Designing Stress Scenarios*

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April 12, 2021

Abstract

We develop a tractable framework to study the optimal design of stress scenarios. A risk-averse principal (e.g., a manager, a regulator) seeks to learn about the exposures of a group of agents (e.g., traders, banks) to a set of risk factors. The principal asks the agents to report their outcomes (e.g., credit losses) under a variety of scenarios that she designs. She can then take remedial actions (e.g., mandate reductions in risk exposures). The principal’s program has two parts. For a given set of scenarios, we show how to apply a Kalman filter to solve the learning problem. The optimal design is then a function of what she wants to learn and how she intends to intervene if she uncovers excessive exposures. The choice of optimal scenarios depends on the principal’s priors about risk exposures, the cost of ex-post interventions, and the potential correlation of exposures across and within agents.

JEL Classification: G2, D82, D83

Keywords: stress test, information, bank regulation, filtering, learning

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*We thank our discussants Itay Goldstein, Florian Heider, Til Schuermann, and Jing Zeng, and Mitchel Berlin, Thomas Eisenbach, Piero Gottardi, Anna Kovner, Ben Lester, Igor Livshits, Tony Saunders, Chester Spatt, Pierre-Olivier Weill, and Basil Williams for their comments. We would also like to thank seminar participants at the NBER Summer Institute, AFA, EFA, SED, NYU, FRB of New York, FRB of Philadelphia, FRB of Boston, University of Wisconsin, Boston College, and the Stress Testing Conference. Abhishek Bhardwaj, Ki Beom Lee, and Luke Min provided excellent research assistance.

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1 Introduction

Stress tests are ubiquitous in risk management and financial supervision. Risk officers use stress tests to set and monitor risk limits within their organizations, and financial regulators around the world use stress tests to assess the health of financial institutions. To give just a few examples: financial firms use stress tests to complement their statistical risk management tools (e.g., Value at Risk); asset managers stress test their portfolios; trading venues stress tests their counter-party exposures; regulators mandate large scale stress tests for banks and insurance companies and use the results to enforce capital requirements and validate dividend policies.\(^1\)

Despite the growing importance of stress testing and the amount of resources devoted to them, there is little theoretical guidance on exactly how one should design the stress scenarios. A theoretical literature has focused on the trade-offs involved in the disclosure of supervisory information (see Goldstein and Sapra, 2014 for a review), which range from concerns about the reputation of the regulator (Shapiro and Skeie, 2015) to the importance of having a fiscal backstop (Faria-e-Castro et al., 2017). Though these papers provide insights as to what to do with the results of asset quality reviews, they do not model stress testing. They are silent about the design of forward-looking scenarios, which are the hallmarks of stress testing.

The goal of our paper is to start filling this void. Stress tests are used for risk management. Risk management always contains (at least) two parts: risk discovery (learning) and risk mitigation (intervention). Stress tests are thus always embedded in a risk management framework. Stress tests belong to the risk discovery phase but we cannot think about the design of the test without analyzing the remedial actions that will be taken based on the results of the test. We therefore model for the risk discovery stage and the risk mitigation stage. In the particular case of stress tests used for capital adequacy in banking there is often a quasi-mechanical link between the test and the remedial actions since banks are required to maintain a pre-specified level of capital buffer across all scenarios. We do not assume such a mechanical mapping. We assume instead that regulators choose optimal actions conditional on the results of the test. This gives them complete freedom to design the most informative scenarios.\(^2\)

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\(^1\)Central banks in the United States, Europe, England, Brazil, Chile, Singapore, China, Australia, and New Zealand, as well as the International Monetary Fund in Japan, have recently used stress tests to evaluate the banking sector’s solvency and guide banking regulation.

\(^2\)Our results shed light on risk exposures and capital ratios, but we do not specify a mechanical link between passing the test and having a particular level of equity. To do so one would need to take a stand on many
We model stress testing as a learning mechanism. We consider a principal and a potentially large number of agents. The agents can be traders within a financial firm, or they can be financial firms within a financial system. The principal can be a regulator designing supervisory tests, or a risk officer running an internal stress test. For concreteness we will use the supervisory stress testing example in much of the paper. Banks are exposed to a set of risk factors, but their exposures to aggregate factors are unknown to the regulator. By exposure we mean the relevant elasticity that determines the loss of a position under a given scenario. An exposure is therefore not the same as the book or market value of a position. Banks and regulators usually agree on the nominal size of positions and on the market value of liquid portfolios. They can disagree about the value of illiquid positions, and in all cases, liquid or not, the impact of a scenario on the loss on that position needs to be estimated. What we call “exposure” combines the position (measured with near certainty) with its value under stress scenarios (computed with error). The regulator is risk averse and worries about the financial system experiencing large losses in some states of the world. The regulator then designs a set of scenarios and asks the banks to report what their losses would be under these scenarios. From their responses, the regulator learns about the underlying exposures of the banks. Based on this information, the regulator can decide to intervene, i.e., she can ask a set of banks to reduce their exposures to some factors.

Our main insight comes from writing the learning problem as a Kalman filter. The filter gives us a mapping from prior beliefs and test results into posterior beliefs. The precision of the mapping depends on the scenarios in the stress test. We can then formulate the regulator’s problem as an information acquisition problem in which the regulator chooses the precision of her signals. By features of financial regulations that are not central to our analysis. For instance, imagine that a bank needs the same level of ex-ante equity to satisfy a 9% capital requirement after scenario 1 or a 7% requirement after scenario 2 (presumably because scenario 2 embodies a higher degree of stress). As far as ex-ante capital adequacy is concerned, these two regulations are equivalent. The law sometimes mandates one of these numbers, in which case our model can shed light on the other, but in general the “level” of the ratio and the “degree” of the stress are not independent and a model is likely to pin down a combination as opposed to a particular value for each. A scenario used for “pass/fail” also needs to be plausible in a way that a learning scenario does not have to be. When stress test results are mechanically linked to capital requirements the choice of scenarios can be used to increase the effective requirement (if one assumes that, for some reason, the baseline requirement is too weak) or to implement counter-cyclical ratios (keeping the level of stress constant as the economy improves leads to larger assumed shocks). Finally, passing the stress test means that regulators deem the institution safe and sound even in the stress scenario, and thus that lender-of-last-resort policies would be appropriate.
explicitly considering the structure of the signals generated by the stress tests, we can map the feasible set of precision choices to the primitive parameters of the model, such as the priors of the regulator regarding the banks’ exposures. If, for instance, the regulator is worried about a particular risk factor, we can derive the stress test that maximizes learning about the exposures to this risk factor.

Will the regulator focus a particular risk factor or will she try and learn about several factors at the same time? We show that the answer depends on her prior beliefs about the banks’ risk exposures and on the marginal cost of intervention. The regulator can choose to mandate a broad risk reduction. This is likely to involve a lot of unnecessary changes and disruptions, but it does not require much information. If the cost of intervention is high, this strategy is not efficient and the regulator will want to learn in order to avoid unnecessary interventions. She can learn by choosing a more extreme scenario along a particular risk dimension, but extreme scenarios lead to less precise answers. The reduction in overall information quality depends on the prior distribution of the risk exposures. Whether or not there is specialization in learning depends therefore on the intervention cost function and on the regulator’s prior beliefs.

More generally, the costs of intervention and the prior beliefs of the regulator are central in determining the optimal scenario design. The effect of intervention costs on the optimal scenario is not monotone. On the one hand, a higher intervention cost along a particular risk dimension makes accurate interventions more appealing. Through this channel, learning about the exposures to factors with higher intervention costs is more valuable. On the other hand, learning about the risk exposures is valuable partly because it allows the regulator to intervene to reduce the risk exposures. This mechanism makes the regulator’s intervention less responsive to the information in the stress test when intervention costs are higher, and hence, make learning less valuable. When the intervention cost is low enough, the first effect dominates and higher intervention costs along a risk dimension increase the weight of that risk factor in the optimal stress scenario. As the intervention cost increases, the second effect becomes stronger. When the intervention cost is large enough, the second effect dominates and the weight of the risk factor in the optimal stress scenario decreases with the cost to intervene along that risk dimension.

