{if } \alpha = 1 \end{cases} \)

No close form for Kendall’s \( \tau \)

3. Normal Copula

\[ C(u, v|\alpha) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi \sqrt{1 - \alpha^2}} \exp \left( \frac{2\alpha uv - u^2 - v^2}{2(1 - \alpha^2)} \right) dudv \]

Spearman’s \( \rho = \frac{6}{\pi} \arcsin \frac{\alpha}{2} \)

Kendall’s \( \tau = \frac{2}{\pi} \arcsin \alpha \)

4. Gumbel Copula

\[ C(u, v|\alpha) = \exp \left\{ -\left( |\log u|^\alpha + |\log v|^\alpha \right)^{\frac{1}{2\alpha}} \right\}, \text{ where } \alpha \in [1, \infty) \]

Kendall’s \( \tau = 1 - \frac{1}{\alpha} \)

No close form for Spearman’s \( \rho \)
In the following section, we will start to build the multivariate model, starting from
the marginal processes.

4 Estimation of the Model

We will adopt a 2-step estimation strategy. Firstly, each of the marginal processes is
estimated by the Bayesian method (MCMC). Secondly, given these estimated parameters
in the margins, we then estimate the dependence parameters in the copula function by
maximum likelihood method. In what follows, we explain the reasoning of this strategy.

The likelihood in a bivariate case is:

\[ L(x, y|\Theta) = \prod_{t=1}^{T} f(x_t, y_t|\mathcal{F}_{t-1}) = \prod_{t=1}^{T} f_1(x_t|\mathcal{F}_{t-1}) f_2(y_t|\mathcal{F}_{t-1}) C_{12}[F_1(x_t|\mathcal{F}_{t-1}), F_2(y_t|\mathcal{F}_{t-1})] \]

\[ \ln(L) = \sum_{t=1}^{T} \ln(f_1(x_t|\mathcal{F}_{t-1})) + \ln(f_2(y_t|\mathcal{F}_{t-1})) + \ln(C_{12}[F_1(x_t|\mathcal{F}_{t-1}), F_2(y_t|\mathcal{F}_{t-1})]) \]

where \( f(x, y) \) is the joint density function and \( \mathcal{F}_{t-1} \) denotes the information set in time \( t-1 \). \( f_1 \) is the marginal density of \( x \) and \( F_1 \) is the distribution function of \( x \). Similarly, \( f_2 \) is
the marginal density of \( y \) and \( F_2 \) is the distribution function of \( y \). The second equality of
the likelihood comes from Sklar’s theorem. In addition, \( C_{12} \) is defined as \( \partial C(u, v)/\partial u \partial v \). Ideally, one would like to maximize the above likelihood function simultaneously over
all parameters, including those in the marginal distribution and those in the copula
function. However, joint estimation may be a formidable task, not only because the
high dimension of the parameters space but also the dependency parameter in the copula
function may be a convoluted expression of the parameters. As it will become clear in
the following section, where a generalized stochastic volatility model is proposed, the
likelihood function is difficult to evaluate even only one marginal process is considered. Therefore, it is more practical to estimate the complete model in two steps. The first step is to estimate the marginal processes and second step is to estimate the parameters in copula function taking the parameters estimated by the first step as given. In fact, the two separate parts in the log likelihood function (one term involves the copula density and its parameter and the other term contains the parameters of the margins) have suggested the more feasible two-step estimation method.

4.1 The Marginal Distribution: Stochastic Volatility Models With Double Jumps and Time-Varying Leverage Effects

The goal in this section is to set up the specification for marginal process in the copula model. Following the seminal work of Engle (1982) and Bollerslev (1986), conditional heteroskedasticity has become a very popular way to model the dynamic process. The so-called ARCH class models are able to capture some stylized empirical phenomena, say, volatility cluster. Before long, the stochastic volatility (hereafter, SVOL) models offer an alternative to ARCH class models. Not only allowing for time-varying volatility, SVOL models also accommodate separate innovation processes in the mean and in the volatility. The basic SVOL models can be extended in several ways. For example, Jacquier, Polson, and Rossi (2004) study fat-tailed distribution of the conditional mean innovation and leverage effect. Also, Eraker, Johannes, and Polson (2003) incorporate jumps in mean and volatility processes. In other words, the basic SVOL with double
jumps can be written as follows:

\[
dY_t = \mu dt + \sqrt{V_t} dW_t^y + \xi_t^y dN_t^y \\
dV_t = \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dW_t^V + \xi_t^V dN_t^V
\]