The regulator’s expected risk exposure to a risk factor also has opposing effects on the optimal stress scenario. First, a higher expected risk exposure increases the value of intervening along that risk dimension and makes accurate interventions more valuable. This effect pushes the regulator towards more learning (and stressing) about the exposure to risk factors with higher expected
risk exposures. Second, the responsiveness of the regulator’s belief, and hence of her intervention policy, to the information contained in the stress test results depends on the level of the prior mean exposure. When the prior mean exposure to a particular factor is low (high), the regulator’s beliefs about it are more (less) sensitive to new information. This effect makes learning about factors with higher expected risk exposures less valuable and dominates the first effect when the expected risk exposure is high. Hence, the weight of a factor in the stress scenario is hump-shaped with respect to its expected risk exposure. In the limit, when the expected risk exposure to a factor is high enough, the regulator’s intervention policy becomes almost independent of the signal about it and the regulator chooses not to stress that risk factor at all.

The regulator’s prior uncertainty, either about the risk exposures or about the risk factors themselves, also shapes the optimal stress scenario. Higher uncertainty about the risk exposure to a particular factor increase the regulator’s willingness to intervene along that dimension. Therefore, the regulator finds it optimal to learn more about exposures to risk factors about which she is more uncertain about. On the other hand, higher uncertainty about the risk factor itself makes the regulator less willing to learn about the risk exposures to it. High uncertainty about a risk factor makes the regulator’s intervention policy less sensitive to new information about the exposure to that factor. In this case, the regulator puts less weight in the stress scenario on risk factors about which she is more uncertain.

Our model also sheds light on the role of factor with correlated risk exposures, within or across banks. When the exposures to two factors within a bank are correlated, learning about the exposure to one factor also provides information about the exposure to the other factor. Hence, the regulator stresses more the factors with correlated exposures and may focus only on these factors if the correlation is high enough. However, due to the convexity of the information acquisition cost, the regulator’s specialization is usually incomplete and she tends to put weight on all factors.

Similarly, the regulator optimally puts more weight on systemic factors in stress scenarios. When different banks have correlated exposures to a (systemic) factor, learning about one bank’s exposure to this factor provides information about the other banks’ exposures. The higher the expected correlation between the exposures to the systemic factor, the more precise the information about the banks’ exposure to it and the less costly it becomes for the regulator to learn about it. Hence, the regulator optimally stresses systemic factors more and may choose to specialize and stress only these factors when they are systemic enough (the banks’ exposures to it are sufficiently correlated).
For expositional purposes we develop our insights in the context of a small number of banks and few independent macroeconomic factors. However, our framework can accommodate correlated macro factors, including non-linear combinations of factors, arbitrary and general correlation structures for risk exposures across and within banks, and different preferences for the regulator. Moreover, the linear structure allows it to be scalable and very easy to extend to a large number of banks, scenarios and risk factors and makes it easy to use as a building block in more complex stress test models.

**Literature Review**

Most of the literature on stress tests focuses on banking. Several recent papers study specifically the trade-offs involved in disclosing stress test results. Goldstein and Leitner (2018) focus on the Hirshleifer (1971) effect: revealing too much information destroys risk-sharing opportunities between risk neutral investors and (effectively) risk averse bankers. These risk-sharing arrangements also play an important role in Allen and Gale (2000). Shapiro and Skeie (2015) study the reputation concerns of a regulator when there is a trade-off between moral hazard and runs. Faria-e-Castro et al. (2017) study a model of optimal disclosure where the government trades off Lemon market costs with bank run costs, and show that a fiscal backstop allows government to run more informative stress tests. Schuermann (2012) analyzes the design and governance (scenario design, models and projection, and disclosure) for more effective stress test exercises. Schuermann (2016) particularly determines how stress testing in crisis times can be adapted to normal times in order to insure adequate lending capacity and other key financial services. Orlov et al. (2017) look at the optimal disclosure policy when it is jointly determined with capital requirements, while Gick and Pausch (2014) and Inostroza and Pavan (2017) do so in the context of Bayesian persuasion. As argued by Goldstein and Leitner (2020), stress test design and disclosure policy are connected. We complement this strand of papers by focusing on the stress scenario design.

While most of the existing literature on stress testing, theoretical and empirical, analyzes the disclosure of stress test results, Leitner and Williams (2018) focus on the disclosure of the regulator’s risk modeling. They examine the trade-offs involved in disclosing the model the regulator uses to perform the stress test to banks. However, none of these papers consider the optimal scenario design, which is the focus of our paper.
Most empirical papers on stress tests focus on the information content at the time of disclosure, using an event study methodology to determine whether stress tests provide valuable information to investors. Petrella and Resti (2013) assess the impact of the 2011 European stress test exercise. For the 51 banks with publicly traded equity, they find that the publication of the detailed results provided valuable information to market participants. Similarly, Donald et al. (2014) evaluate the 2009 U.S. stress test conducted on 19 bank holding companies and find significant abnormal stock returns for banks with capital shortfalls. Candelon and Sy (2015), Bird et al. (2015), and Fernandes et al. (2015) also find significant average cumulative abnormal returns for stress tested BHCs around many of the stress test disclosure dates. Flannery et al. (2016) find that U.S. stress tests contain significant new information about assessed BHCs. Using a sample of large banks with publicly traded equity, the authors find significant average abnormal returns around many of the stress test disclosures dates. They also find that stress tests provide relatively more information about riskier and more highly leveraged bank holding companies. Glasserman and Tangirala (2016) evaluate one aspect of the relevance of scenario choices. They show that the results of U.S. stress tests are somewhat predictable, in the sense that rankings according to projected stress losses in 2013 and 2014 are correlated. Similarly, the rankings across scenarios in a given year are also correlated. They argue that regulators should experiment with more diverse scenarios, so that it is not always the same banks that project the higher losses. Acharya et al. (2014) compare the capital shortfalls from the stress tests with the capital shortfalls predicted using the systemic risk model of Acharya et al. (2016) based on equity market data. Camara et al. (2016) study the quality of the 2014 EBA stress tests using the actual micro data from the tests.

Finally, our paper is related to the large theoretical literature on information acquisition following Verrecchia (1982), Kyle (1989), and especially Van Nieuwerburgh and Veldkamp (2010). In this class of models, the cost of acquiring information pins down the set of feasible precisions and determines whether there are signals are complement or substitutes. Vives (2008) and Veldkamp (2009) provide a comprehensive review of this literature. These papers take the information processing constraint on the signal precisions as given. In contrast, our paper focuses on the design of the signals that the regulator receives and endogeneizes the information processing constraint.

The rest of the paper is organized as follows. Section 2 describes the environment. Section 3 introduces the notion of stress test. Sections ?? and 5 respectively provide an application of the general environment to linear quadratic preferences for the regulator and characterize the optimal
stress scenarios for this case. Section 6 concludes.

2 Environment

We consider the problem of a principal who wants to learn about the risk exposures of a set of agents in order to take remedial actions. The model has several natural interpretations. The principal could be a chief risk officer and the agents could be traders in her firm. The remedial actions could be hedging or downsizing the traders’ positions. Alternatively, the principal could be a regulator and the agents could be a set of banks. The remedial actions could be hedging, reducing new deal flows, selling non-performing assets, or raising capital.

To be concrete we use the regulator/banks metaphor when describing the model. The regulator elicits information from the banks in the form of stress tests. In our model, a stress test is a technology used by regulators to ask questions about profits and losses under hypothetical scenarios. The banks cannot evade the questions and have to answer to the best of their abilities. Banks in our model can only lie by omission: they do not have to volunteer information, but they have to provide estimates of their losses under various scenarios.

2.1 Banks and Risks

There is one regulator overseeing $N$ banks indexed by $i \in [1, \ldots, N]$ exposed to systematic and idiosyncratic risks. The macro-economy is described by a vector of $J$ systematic factors. We denote by $s_j$ the value of factor $j$. The macroeconomic state of the economy is

$$s = \begin{bmatrix}
    s_1 \\
    \vdots \\
    s_J
\end{bmatrix}.$$

The risks of bank $i$ are captured by a vector of $J$ exposures

$$x_i = \begin{bmatrix}
    x_{i,1} \\
    \vdots \\
    x_{i,J}
\end{bmatrix},$$

where $x_{i,j}$ represents the exposure of bank $i$ to factor $j$. We use the term “exposure” to denote the relevant elasticity that determines losses under a given realization of the macroeconomic state.
An exposure is therefore not the same as the nominal value of a position. In many cases (e.g., a commercial loan) the size of the position is unambiguous but the impact of a realization of the macro state on the loss on that position needs to be estimated. What we call “exposure” combines the position (measured with near certainty) with its value in a particular macro state (computed with error).

The losses of bank $i$ in state $s$ are given by

$$y_i(s) = s \cdot x_i + \eta_i = \sum_{j=1}^{J} x_{ij}s_j + \eta_i,$$

(1)

where $\eta_i$ is a random idiosyncratic (i.e., bank specific) shock. We normalize the baseline state to $s = 0$ and $\mathbb{E}[s] = 0$, so that the realizations of the macro state should be interpreted as deviations from the baseline. Our model has only one period so $y_i(s)$ should be interpreted as the cumulative losses in state $s$. The net worth of bank $i$ is then given by

$$w_i(s) = \bar{w}_i - y_i(s),$$

(2)

where $\bar{w}_i$ is the mean level of net worth. Given Equation (1) and Equation (2), the aggregate net worth of the banking system is

$$W(s) = \sum_{i=1}^{N} w_i = \bar{W} - s \cdot \bar{x} + \sum_{i=1}^{N} \eta_i,$$

(3)

where $\bar{W}$ and $\bar{x}$ are the sum of the corresponding variables across the $N$ banks in the economy.

In keeping with actual stress tests, we usually interpret the systematic factors as innovations to the relevant macro variables. There remains, however, a tension between this interpretation and modern macroeconomic theory. In actual stress tests regulators use traditional macroeconomic variables such as GDP, unemployment, and house prices. In DSGE models, on the other hand, these macro variables would themselves be functions of underlying structural shocks such as productivity, beliefs, risk aversion, etc.\(^3\) Formally, let $\epsilon^s$ be the structural shocks and $H$ the

\[^3\]The typical DSGE model contains Euler equations, production functions and resource constraints that lead to a set of equations such as

$$A s_t = C + BS_{t-1} + D E_t s_{t+1} + F \epsilon^s_t.$$  

When we solve the model we invert the mapping to obtain a VAR representation

$$s_t = J + Q s_{t-1} + H \epsilon^s_t.$$
solution matrix of the DSGE model, so that \( s = H\epsilon^s \). In a fully specified model, banks’ losses would also be functions of the structural shocks: \( y_i(\epsilon^s) = \tilde{x}_i^\prime \epsilon^s + \eta_i \), where \( \tilde{x}_i \) are structural exposures. This equation is equivalent to (1) when \( H \) is invertible. In that case we can write \( \epsilon^s = H^{-1}s \) and define \( x_i = H^{-1}\tilde{x}_i \), and we obtain \( y_i(s) = x_i^\prime s + \eta_i \).

In theory the regulator could supply the structural shocks \( \epsilon^s \) and ask for estimated losses. In practice regulators supply directly the macro variables \( s \). This reflects the fundamental issue of model ambiguity. Even if \( H \) is invertible, models for \( H \) would likely differ across banks as well as between banks and regulators. By contrast, a handful of macro-economic variables (GDP, credit spreads, house and stock prices, etc.) are well-understood by all participants and capture much of the macro-economic dynamics that matter for expected losses. This is why stress tests are written in terms of \( s \) and not \( \epsilon^s \).

In most of our applications we will assume that \( H \) is invertible and that the regulator feels confident about estimating \( H^{-1} \). In that case there is no real difference between estimating \( x_i \) or \( \tilde{x}_i \) and we can assume that the factors are independently distributed. Our main propositions, however, do not require this assumption.

### 2.2 Regulator’s Preferences and Interventions

We model the regulator as a risk manager. As in Acharya et al. (2016), we assume that the regulator has preferences \( U(W) \) over the total net worth of the banking system \( W \). This specification makes sense when there is a relatively efficient way to relocate assets and equity across banks, e.g. when healthy banks can take over failed ones.\(^4\) If the regulator believes that the risks in the system are too high she can intervene to force the banks to take remedial actions. In our context, an intervention is any requirement imposed on a bank to decrease its risk exposures. The regulator could force the bank to raise capital. She could ask the banks to sell assets or divest a particular line of business, thereby reducing several exposures at the same time. But she could also use targeted interventions such as imposing higher regulatory ratios against specific types of

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\(^4\) More generally, we could have \( U([w_i]_{1:N}) \). This would capture the case where the idiosyncratic failure of bank \( i \) matters regardless of the health of the banking sector as a whole. As in the systemic risk literature, we impose the restriction that only \( W \) matters. As a result, a financial crisis only happens when the financial system as a whole is under-capitalized.
loans or against specific borrowers.

The most broad based intervention would be to ask banks to increase their capital by \( \Delta \). The most granular description of intervention is at the bank-exposure level. If the regulator takes action \( a_i = \{a_{i,j}\}_{j=1..J} \) on bank \( i \), the exposure of bank \( i \) to factor \( j \) becomes \((1 - a_{i,j}) x_{i,j}\). To shorten the notation, we define the cell-by-cell multiplication operator \( \circ \) as

\[
(1 - a_i) \circ x_i \equiv [(1 - a_{i,1}) x_{i,1}, \ldots, (1 - a_{i,J}) x_{i,J}].
\]

Given an intervention policy \((\Delta, a)\), the total net worth of the financial system is \( W = \bar{W} + \Delta - \eta - s \cdot \left( \sum_{i=1}^{N} (1 - a_i) \circ x_i \right) \). Interventions are costly. There are direct costs born by the regulators and the banks, as well as indirect costs from the disruption of valuable activities. We denote by \( \kappa(\Delta) \) the cost of requiring banks to increase their capital by \( \Delta \) and by \( \Phi(a) \) be the cost of action \( a = \{a_i\}_{i=1..N} \).

Let \( \mathcal{S} \) denote the information set of the regulator at the time when she chooses her intervention policy. The regulator’s problem is then to choose an intervention policy to maximize her expected utility given by

\[
\mathbb{E} \left[ U \left( \bar{W} + \Delta - \eta - s \cdot \left( \sum_{i=1}^{N} (1 - a_i) \circ x_i \right) \right) \mid \mathcal{S} \right] - \Phi(a) - \kappa(\Delta).
\]

The regulator will choose the optimal intervention policy to solve

\[
\max_{\Delta, a} \mathbb{E} \left[ U \left( \bar{W} + \Delta - \eta - s \cdot \left( \sum_{i=1}^{N} (1 - a_i) \circ x_i \right) \right) \mid \mathcal{S} \right] - \kappa(\Delta).
\]

Conditional on the regulator information set \( \mathcal{S} \), the optimal increase in capital required by the regulator is given by

\[
\kappa'(\Delta) = \mathbb{E} \left[ U' \left( W \right) \mid \mathcal{S} \right].
\]

The regulator will require an increase in capital such that the marginal cost of raising aggregate banking capital equals the expected marginal utility of doing so. The optimal targeted intervention in bank \( i \) along dimension \( j \) is given by

\[
\frac{\partial \Phi(a)}{\partial a_{i,j}} = \mathbb{E} \left[ x_{ij}s_j U' \left( W \right) \mid \mathcal{S} \right].
\]

In an interior solution, the first order condition (FOC) equates the marginal cost of an intervention to its expected marginal benefit. The expected marginal benefit of reducing the
risk exposure to factor \( j \) in bank \( i \) depends on the covariance between the marginal social utility \( U'(W) \) and the contribution of factor \( j \) to bank \( i \)'s losses, \( x_{ij}s_j \). Risk reduction is more valuable when we expect \( U' \) to be large when \( x_{ij}s_j \) is positive.

Targeted interventions might require more information than non-targeted ones. For instance, the regulator could mandate a uniform reduction is all risky activities: this would not require much information, but this strategy would be costly since it would impact socially valuable activities. In that case a capital requirement might be more efficient. On the other hand, if the regulator had access to adequate information, she could mandate reductions only for the activities that create significant systematic risk. We think of stress tests as a way of eliciting information to determine what these activities are. The same logic applies to interventions across banks. Asking all banks to sell assets is simple but it might lead to fire sale prices. With better information the regulator might be able to ask only a subset of banks to reduce their exposures.

In the remainder of the text, we assume that the regulator’s preferences are linear quadratic given by \( U(W) = W - \frac{\gamma}{2}W^2 \). We also consider a particularly simple cost function for the targeted interventions that assumes that the costs are convex (quadratic) and separable across banks. The cost of intervening in bank \( i \) is then simply \( \frac{1}{2} \sum_j \phi_j a_{ij}^2 \), and the total costs are \( \frac{1}{2} \sum_{i=1}^N \| \phi \circ a_i \|^2 \) where \( \phi = (\sqrt{\phi_1}, \ldots, \sqrt{\phi_J}) \) is an \( N \times 1 \) vector that modulates the marginal cost of intervening to reduce risk exposures along each dimension \( j = 1, \ldots, J \).