Where \(\text{cov}(dW_t^y, dW_t^V) = \rho\), and \(N\) is the Possion process. It can be re-parameterized to take care of the leverage effects:

\[
dY_t = \mu dt + \sqrt{V_t} dB_t^{(1)} + \xi_t^y dN_t^y \\
dV_t = \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dB_t^{(1)} + \sqrt{1 - \rho^2} dB_t^{(2)} + \xi_t^V dN_t^V
\]

Jones (2003) considers time-varying leverage effects (but without jumps). His model can be written as:

\[
dY_t = \mu dt + \sqrt{V_t} dB_t^{(1)} \\
dV_t = \kappa(\theta - V_t) dt + aV_t^{\gamma_1^1} dB_t^{(1)} + bV_t^{\gamma_2^2} dB_t^{(2)}
\]

The time-varying correlation \((\rho_t)\) can be expressed as:

\[
\rho_t = \frac{aV_t^{\gamma_1}}{\sqrt{a^2V_t^{2\gamma_1^1} + b^2V_t^{2\gamma_2^2}}}
\]

Motivated by these authors, we present SVOL models with double jumps and time-varying leverage effects, namely,

\[
dY_t = \mu dt + \sqrt{V_t} dB_t^{(1)} + \xi_t^y dN_t^y \\
dV_t = \kappa(\theta - V_t) dt + aV_t^{\gamma_1} dB_t^{(1)} + bV_t^{\gamma_2} dB_t^{(2)} + \xi_t^V dN_t^V
\]
4.2 Estimation of The Marginal Processes

Markov Chain Monte Carlo (MCMC) methods are used to estimate the parameters as well as the latent variables (including $V$, $\xi_y$, $\xi^V$, and jumps). As mentioned earlier, likelihood function are hard to evaluate in such high dimensional cases, and MCMC may be promising for its efficiently and tractability. For the convenience of reference, we restate the model again.

$$dY_t = \mu dt + \sqrt{V_t} dB_t^{(1)} + \xi_y^y dN_t^y$$
$$dV_t = \kappa (\theta - V_t) dt + aV_t^{\gamma_1} dB_t^{(1)} + bV_t^{\gamma_2} dB_t^{(2)} + \xi^V dN_t^V$$

Where $Y$ is the log asset price. To perform the estimation, we discretize the SDE as (intervals are equally set to be 1):

$$Y_{t+1} - Y_t = \mu + \sqrt{V_t} \varepsilon_t^{(1)} + \xi_y^y J_{t+1}^y$$
$$V_{t+1} - V_t = \kappa (\theta - V_t) + aV_t^{\gamma_1} \varepsilon_t^{(1)} + bV_t^{\gamma_2} \varepsilon_t^{(2)} + \xi^V J_{t+1}^V$$

Where $Y_{t+1} - Y_t$ is the log return on time $t+1$. In order to get the posterior distributions, we need to specify the relation between jump components. Two assumptions are commonly used in the literature. One assumes jumps in returns, $\xi_y^y \sim N(\mu_y, \sigma_y^2)$ and jumps in volatility, $\xi^V \sim \exp(\mu_v)$, arrive independently [i.e. $J_{t}^y \sim Bernoulli(\lambda_y)$ and

---

$^3$It may be natural to think the possibility of applying MCMC to estimate the whole model, instead of two-step estimation. However, the conditional distribution of dependency parameters given data and other parameters in the marginal processes is unknown unless additional assumption were imposed on the model.