### 3 Learning from stress tests

The information set of the regulator is a crucial input in choosing how to intervene and it depends on the regulator’s prior beliefs and on the information she acquires. The main friction in our model is that the banks’ exposures to the macro factors are not known by the banks nor by the regulator. However, the banks have imperfect (noisy) models to project their losses in a given state. We model stress tests as a mechanism for the regulator to elicit this information from the banks and acquire information. The results of the stress tests performed by the regulator allow her to learn about banks’ risk exposures \( \{x_i\}_{i=1}^N \) and shape the regulator’s optimal intervention policy.
3.1 Prior beliefs and stress tests

The regulator has prior beliefs over the distribution of exposures within banks and across banks. These prior beliefs come from historical experiences and the regulator’s own risk models. We stack the banks’ exposures in one large $NJ \times 1$ vector as follows

$$
\mathbf{x} \equiv \begin{bmatrix}
    x_1 \\
    \vdots \\
    x_N
\end{bmatrix},
$$

and we summarize the regulator’s prior over the vector of exposures $\mathbf{x}$ as

$$
\mathbf{x} \sim N(\mathbf{x}, \Sigma_x),
$$

where the $NJ \times 1$ vector of unconditional means and the $NJ \times NJ$ covariance matrix are, respectively,

$$
\mathbf{x} = \left( \begin{array}{c}
    x_1 \\
    \vdots \\
    x_N
\end{array} \right) \quad \text{and} \quad \Sigma_x = \begin{bmatrix}
    \Sigma_x^{1,1} & \Sigma_x^{1,2} & \cdots & \Sigma_x^{1,N} \\
    \Sigma_x^{2,1} & \Sigma_x^{2,2} & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    \Sigma_x^{N,1} & \cdots & \Sigma_x^{(N-1),N} & \Sigma_x^{N,N}
\end{bmatrix},
$$

with $\Sigma_x^i = \nabla(x_i)$ for all $i$ and $\Sigma_x^{i,j} = \text{Cov}(x_i, x_j)$ for all $i \neq j$. If $\Sigma_x^i$ is diagonal the regulator expects the exposures of bank $i$ to the different factors to be independent of each other. If $\Sigma_x^{i,j} = 0$, the regulator’s prior is that the risk exposures of banks $i$ and $j$ are independent. In almost all empirically relevant cases the covariance matrices are not diagonal.

To learn about the banks’ risk exposures, the regulator asks the banks to estimate and report their losses under a particular realization of the macroeconomic state. This choice of macroeconomic state is a scenario $\hat{s}$.

**Definition 1.** A scenario $\hat{s}$ is a realization of the vector of states $s$.

A scenario $\hat{s}$ is a row-vector of size $J$ that represents an aggregate state of the economy. We entertain two interpretations of the size of the state space, $J$. The simplest way is to think of $J$ as exogenously given. There might be a limited number of macroeconomic variables (GDP, unemployment, house prices) that everyone agrees need to be included in the test. The other way to think about $J$ is as a large number capturing the set of all possible risk factors and in any
given tests many have zero loadings. A non-zero weight is then a statement about whether that risk factor is included in the particular stress test. Our model can then shed light on which risk factors should be used.

Given our normalization of the baseline state to \( s = 0 \), a scenario close to 0 is a scenario close to the baseline of the economy. A scenario \( \hat{s} \) in which element \( \hat{s}_j \) is large, represents a large deviation from the baseline along the dimension of factor \( j \). The larger \( |\hat{s}_j| \), the more extreme the scenario along dimension \( j \). When designing a stress test, the regulator specifies a set of scenarios for which the banks need to report their losses.

**Definition 2.** A stress test is a collection of \( M \) scenarios \( \{\hat{s}_m\}_{m=1}^M \) presented by the regulator, and a collection of estimated losses \( \{\hat{y}_{i,m}\}_{i=1}^N \) reported by the banks.

For each scenario \( m \), each bank \( i \) estimates and reports its net losses \( \hat{y}_{i,m} \) given the input parameters in scenario \( \hat{s}_m \).

### 3.2 Stress test results

Banks use imperfect models to predict their losses under the stress test scenarios. More precisely, we assume that bank \( i \) estimates its losses under scenario \( \hat{s} \) as

\[
\hat{y}_i (\hat{s}, M) = \hat{s} \cdot x_i + \hat{\epsilon}_i (\hat{s}, M),
\]

where the error term \( \hat{\epsilon}_i (\hat{s}, M) \) plays an important role in our analysis. We use the following specification

\[
\hat{\epsilon}_i (\hat{s}, M) = \alpha_i (M) \epsilon_{i,0} + \beta_i \|\hat{s}\|^{1+\theta} \epsilon_{i,1},
\]

where \( \alpha_i (M) \) is a positive scalar increasing in the number of scenarios, \( \beta_i \) is a positive number, \( \epsilon_{i,0} \) and \( \epsilon_{i,1} \) are independent \( N(0,1) \) random scalars. We assume \( \theta > 0 \). The error terms \( \epsilon_{i,0} \) and \( \epsilon_{i,1} \) can be correlated across banks.

There are two sources of noise in a bank’s report, \( \epsilon_{i,0} \) and \( \epsilon_{i,1} \), capturing different kinds of measurement errors and model uncertainty. The error term \( \alpha_i (M) \epsilon_{i,0} \) captures bank level uncertainty that is present even in the baseline scenario. The assumption of an increasing \( \alpha_i (M) \) captures capacity constraints either on the bank side or on the regulator side. On the bank side it captures the fairly uncontroversial idea that if a regulator asks too many questions, the average quality of the responses decreases. On the regulator side it also captures the difficulty of designing
meaningful scenarios. Macroeconomic stress tests rely on few scenarios and a limited set of risk factors. For example, the Fed currently uses 28 risk factors and a couple of scenarios for the US economy and the rest of the world. The relevant value for $M$ clearly depends on the context. In non-supervisory settings, stress tests of trading books routinely involve tens of thousands of risk factors and scenarios. The second source of noise is modulated by the norm of the scenario and it captures bank level risks that are increasingly more difficult to forecast in more extreme scenarios. These include extreme credit losses as well as potential operational, liquidity and funding risks. Finally, note that in our model the banks are the ones reporting their losses. Often the regulator also has models to estimate the risk exposures. We assume that these models are embedded in the priors of the regulator, but one could also easily add them as additional signals.\footnote{The regulator does not need to know the banks’ models. The analysis remains unchanged as long as the regulator’s beliefs over the model used by the banks are consistent with Equation (6).}

The results of the stress test for one bank can be summarized in the $M \times 1$ vector

$$\hat{y}_i(\hat{S}) = \hat{S}x_i + \hat{\epsilon}_i,$$

where $\hat{y}_i(\hat{S})$ represents the results of the test for bank $i$, the $M \times J$ matrix $\hat{S}$ gather the scenarios in the stress test, and the errors in bank $i$’s reported losses are gathered in the $M \times 1$ vector $\hat{\epsilon}_i$, i.e.,

$$\hat{y}_i(\hat{S}) = \begin{bmatrix} \hat{y}_i(\hat{s}_1, M) \\ \vdots \\ \hat{y}_i(\hat{s}_M, M) \end{bmatrix}, \quad \hat{S} \equiv \begin{bmatrix} \hat{s}_1' \\ \vdots \\ \hat{s}_M' \end{bmatrix}, \quad \text{and} \quad \hat{\epsilon}_i = \begin{bmatrix} \hat{\epsilon}_i(\hat{s}_1, M) \\ \vdots \\ \hat{\epsilon}_i(\hat{s}_M, M) \end{bmatrix}.$$

The structure of the bank’s internal models are the outcome of learning about risk exposures from historical data. Therefore, the mistakes a bank makes in computing its expected revenues may be correlated across scenarios. The variance-covariance matrix of the errors made by bank $i$ in computing its stress test results is given by

$$\Sigma_i^\epsilon \equiv Var [\hat{\epsilon}_i].$$

Differences in $\Sigma_i^\epsilon$ across banks reflect differences in information (priors), in the amount or quality of data available to each bank, or in the bank’s information processing capacity. We assume that $x_i$ and $\hat{\epsilon}_i$ are independent, but we allow banks to make correlated mistakes.
3.3 A Kalman Filter

By implementing stress tests, the regulator elicits information from the banks which she uses to learn about their risk exposures. More specifically, the stress test results can be interpreted as signals about the banks’ risk exposures by defining the error terms in the signals appropriately.

Let us briefly summarize all the sources of risks and information in our model:

1. Prior correlation among exposures within and across banks: These correlations are summarized by the $NJ \times NJ$ covariance matrix $\Sigma_x$. The covariance matrix $\Sigma_x$ is independent of the scenarios and reflect the fundamental common exposures in the banking system as perceived by the regulator.

2. Correlation among estimation errors made by bank $i$ across the $M$ scenarios: These correlations are measured by the $M \times M$ matrix $\Sigma_i \equiv \text{Var}[\hat{\varepsilon}_i]$ where $\hat{\varepsilon}_i$ is an $M \times 1$ vector of errors that depends on the underlying scenarios: element $m$ of $\hat{\varepsilon}_i$ is given by $\hat{\varepsilon}_{i,m} = \alpha_i(M) \epsilon_{i,0} + \beta_i \| \hat{S}^{(m)} \|^{1+\theta} \epsilon_{i,1}$. This part crucially depends on the scenarios and captures the idea that the consequences of extreme scenarios are harder to estimate.

3. Reported losses for each bank $i$ for each scenario in the stress test: $\hat{y}_i(\hat{S})$ is an $M \times 1$ vector that summarizes the results of the stress test for bank $i$. These results depend on the scenarios by design.

For $N$ banks, we respectively stack the reported losses of the banks in the stress test and the mistakes in them in the $NM \times 1$ vectors

$$\hat{y} \equiv \begin{bmatrix} \hat{y}_1(\hat{S}) \\ \vdots \\ \hat{y}_N(\hat{S}) \end{bmatrix} \quad \text{and} \quad \hat{\varepsilon} = \begin{bmatrix} \hat{\varepsilon}_1 \\ \vdots \\ \hat{\varepsilon}_N \end{bmatrix}.$$  

Then, the information available to the regulator after seeing the results of the stress test can then be summarized in the following state space representation

$$\hat{y} = \hat{S}x + \hat{\varepsilon},$$

(8)

where $\hat{S} \equiv \left( I_N \otimes \hat{S} \right)$ simply repeats $\hat{S}$ on its diagonal, and $\hat{\varepsilon} \sim N(0, \Sigma_\varepsilon)$. Remember that the regulator observes $\hat{y}$ and wants to learn about $x$. Expressing the stress test as in equation (8) allows us to apply the Kalman filter and to obtain a full characterization of the posteriors beliefs of the regulator.
Proposition 1. After observing the results $\hat{y}$ of the stress test, the posterior beliefs of the regulator regarding the banks’ risk exposures are

$$ x | \hat{y} \sim N\left( \hat{x}, \hat{\Sigma}_x \right), $$

where the posterior mean $\hat{x}$, the Kalman gain $K$, and the residual covariance matrix $\hat{\Sigma}_x$ are given by

$$ \hat{x} = \left( \mathbf{I}_{NJ} - K \hat{S} \right) \bar{x} + K \hat{y}, \quad (9) $$
$$ K = \Sigma_x \hat{S}' \left( \hat{S} \Sigma_x \hat{S}' + \Sigma_\xi \right)^{-1}, \quad (10) $$
$$ \hat{\Sigma}_x = \Sigma_x - K \hat{S} \Sigma_x. \quad (11) $$

The proof of Proposition 1 is a direct application of the standard Kalman filter. The Kalman gain $K$ is an $NJ \times MN$ matrix. A few special cases can give some intuition. With one bank ($N = 1$), then $K_{j,m}$ is a measure of the amount of information about the exposure to risk factor $j$ contained in the results from scenario $m$. With one scenario ($J = 1$) and uncorrelated exposures among banks, $K_{j,m}$ also measures the reduction in uncertainty about bank $j$’s exposure to the risk factor.

The posterior covariance matrix $\hat{\Sigma}_x$ plays a critical role in our analysis. $\hat{\Sigma}_x$ measures the residual uncertainty that persists after observing the results of the stress test. The true exposures are distributed around $\hat{x}$ with covariance $\hat{\Sigma}_x$. The goal of the stress test is to reduce this residual uncertainty as much as possible, along dimensions that depend on the objective function and on the priors of the regulator.

In the standard state-space representation in Equation (8), the stress test scenarios determine the structure of the signals observed by the regulator. The scenarios also determine the precision of the banks’ reported losses in Equation (7). Increasing $|s_j|$ in a scenario makes the results more informative about exposures to factor $j$, but extreme scenarios reduce the precision of the banks’ estimates and the noise might spill over to the measurement of other exposures. On the other hand, the regulator can improve her learning by taking into account the fact that true exposures are correlated across positions and across banks.

When designing the scenarios, the regulator must anticipate how she will interpret and use the results of the test. The extent to which learning takes place is captured by the expected distribution of the posterior mean, given by

$$ \hat{x} \sim N\left( \bar{x}, \Sigma_\hat{x} \right), \quad (12) $$
where the ex-ante variance of the posterior mean is given by

\[ \Sigma_{\hat{x}} \equiv \Sigma_{x} - \hat{\Sigma}_{x} = K \left( I_{N} \otimes \hat{S} \right) \Sigma_{x}. \]

The matrix \( \Sigma_{\hat{x}} \) represents the expected amount of learning from stress test \( \hat{S} \). If the stress test is pure noise, then \( K = 0, \hat{\Sigma}_{x} = \Sigma_{x} \) and the regulator learns nothing, i.e., \( \Sigma_{\hat{x}} = 0 \). If the test is fully informative, then \( \hat{\Sigma}_{x} = 0 \) and the regulator learn exactly all the exposures, i.e., \( \Sigma_{\hat{x}} = \Sigma_{x} \). One goal of the regulator is to maximize \( \Sigma_{\hat{x}} = K \left( I_{N} \otimes \hat{S} \right) \Sigma_{x} \) – or equivalently to minimize the residual uncertainty \( \hat{\Sigma}_{x} \) – to design a more parsimonious and accurate intervention policy. When choosing how much to learn, the regulator takes into account that the Kalman gain \( K \) is itself a function of the scenarios, given by equation (10).

**Remark.** A goal of the regulator is to maximize the amount of learning \( \Sigma_{\hat{x}} = K \left( I_{N} \otimes \hat{S} \right) \Sigma_{x} \) to intervene more accurately. Without ex-post interventions, the regulator does not care about learning.

**A Simple Example: Scenario Choice and Learning**

Consider the case of one bank \((N = 1)\), one scenario \((M = 1)\), and two risk factors \((J = 2)\). The Kalman gain is then a \(2 \times 1\) vector \( K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \), and we simplify the notations by using \( \sigma^2_{1} \equiv \Sigma_{x,11}, \rho \sigma_{12} \equiv \Sigma_{x,12} \) and \( \sigma^2_{\epsilon} (\hat{s}) \equiv \text{Var} \left[ \hat{\epsilon} \mid \hat{s}, 1 \right] \). The posterior mean is then

\[ \hat{x} = x + K (\hat{y} - \hat{s}' x) \]

and the amount of learning is given by

\[ \Sigma_{\hat{x}} = \begin{bmatrix} k_1 \hat{s}_1 & k_1 \hat{s}_2 \\ k_2 \hat{s}_1 & k_2 \hat{s}_2 \end{bmatrix} \begin{bmatrix} \sigma^2_{1} & \rho \sigma_{12} \\ \rho \sigma_{12} & \sigma^2_{2} \end{bmatrix} = \begin{bmatrix} k_1 \hat{s}_1 \left( \sigma^2_{1} + \frac{\hat{s}_2}{\hat{s}_1} \rho \sigma_{12} \right) & k_1 \hat{s}_1 \left( \rho \sigma_{12} + \frac{\hat{s}_2}{\hat{s}_1} \sigma^2_{2} \right) \\ k_2 \hat{s}_2 \left( \rho \sigma_{12} + \frac{\hat{s}_2}{\hat{s}_1} \sigma^2_{1} \right) & k_2 \hat{s}_2 \left( \sigma^2_{2} + \frac{\hat{s}_2}{\hat{s}_1} \rho \sigma_{12} \right) \end{bmatrix}, \]

where the Kalman gain is given by

\[ K = \begin{bmatrix} \sigma^2_{1} & \rho \sigma_{12} \\ \rho \sigma_{12} & \sigma^2_{2} \end{bmatrix} \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} \left( \begin{bmatrix} \sigma^2_{1} & \rho \sigma_{12} \\ \rho \sigma_{12} & \sigma^2_{2} \end{bmatrix} \right)^{-1} \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} + \sigma^2_{\epsilon} (\hat{s})^{-1}. \]
Therefore, we have that

\[ k_1 = \frac{\sigma_1^2 \hat{s}_1 + \rho \sigma_1 \sigma_2 \hat{s}_2}{\sigma_1^2 \hat{s}_1^2 + 2\rho \sigma_1 \sigma_2 \hat{s}_1 \hat{s}_2 + \sigma_2^2 \hat{s}_2^2 + \sigma_\epsilon^2(\hat{s})} \] (13)

\[ k_2 = \frac{\sigma_2^2 \hat{s}_2 + \rho \sigma_1 \sigma_2 \hat{s}_1}{\sigma_1^2 \hat{s}_1^2 + 2\rho \sigma_1 \sigma_2 \hat{s}_1 \hat{s}_2 + \sigma_2^2 \hat{s}_2^2 + \sigma_\epsilon^2(\hat{s})} \] (14)

These equations for the components of the Kalman gain capture all the dimensions of learning. The scenario choice \( \hat{s} \) affects the Kalman gain in two ways: indirectly, through the amount of noise in the stress test result, \( \sigma_\epsilon^2(\hat{s}) \), and directly, by determining the correlation between the stress test result and unknown risk exposures.