$^4$12 parameters and $4 \times T$ latent variables are involved in the marginal process if we assume the $N_t^y = N_t^V$; otherwise, we have T extra latent variables
$J_t^V \sim Bernoulli(\lambda^V)$. The other assumes they arrive at the same time \( i.e. J_t^y = J_t^V = J_t \sim Bernoulli(\lambda) \) and the jump sizes are correlated, namely, $\xi^V \sim Exponential(\mu_v)$ and $\xi_t^y|\xi_t^V \sim Normal(\mu_y + \rho_J \xi_t^V, \sigma^2_y)$.

For demonstration purpose, we only explain the MCMC setting in the case of independent jumps. The priors are: $\mu \sim Normal$, $\kappa \theta \sim Normal$, $\kappa \sim Normal$, $\lambda^y \sim Beta$, $\lambda^V \sim Beta$, $\mu_y \sim Normal$, $\sigma^2_y \sim Inverse\ Gamma$, and $\mu_v \sim Gamma$.

Let $\Theta \equiv \{\mu, \lambda^y, \lambda^V, \sigma^2_y, \mu_y, \mu_v, \kappa, \theta, \gamma^1, \gamma^2, a, b\}$; $V \equiv \{V_t, t = 0 \ldots T - 1\}$; $\xi^V \equiv \{\xi_t^V, t = 1 \ldots T\}$; $\xi^y \equiv \{\xi_t^y, t = 1 \ldots T\}$; $J^y \equiv \{J_t^y, t = 1 \ldots T\}$; $J^V \equiv \{J_t^V, t = 1 \ldots T\}$; $J \equiv \{J^y, J^V\}$, $Y \equiv \{Y_t, t = 0 \ldots T\}$ (Note there is only T log return data, from 1 to T).
The Gibbs sampler is outlined as below.

\[
P(\mu | \Theta^-, Y, V, \xi^y, \xi^V, J) \sim \text{Normal}
\]
\[
P(\lambda^y | \Theta^-, Y, V, \xi^y, \xi^V, J) \sim \text{Beta}
\]
\[
P(\lambda^V | \Theta^-, Y, V, \xi^y, \xi^V, J) \sim \text{Beta}
\]
\[
P(\sigma^2_y | \Theta^-, Y, V, \xi^y, \xi^V, J) \sim \text{Inverse Gamma}
\]
\[
P(\mu_y | \Theta^-, Y, V, \xi^y, \xi^V, J) \sim \text{Normal}
\]
\[
P(\mu_v | \Theta^-, Y, V, \xi^y, \xi^V, J) \sim \text{Gamma}
\]
\[
P(\kappa \theta, \kappa | \Theta^-, Y, V, \xi^y, \xi^V, J) \sim \text{Normal}
\]
\[
P(\gamma_1, \gamma_2 | \Theta^-, Y, V, \xi^y, \xi^V, J) \sim \text{Random Walk Metropolis}
\]
\[
P(a | \Theta^-, Y, V, \xi^y, \xi^V, J) \sim \text{Normal}
\]
\[
P(b | \Theta^-, Y, V, \xi^y, \xi^V, J) \sim \text{Inverse Gamma}
\]
\[
P(\xi^V_t | \Theta, Y, V, \xi^y, J^V_t = 1) \sim \text{Truncated Normal}
\]
\[
P(\xi^V_t | \Theta, Y, V, \xi^y, J^V_t = 0) \sim \text{Exponential}
\]
\[
P(\xi^V_t | \Theta, Y, V, \xi^y, J^V_t = 1) \sim \text{Normal}
\]
\[
P(\xi^V_t | \Theta, Y, V, \xi^y, J^V_t = 0) \sim \text{Normal}
\]
\[
P(J^V_t | \Theta, Y, V, \xi^y, \xi^V) \sim \text{Bernulli}
\]
\[
P(J^y_t | \Theta, Y, V, \xi^y, \xi^V) \sim \text{Bernulli}
\]
\[
P(V_t | \Theta, Y, V_{t+1}, V_{t-1}, \xi^y, \xi^V, J) \sim \text{Random Walk Metropolis}
\]

Where \( \Theta^- \) can be understood as some subset of \( \Theta \), with the parameters under consid-
We use daily log returns of Standard and Poor’s 500 and Dow Jones Industrial Average as the marginal processes. The time period ranges from January 3, 2000 to May 31, 2005. After excluding weekends and holidays, we get a paired sample with 1358 observations. Some statistics of the data are shown in table 1 and the estimation results are shown in table 2.