It is clear that

\[ \frac{\partial k_1}{\partial \sigma_\epsilon^2} < 0, \]

which simply says that more noisy signals (higher \( \sigma_\epsilon^2(\hat{s}) \)) reduce the Kalman gain. Moreover, since

\[ \hat{\epsilon}(\hat{s}) = \alpha \epsilon_0 + \beta \|\hat{s}\|^{1+\theta} \epsilon_1, \]

we have

\[ \sigma_\epsilon^2(\hat{s}) = \alpha^2 + \beta^2 \left( \hat{s}_1^2 + \hat{s}_2^2 \right)^{1+\theta}, \]

which is decreasing in \( \hat{s}_j^2 \). Therefore, making the scenario more extreme, by increasing \( \hat{s}_j^2 \), increases the noise in the signal and reduces the amount of learning from stress tests.

Let us now focus on the effect of \( \hat{s} \) on the amount of learning about \( x_1 \). We see that

\[ \Sigma_{\hat{x}}(1, 1) = k_1 \hat{s}_1 \left( \sigma_1^2 + \frac{\hat{s}_2}{\hat{s}_1} \rho \sigma_1 \sigma_2 \right) \]

and, using Equation (6), we have

\[ k_1 \hat{s}_1 = \frac{\sigma_1^2 \hat{s}_1^2 + \rho \sigma_1 \sigma_2 \hat{s}_1 \hat{s}_2}{\sigma_1^2 \hat{s}_1^2 + 2\rho \sigma_1 \sigma_2 \hat{s}_1 \hat{s}_2 + \sigma_2^2 \hat{s}_2^2 + \sigma_\epsilon^2(\hat{s})} = 1 - \frac{\rho \sigma_1 \sigma_2 \hat{s}_1 \hat{s}_2 + \sigma_2^2 \hat{s}_2^2 + \sigma_\epsilon^2}{\sigma_1^2 \hat{s}_1^2 + 2\rho \sigma_1 \sigma_2 \hat{s}_1 \hat{s}_2 + \sigma_2^2 \hat{s}_2^2 + \sigma_\epsilon^2(\hat{s})} \] (15)

For illustration purposes, suppose that \( \rho = 0 \). When \( \rho = 0 \) we have

\[ k_1 \hat{s}_1|_{\rho=0} = 1 - \frac{\sigma_2^2 \hat{s}_2^2 + \sigma_\epsilon^2}{\sigma_1^2 \hat{s}_1^2 + \sigma_2^2 \hat{s}_2^2 + \sigma_\epsilon^2(\hat{s})} \]

which is an increasing function of \( \hat{s}_1^2 \) if \( \sigma_\epsilon^2 \) is fixed, i.e.,

\[ \frac{\partial (k_1 \hat{s}_1)}{\partial (\hat{s}_1)^2} \bigg|_{\rho=0; \sigma_\epsilon^2=\bar{\sigma}} > 0. \]
Therefore, keeping the amount of noise fixed, putting more weight on factor one in the scenario leads to the regulator putting more weight on the stress test results to update her beliefs about the exposure to factor one. This clarifies the partial effect of the scenario choice holding constant the noise.

**Lemma 1.** Learning an uncorrelated risk with exogenous noise improves with the size of the scenario

\[
\frac{\partial \Sigma_{\hat{x}} (1, 1)}{\partial \left( \hat{s}_1 \right)^2}_{\rho=0, \sigma^2_\epsilon=\bar{\sigma}} > 0.
\]

The effect of the scenario choice on learning is a lot more complex when the risk exposures are correlated. In particular, when \(\sigma^2_\epsilon\) is sharply increasing in \(s_1\) and \(\rho\) is large, it might be more efficient for the planner to learn about \(x_1\) by increasing \(\hat{s}_2\) instead of \(\hat{s}_1\).

More generally, the overall effect of an increase in \(\hat{s}_1\) on the amount of learning about \(x_1\) depends on the trade off between the increase in noise and the increase in correlation between the stress test result and \(x_1\) that are generated by increasing \(\hat{s}_1\). Using Equation (15) and the definition of \(\sigma^2_i(\hat{s})\), we have that

\[
k_1 \hat{s}_1 = \frac{\sigma^2_1 \hat{s}_2^2 + \rho \sigma_1 \sigma_2 \hat{s}_1 \hat{s}_2}{\sigma^2_1 \hat{s}_1^2 + 2 \rho \sigma_1 \sigma_2 \hat{s}_1 \hat{s}_2 + \sigma^2_2 \hat{s}_2^2 + \alpha^2 + \beta^2 (\hat{s}_1^2 + \hat{s}_2^2)^{1+\theta}}.
\]

The assumption that \(\theta > 0\) guarantees that \(k_1 \hat{s}_1\) is eventually decreasing in \(\hat{s}_1^2\).

### 3.4 Scenario choice as information precision choice

The regulator cares about the scenarios \(\hat{S}\) only through the distribution of \(\hat{y}\). Given the Gaussian structure of the random variables and the linearity of the signals, this distribution only depends on the posterior precision of the exposures, \(\hat{\Sigma}_x\). Every set of scenarios \(\hat{S}\) has a unique posterior covariance matrix associated with it. Therefore, choosing the scenarios \(\hat{S}\) is equivalent to choosing a posterior covariance matrix \(\hat{\Sigma}_x\) (or alternatively, choosing how much to learn \(\Sigma_x\)). Each scenario \(\hat{s}\) determines the structure of the signals about the banks’ risk exposures contained in the stress test results \(\hat{y}\). This signal structure determines the amount of information that can be extracted from the stress test results about each risk exposure \(x_{ij}\) and, hence, the amount of residual uncertainty that the regulator faces after observing \(\hat{y}\), which is measured by \(\hat{\Sigma}_x\).

We can therefore rewrite the scenario choice as choosing a residual covariance matrix \(\hat{\Sigma}_x \in \Sigma\). The feasible set \(\Sigma\) is closed and convex and its shape is determined only by the regulator’s priors.
\( \Sigma_x \) and the mistakes in the banks’ models, \( \Sigma_\varepsilon \). The set \( \Sigma \) restricts how much information can be acquired by the regulator and it plays the role of a capacity constraints in models of information acquisition. Figures (1) shows the feasible set of posterior precisions \( \{ \hat{\Sigma}_{x,1}, \hat{\Sigma}_{x,2} \} \) in a model with one representative bank and two risk factors, and for different values of prior correlations among risk exposures.

As illustrated by the example above, choosing a more extreme scenario has two effects on the amount of information that the regulator can acquire. On one hand, a higher value of \( \hat{s}_i \) increases the weight the bank’s stress test results put on the bank’s exposure to factor \( i \). On the other hand, more extreme scenarios are harder to predict and the stress test results become more noisy, i.e., the variance of the error term, \( \Sigma_\varepsilon \), is larger. For scenarios that are close to the benchmark, small values of \( (\hat{s}_1, \hat{s}_2) \), the first effect dominates and making the scenario more extreme in one dimension translates into lower posterior variances (acquiring more information) in that dimension. For more extreme scenarios, the second effect dominates and moving away from the benchmark reduces the amount of information the regulator can get from the stress test. These countervailing effects limit how much the regulator can learn from the stress test, as can be seen in Figures 1.