As described in Eraker, Johannes and Polson (2003), the posterior of residuals can provide a diagnostics of the model. The residuals in our model is:

\[ \varepsilon^{(1)}_t = \frac{Y_{t+1} - Y_t - \mu - \xi^y_{t+1} F^y_{t+1}}{\sqrt{V_t}} \approx N(0,1) \]

Since we discretize the original model, these residuals need not be exactly distributed as standard normal, but they should be close to standard normal. Figure 1 shows the QQ plot of S&P500 and DJIA. The diagnostics shows no evidence against our model.

The time-varying leverage effect are demonstrated in figure 2. The average of \( \rho_t \) is -0.49 for S&P500 and -0.45 for DJIA. Our results are similar to those obtained by Bates (2000) and Pan (2002) who use option data to conduct the estimation, and similar to the results obtained by Eracker, Johannes and Polson (2003) in their SVIJ model. However, our average of \( \rho_t \) is smaller than those in Jones (2003) (His estimation of \( \rho_t \) is around...
Table 2: Parameter Estimation Results For The Two Equity Indices. Independent jumps are assumed. The standard deviation for the posterior is in the parentheses.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>S&amp;P500</th>
<th>DJIA</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-3.7627 (0.05181)</td>
<td>-3.6492 (0.05133)</td>
</tr>
<tr>
<td>b</td>
<td>0.8712 (0.1013)</td>
<td>0.0146 (0.0071)</td>
</tr>
<tr>
<td>γ¹</td>
<td>1.4965 (0.3876)</td>
<td>1.478 (0.1228)</td>
</tr>
<tr>
<td>γ²</td>
<td>1.2702 (0.2515)</td>
<td>0.792 (0.04128)</td>
</tr>
<tr>
<td>μ</td>
<td>-0.0001908 (0.0002736)</td>
<td>-0.00011489 (0.0002647)</td>
</tr>
<tr>
<td>θ</td>
<td>1.6503×10⁻⁴ (2.3661×10⁻⁵)</td>
<td>1.2032×10⁻⁴ (7.2852×10⁻⁵)</td>
</tr>
<tr>
<td>κ</td>
<td>0.004987 (0.001451)</td>
<td>0.001904 (0.000752)</td>
</tr>
<tr>
<td>λ⁹</td>
<td>0.008735 (0.004621)</td>
<td>0.006971 (0.003881)</td>
</tr>
<tr>
<td>λ¹⁰</td>
<td>0.001595 (0.001121)</td>
<td>0.001602 (0.001137)</td>
</tr>
<tr>
<td>µᵥ</td>
<td>0.0001620 (3.2×10⁻⁵)</td>
<td>0.0001768 (2.83×10⁻⁵)</td>
</tr>
<tr>
<td>σᵧ</td>
<td>0.03039 (0.006762)</td>
<td>0.03227 (0.006598)</td>
</tr>
<tr>
<td>µᵧ</td>
<td>-0.01581 (0.01659)</td>
<td>-0.03594 (0.01872)</td>
</tr>
<tr>
<td>mean(ρₜ)</td>
<td>-0.4906</td>
<td>-0.4561</td>
</tr>
</tbody>
</table>
Figure 1: QQ plot of the model under independent-jump assumption
-0.7). The estimated daily jump intensity in returns, \( \lambda_y \), is around 0.007, which means jumps arrive at a rate of about 1.7 per year, but the jump intensity in volatility is much smaller. In turn, the proportion of total variance of returns coming from jumps\(^5\) is 2.24% for S&P500 and 2.08% for DJIA.