The prior correlation between the risk exposures also determines the shape of the feasible set. When the regulator’s prior is such that risk exposures are correlated, the cost of increasing the value of \( \hat{s}_i \) in terms of decreasing the information about risk exposure \( j \) is lower. Hence, the regulator can learn more from the stress tests and reduce the posterior variances. At the same time, the regulator cannot learn about the bank’s exposure to factor 1 without learning about the bank’s exposure to factor 2. Hence, as it can be seen from panels a, b, c and d in Figure 1, the boundary of set of feasible posterior precisions, \( \Sigma \), becomes more convex. When the correlation is high, the boundary of the feasible set slopes up in the tails. This is in line with the discussion in the example: when \( s_1 \) is already large, it becomes more efficient to learn about \( x_1 \) by increasing \( s_2 \) instead of \( s_1 \).

4 Taking action

The regulator values the information provided by the stress test because it allows her to intervene in a more accurate and parsimonious manner. How the regulator’s actions depend on the stress test results will determine how valuable it is for the regulator to learn and, therefore, is a key determinant of the optimal scenario choice.
(a) Prior correlation in exposures $\Sigma_{x,12} = 0$

(b) Prior correlation in exposures $\Sigma_{x,12} = 0.2$

(c) Prior correlation in exposures $\Sigma_{x,12} = 0.5$

(d) Prior correlation in exposures $\Sigma_{x,12} = 0.8$

Figure 1: Feasible set of posterior variances, $(\hat{\Sigma}_{x,11}, \hat{\Sigma}_{x,22})$ when there are two factors and one representative bank for different values of the regulator’s prior correlation among the bank’s risk exposures to factors 1 and 2.

Note: Figures 1 illustrates the set of feasible posterior variances, $\Sigma$ for different values of prior correlations among risk exposures. The parameters used are $N = 1$, $J = 2$, $\gamma = 1$, $\phi_1 = 1$, $x = [1, 1]$, $\Sigma_x = I_J$, $\alpha_1 (M) = 0.8M$, $\beta_1 = 1$, $M = 1$, $E [\epsilon_{1,0}^2] = 2$ and $E [\epsilon_{1,1}^2] = 1$.  

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Suppose that the regulator has two ways of intervening in the banking sector. It can mandate an increase of capital of $\Delta$ or target interventions at the bank-risk level by requiring each bank $i$ to reduce their exposure to factor $j$ in $a_{i,j}$. Recall that, in this case, the aggregate banking wealth is given by

$$W(a, \Delta) = \bar{W} + \Delta - \eta - \sum_{i=1}^{N} \sum_{j=1}^{J} s_j (1 - a_{i,j}) x_{i,j}.$$ 

The regulator takes action after observing the results of the test. Therefore, the regulator’s expected utility is

$$V(\mathcal{S}) = \max_{\Delta, a} \mathbb{E} [U(W(a, \Delta)) | \mathcal{S}] - \Phi(a) - \kappa(\Delta),$$

where $\mathcal{S}$ denotes the total information from the stress test, which includes the matrix of scenarios and the actual outcomes:

$$\mathcal{S} \equiv \{\hat{S}, \hat{y}\}.$$ 

Since $\Phi(a) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{J} \phi_j a_{i,j}^2$ then we have that the optimal intervention policies for the regulator are given by

$$\kappa'(\Delta) = \mathbb{E} [U'(W) | \mathcal{S}]$$

and

$$\phi_j a_{i,j} = \mathbb{E} [s_j x_{i,j} U'(W) | \mathcal{S}].$$

These expressions are the same as the ones in Equations (4) and (5). While the FOC for $\Delta$ equates the marginal cost of raising aggregate banking capital to the expected marginal utility, the FOC for the targeted interventions also depends on the covariance between scenario-weighted exposures and marginal utility.

**Lemma 2.** With linear quadratic preferences, the capital requirement only depends on first order expected losses and solves

$$\kappa'(\Delta^*) = 1 - \gamma \left( \bar{W} + \Delta^* + \mathbb{E} [s] \cdot \left( \sum_{i=1}^{N} (1 - a_i) \circ \hat{x}_i \right) \right).$$

This Lemma is a reflection of the certainty equivalent. It highlights the limits of linear quadratic preferences, but it also emphasizes the fact that broad capital requirements are less likely to depend on the details of the stress test results. Since we define $s$ as deviation from a central scenario we have $\mathbb{E} [s_j] = 0$ unconditionally, but we may want to consider cases where the planner happens to have information about the expected value of $s$. In this case, we can also interpret the capital requirement as capital adequacy under a severe but realistic scenario so that $\mathbb{E} [s] > 0$. 

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Lemma 3. With linear quadratic preferences, the targeted actions are given by

\[ a_j^* = \left( \phi_j I_N + \gamma \mathbb{E} \left[ x_j^2 | \mathcal{S} \right] \mathbb{E} \left[ s_j^2 \right] \right)^{-1} \gamma \mathbb{E} \left[ x_j^2 x_j^T | \mathcal{S} \right] \mathbf{1}_{N \times 1} \mathbb{E} \left[ s_j^2 \right] \quad \forall j = 1, ..., J, \quad (16) \]

where \( x_j = [x_{1,j}, ..., x_{N,j}]^T \).

The optimal intervention along dimension \( j \) therefore depends on the posterior means of the banks’ exposures to factor \( j \), \( \hat{x}_j \equiv \mathbb{E} \left[ x_j | \mathcal{S} \right] \), as well as on the residual uncertainty along this dimension, \( \hat{\Sigma}_x \equiv \text{Var} \left[ x_j | \mathcal{S} \right] \), since \( \mathbb{E} \left[ x_j x_j^T | \mathcal{S} \right] = \hat{\Sigma}_x + (\hat{x}_j \hat{x}_j^T) \). The residual uncertainty is known in advance since the evolution of the covariance matrix is deterministic, but the posterior mean depends on the random realization the test itself, since \( \hat{x} = \pi + K (\hat{y} - \hat{s}^T \pi) \).

Proposition 2. The regulator intervenes more in bank \( i \) along dimension \( j \) when she is more risk averse, when factor \( j \) is more volatile, when \( \mathbb{E} \left[ x_{i,j}^2 | \mathcal{S} \right] \) is high, and when other banks have correlated exposures.

A risk averse regulator will intervene more to reduce risk along dimension \( j \) in bank \( i \) when she perceives a higher risk, which has five components: risk aversion \( \gamma \); the variance of risk factor \( j \), \( \mathbb{E} \left[ s_j^2 \right] \); the banks own expected exposure \( \hat{x}_{i,j} \); the residual uncertainty and the correlation with other banks’ exposures, captured by the diagonal and non-diagonal elements of \( \hat{\Sigma}_x \), respectively.

Once we account for the optimal actions the interim utility of the planner is

\[ V (\mathcal{S}) = \mathbb{E} \left[ U \left( W (a^*, \Delta^*) \right) | \mathcal{S} \right] - \Phi (a^*) \] \[ - \kappa (\Delta^*) \]

which depends on the scenarios in the stress test, \( \hat{S} \), and on the stress test results, \( \hat{y} \).

5 Optimal scenario design

The regulator will choose the stress test scenarios \( \hat{S} \) taking into account their impact on her ex-post interventions. Therefore, the regulator’s problem can be written as

\[ \max_{\hat{S} \in \mathbb{R}} \mathbb{E}_g \left[ V (\mathcal{S}) | \hat{S} \right]. \]

Depending on the particular application, we could incorporate a cost of creating additional scenarios for the regulator: choosing \( M \) scenarios for the stress test could have a cost \( C (M) \). In that case the objective function would simply be \( \mathbb{E}_g \left[ V (\mathcal{S}) | \hat{S} \right] - C (M) \) and the regulator would also choose the number of scenarios to include in the stress test.
Proposition 3. The stress scenario design problem is equivalent to choosing a posterior variance-covariance matrix \( \hat{\Sigma}_x \) to minimize the expected targeted intervention cost. Formally, the regulator’s scenario design problem is

\[
\min_{\Sigma_x \in \Sigma} \mathbb{E}_x \left[ \sum_{j=1}^{J} \sum_{i=1}^{N} \phi_j a_{i,j}^* (\hat{\mathbf{y}}) \right],
\]

where \( a_{i,j}^* \) is given by Equation 16.

Since ex-post interventions are costly for the regulator, she chooses posterior variances to minimize the total amount of expected interventions weighted by the average marginal cost of intervening. Even though the objective function is separable in the dimension of the factors, the precision choices along the factor dimensions are not independent of each other. The precision budget set \( \Sigma \) restricts the amount of information about the exposures to different factors that the regulator can choose. For example, when the regulator’s prior is that risk exposures to different factors are uncorrelated, increasing the precision of the stress test results as a signal of the exposure to one factor comes at the expense of receiving less precise information about all other factors to which the bank is exposed.