The average annualized volatility, \( \sqrt{252} \theta \), is 20.39% for S&P500 and is 17.41% for DJIA. Parameter \( \mu_v \) indicates how much the volatility will increase when a jump arrives. For example, assume volatility is at their average annualized level, a average jump size increase volatility to 28.71% for S&P500 and to 27.36% for DJIA.

The estimation shows negative shock in returns are more common when a jump arrives since \( \mu_y \) are negative in both series. The estimated \( \kappa \), about 0.005 for S&P 500 and about 0.002 for DJIA, indicates volatility is highly persistent for both series. In addition, the magnitude of \( \kappa \), which is smaller than that of Eracker et al, shows volatility mean reverts slower after year 2000.

### 4.3 Estimation of Copula Function

It has been known from the work of Sklar (1959) that any multivariate distribution function can be uniquely factored into a copula and its margins. However, Sklar’s theorem does not specify which copula should be used. Clearly, the suitable parametric copula is dependent on the data and the marginal specification. Although one can take a nonparametric approach, it is generally not suitable for the purpose of forecasting. In the other hand, a parametric approach has a distinct advantage, as Goorbergh and Genest (2004) said, that we can easily verify the robustness of the conclusion by repeating

\(^5\)this ratio is calculated as \( \frac{E[|\xi_t|^2]|_\lambda}{\sqrt{V+E[|\xi_t|^2]|_\lambda}} \), according to Eraker, Johannes and Polson(2003).
Figure 2: time-varying leverage in both indices
the analysis for as many different copula families as desired. To mitigate the problem of model misspecification in parametric copula, we employ two goodness-of-fit tests by Chen, Fan and Patton (2004) to choose the copula function. The outline of the tests is repeated here for the sake of completeness.

The basic idea under these two tests are illuminated by Rosenblatt (1952); that is, the copula under consideration is the correct one if and only if the probability integral transformed random variables are independent and identically distributed as a Uniform (0,1) random variables. In the bivariate case, we are interested in testing the null hypothesis:

\[ H_0 : P \{ C(u, v) = C_0(u, v|\alpha_0) \} = 1 \]

against the alternative hypothesis:

\[ H_1 : P \{ C(u, v) = C_0(u, v|\alpha_0) \} < 1 \]

The probability integral transformed random variables are defined as:

\[ U_1 \equiv u \]
\[ U_2 \equiv C_0(v|u; \alpha_0) = \frac{\partial C_0(u, v|\alpha_0)}{\partial u} / \frac{\partial C_0(u, 1|\alpha_0)}{\partial u} \]

\( U_1 \) and \( U_2 \) should be independently and identically distributed as a Uniform[0,1] under the null hypothesis. The intuition is that conditional on \( u \), \( v|u \) has used up all information from \( u \); therefore, \( U_2 \) are independent of \( U_1 \). In other words, the null hypothesis is equivalent to \( H_0 : P \{ g(U_1, U_2) = 1 \} = 1 \), where \( g \) is the joint density function of \( U_1 \) and \( U_2 \).

\( U_1 \) and \( U_2 \) can be computed as \( \hat{F}(X) \) and \( \hat{F}(Y) \), respectively, where \( \hat{F}(X) \) and \( \hat{F}(Y) \) come from the estimation of the marginal processes of \( X \) and \( Y \).
Table 3: Parameter Estimation and Test Results of Several Copula Functions.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Frank</th>
<th>Plackett</th>
<th>Normal</th>
<th>Gumbel</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-16.4604$</td>
<td>$87.7596$</td>
<td>$0.9410$</td>
<td>$4.4648$</td>
</tr>
<tr>
<td></td>
<td>$(0.932)$</td>
<td>$(2.2077)$</td>
<td>$(0.0162)$</td>
<td>$(0.1026)$</td>
</tr>
<tr>
<td>Spearman $\rho$</td>
<td>$0.9401$</td>
<td>$0.9187$</td>
<td>$0.9356$</td>
<td>$0.9292$</td>
</tr>
<tr>
<td>Kendall $\tau$</td>
<td>$0.7813$</td>
<td>$0.4437$</td>
<td>$0.7802$</td>
<td>$0.7760$</td>
</tr>
<tr>
<td>Test 1</td>
<td>$6.5235$</td>
<td>$1.8112$</td>
<td>$3.0608$</td>
<td>$2.9007$</td>
</tr>
<tr>
<td>Test 2</td>
<td>$0.8974$</td>
<td>$9.5808$</td>
<td>$4.5752$</td>
<td>$1.4958$</td>
</tr>
</tbody>
</table>

In Chen, Fan and Patten(2004), test 1 constructs a test statistics which converges to Normal $(0,1)$. The test statistics is based on the joint density $g$, which is estimated by a kernel function. Test 2 also constructs a test statistics which converges to Normal $(0,1)$, but the derivation is via the $\chi^2$ distribution. Both test 1 and test 2 are two-tailed tests, which allow us to judge whether the correct copula function is chosen.

Four widely-used copula are proposed and tested: Frank, Plackett, Normal and Gumbel copula. We use maximum likelihood methods to estimate the parameter in the copula (taking the parameters estimated from marginal processes as given). Note in the bivariate case, only one parameter needs to be estimated. As shown in table 3, all parameters are significant. The nonparametric measures are calculated either by close form formula or by numerical integral. All Spearman $\rho$ are close to sample correlation (0.9434). Another nonparametric measure, Kendall’s $\tau$ is about 0.76. After the parameters are determined, we are ready to perform the goodness-of-fit tests. The results are also summarized in table 3.
Table 4: Parameter Estimation of Mixed Copula

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$p$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>0.3243</td>
<td>-30.9102</td>
<td>3.9215</td>
</tr>
<tr>
<td>Standard Error</td>
<td>0.0777</td>
<td>0.6063</td>
<td>0.3289</td>
</tr>
<tr>
<td>test 1</td>
<td>1.8198</td>
<td></td>
<td></td>
</tr>
<tr>
<td>test 2</td>
<td>0.9189</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The numbers beside test 1 and test 2 are the values of test statistics, which is distributed as $\text{Normal}(0,1)$ under null hypothesis. As we can see from the table, almost all proposed copulas are rejected in Test 1 except Plackett copula (under 5% significant level). Only Frank and Gumbel copula pass test 2. The concern is, as mentioned in Chen et al (2004), test 2 may have no power against certain deviation from the null model (most of the time, test 2 are equally powerful as test 1). In order to work out a more suitable model, we consider a mixed copula as follows: (and the results is encouraging!)

$$p \times \text{Frank}(\alpha_1) + (1 - p) \times \text{Gumbel}(\alpha_2)$$

Where $p$ is the probability for Frank Copula and $1 - p$ is for Gumbel Copula. $\alpha_1$ and $\alpha_2$ are the dependence parameters in Frank and Gumbel copula, respectively. The estimation results are shown in table 4. Also, figure 3 shows the density and contours of the mixed copula and its components.

Clearly, the density and contours concentrate on the upper and lower quadrant tails in the mixed copula. It indicates that S&P500 and DJIA are likely to move together when extreme events happen. The long and narrow shaped contours in Frank component
Figure 3: The density and contours of Frank Copula (upper), Gumbel Copula (middle) and Mixed Copula (bottom)
also confirm the high correlation between these two series. Figure 3 also indicates how
the mixed copula works: Although the Frank Copula already exhibits strong tendency
for the two series to move together, the tail dependence is not sufficiently captured. This
is where the Gumbel copula comes into play. A carefully-picked ratio combines these
two desired properties and best describes the association between S&P500 and DJIA.

Until now, we assume the dependence parameter in copula is a constant or two pos-
sible values if mixed. It amounts to mean the dependence between the two marginal
processes remains unchanged during the sampling period. However, it is more plausible
that they have a dynamic relation. To materialize this idea, we propose two specification
for the dependency parameter in the copula. Firstly, we set $\alpha_t$ in the Frank copula to be:

$$\alpha_t \text{(Frank Copula)} \mid \mathcal{F}_{t-1} = c_0 + c_1 \alpha_{t-1} + c_2 \alpha_{t-2} + c_3 \alpha_{t-3} + c_4 \max(\ln(|S&P500_{t-1}|), \ln(|DJIA_{t-1}|))$$

Secondly, we set $\alpha_t$ in Gumbel copula to be:

$$\alpha_t \text{(Gumbel Copula)} \mid \mathcal{F}_{t-1} = d_0 + d_1 \alpha_{t-1} + d_2 \alpha_{t-2} + d_3 \alpha_{t-3} + d_4 |S&P500_{t-1}| + d_5 |DJIA_{t-1}|$$

These specifications are motivated by the empirical facts that financial prices tend to
move together when market have large innovation. In addition, any persistence of the
dependence measure is captured by lag terms of $\alpha$. Estimation results are reported in
table 5 and table 6. As shown in the tables, most parameters are significant (except
for some lag terms) and both specifications pass test 2. The estimation suggests that
the dependence in Frank copula is very persistent as $c_1$ is as high as 0.86. The dynamic
dependence measures (Spearman $\rho$ in Frank copula) between S&P500 and DJIA are
demonstrated in figure 4. The average level is also close to the sample correlation.

Actually, there are a lot of possible specifications of the dependence parameters.
Meaningful explanatory variables and reasonable lags of $\alpha$ can do the trick. This exercise
Table 5 : Parameter Estimation of Dynamic Frank Copula

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$c_0$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>4.7491</td>
<td>0.8599</td>
<td>0.03</td>
<td>0.01</td>
<td>1.334</td>
</tr>
<tr>
<td>Standard Error</td>
<td>0.04522</td>
<td>0.0418</td>
<td>0.021</td>
<td>0.015</td>
<td>0.536</td>
</tr>
<tr>
<td>test 1</td>
<td>5.3013</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>test 2</td>
<td>-0.1063</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6 : Parameter Estimation of Dynamic Gumbel Copula

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$d_0$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>1.3069</td>
<td>1.9213</td>
<td>-1.7166</td>
<td>0.5097</td>
<td>0.0184</td>
<td>-2.9479</td>
</tr>
<tr>
<td>Standard Error</td>
<td>0.07436</td>
<td>0.0519</td>
<td>1.125</td>
<td>1.361</td>
<td>0.00927</td>
<td>1.325</td>
</tr>
<tr>
<td>test 1</td>
<td>3.286</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>test 2</td>
<td>1.1889</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

shows the copula approach can be extended to accommodate time-varying dependence.

5 Conclusion

We have proposed a new class of multivariate continuous time models. Each marginal process is described by stochastic volatility model with jumps in both returns and in volatility. Moreover, motivated by Jones (2003), we modify the volatility dynamic to accommodate the time varying leverage effects. The margins are further linked by copula. We prefer parametric copula to nonparametric copula since the former provide
Figure 4: Spearman $\rho$ in Frank Copula
analytical tractability which is not shared by the later. We employ the goodness-of-fit test of Chen, Fan and Patton (2004) to pick up the suitable parametric copula. Since there is no "traditional" copula can completely pass the tests, we further work out a mixed copula. Not only studying the copula with fixed dependence parameter, we also extend it to allow dynamic structure.

Marginal processes are estimated by MCMC and the copula function is estimated by maximum likelihood methods given the (estimated) parameters in the margins. MCMC methods are suitable to deal with complicated models especially when latent variables are included. The estimation results under independent jumps are reported in the body of the text. We also estimate the marginal processes under the assumption of contemporaneous jumps and correlated jump sizes; however, the modeling check (QQ plot) is not so good. Therefore, we keep the assumption of independent jumps while estimating copula functions.

In sum, this paper presents a very general and flexible approach to model multivariate financial series. Further research will include several applications of this model. For example, derivative pricing, portfolio selection, and risk management.
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