Given that \( \hat{\mathbf{y}} \) is normally distributed and the macro factors are independently distributed with mean zero, we can integrate the indirect value function \( \mathbb{E}_y \left[ V \left( \mathcal{S} \right) | \hat{\mathbf{y}} \right] \) and express it as function of the covariance matrices along each risk dimension \( j \). To see this, note that the regulator’s objective depends on the result of the stress test \( \hat{\mathbf{y}} \) only through \( \hat{\mathbf{x}} \). Moreover, the optimal targeted interventions along dimension \( j \) only depend on \( \hat{\mathbf{x}} \) through \( \hat{x}^j \) and on the posterior variance through \( \hat{\Sigma}_x^j \). Then, in a slight abuse of notation we can write the regulator’s problem as

\[
\min_{\left\{ \Sigma_x^j \right\} \in \Sigma} \sum_{j=1}^{J} \sum_{i=1}^{N} \phi_j \mathbb{E}_x \left[ a_{i,j}^* \left( \hat{x}^j, \hat{\Sigma}_x^j \right) \right],
\]

where \( \hat{\Sigma} \) is the set of feasible posterior variances along each dimension set that result from the Kalman filter in XX. Figure (2) illustrates the indifference curves of a regulator when there is one representative bank and two factors. Intuitively, a risk-averse regulator has higher utility the lower the posterior uncertainty about the bank’s risk exposures.

To fix ideas, we will focus on the case in which there is one bank and \( N = 1 \). From Proposition Equation (16) we know that the optimal intervention to reduce the exposure to factor \( j \) in bank \( i \), \( a_{i,j}^* \left( \hat{x}^j, \hat{\Sigma}_x^j \right) \), is increasing in the uncertainty about the bank’s exposure to factor \( j \), \( \hat{\Sigma}_x^j \), and in
Figure 2: Indifference curves of the regulator with linear quadratic preferences

Note: Figures 2 illustrates the regulator’s indifference curves as a function of the posterior variances, \( \{ \Sigma_{x,jj} \}_j \).

The parameters used are \( N = 1, J = 2, \gamma = 1, \) and \( \phi_1 = \phi_2 = 1 \).

the bank’s expected exposure to that factor, \( \hat{x}^j \). Moreover, since \( \hat{x}^j \sim N (\bar{x}^j, (\Sigma_x - \hat{\Sigma}_x^j)) \) we can write

\[
\hat{x}^j = (\Sigma_x - \hat{\Sigma}_x^j)\frac{1}{2} z^j + \bar{x}^j
\]

where \( z^j \sim N (0, 1) \). The derivative of the expected intervention in dimension \( j \) with respect to \( \hat{\Sigma}_x^j \) is

\[
\phi_j \frac{dE_x [a_{i,j}^* (\hat{x}^j, \hat{\Sigma}_x^j)]}{d\hat{\Sigma}_x^j} = \phi_j E_x \left[ \frac{\partial a_{i,j}^* (\hat{x}^j, \hat{\Sigma}_x^j)}{\partial \Sigma_x^j} + \frac{\partial a_{i,j}^* (\hat{x}^j, \hat{\Sigma}_x^j)}{\partial \hat{x}^j} \frac{\partial \hat{x}^j}{\partial \Sigma_x^j} \right]. \tag{18}
\]

The first term inside the expectation on the right-hand side of Equation (18) represents the change in the targeted intervention policy due to precautionary motives: the regulator intervenes more aggressively along dimensions she is more uncertain about. This effect shifts the optimal intervention policy in the same way for all realizations of the stress test results: a decrease in \( \hat{\Sigma}_x^j \) shifts the optimal policy function to the left.
The second term captures the value of intervening more accurately and it depends on the sensitivity of the targeted intervention to the ex-post expected exposures $\hat{x}$, which given by $a_{i,j}(\hat{x})$, and on how the new information from the stress test changes this posterior mean, which is determined by $\frac{\partial \hat{x}}{\partial \Sigma}$. Suppose that the regulator’s prior beliefs were very precise. In this case, only very extreme realizations of $\hat{y}$ would move the regulator’s priors and the sensitivity of the intervention policy to new information would be very low, i.e., $\frac{\partial \hat{x}}{\partial \Sigma}$ would be small. In this case, increasing the precision of the stress test along dimension $j$ would not improve the accuracy of the intervention much and the value of learning along dimension $j$ would be low. Similarly, if the intervention policy was not very responsive to information, reducing the residual uncertainty of the regulator along dimension $j$ would not have a large effect on the intervention policy and the value of information would be low.

The regulator is always better off with more precise information. What limits her information choice is the feasibility set $\hat{\Sigma}$, which is determined by the loss model used by the bank and by the regulator’s beliefs. While the regulator will always choose posterior variances on the frontier of $\hat{\Sigma}$, the exact choice of information depends on the marginal benefits of learning across the different dimensions. If the solution to the regulator’s problem is interior, the regulator chooses to diversify her learning and designs a stress test that provides her with information along multiple dimensions. However, the regulator may prefer to specialize her learning and focus only on reducing her uncertainty about a subset of the bank’s risk exposures. In the remainder of this section we provide numerical examples to illustrate how the relative intervention cost and the regulator’s prior beliefs about the bank’s risk exposures determine her optimal scenario choice.

**Specialization in learning**

Whether the regulator chooses to diversify or specialize her learning about the bank’s risk exposures depends on the convexity of the intervention cost function relative to the risk aversion of the regulator, $\gamma$ and on the shape of the feasibility set for posterior precisions, $\Sigma$. More specifically, the regulator will choose to specialize or diversify her learning depending on the convexity of the indifference curves relative to the convexity of $\Sigma$. The ratio $\frac{\phi}{\gamma}$ determines the sensitivity of the ex-post intervention to new information and, through it, the shape of the regulator’s indifference curves over the posterior precisions. The prior correlation between the exposures determines the concavity of the set $\Sigma$. 

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Figure 3: Optimal scenario and information choice as a function the intervention cost to reduce the exposure to factor 2, $\phi_2$.

Note: Figure 3 illustrates the regulator’s optimal choice of scenario and the implied posterior variance as a function of the intervention cost along the dimension of factor 2. The parameters used are $N = 1$, $J = 2$, $\gamma = 0.3$, $\bar{x} = [1,1]$, $\Sigma_x = I_J$, $\alpha_1(M) = 0.8M$, $\beta_1 = 1$, $M = 1$, and $\theta = 2$, $E[s_k^2] = 1$, $E[\epsilon_{1,0}^2] = 1$ and $E[\epsilon_{1,1}^2] = 1$.

Figures (3) show the optimal scenario and optimal posterior variance as a function of $\frac{\phi}{\gamma}$ when there are two symmetric risk factors and one bank whose priors over the risk exposures are uncorrelated. Intuitively, for low values of $\frac{\phi}{\gamma}$ the intervention action $a_j^*$ is very sensitive to the precision of the information in the stress test about the exposure to factor $j$, measured by $\hat{\Sigma}_{x,jj}$. In this case, the regulator can reduce her intervention cost significantly by intervening accurately on both risk dimensions and chooses to diversify her learning, i.e., the optimal scenario puts weight on both risk factors. As $\frac{\phi}{\gamma}$ increases, the sensitivity of the optimal intervention to the precision of the stress test result decreases. Now, the regulator needs very precise information to improve the accuracy of her intervention enough to reduce her intervention costs significantly. In this case, the regulator chooses to specialize her learning and learn only about one risk factor (only one factor has positive weight in the optimal scenario). When $\frac{\phi}{\gamma}$ is high enough, the regulator finds it optimal to intervene accurately in one dimension and having a blanket intervention in the other rather than intervening with low accuracy in both dimensions.
**Intervention costs**

The effect of intervention costs on the optimal scenario is not monotone. When the intervention costs are low, the regulator can intervene substantially ex-post without creating significant dead-weight losses. In this case, intervening inaccurately is not too costly and the regulator cares little about learning about that factor. When the intervention costs are intermediate, interventions are sensitive to the information of the stress test and their cost is significant. In this case, the regulator values learning more since more precise information allows her to avoid wasteful interventions. Finally, when the intervention costs are very high, the ex-post interventions are small irrespective of what the regulator might learn. In this case, the sensitivity of the ex-post intervention with respect to the new information is low and learning is less valuable for the regulator.

Figure (4) illustrates the optimal scenario choice and the implied posterior precisions as the cost of intervening to reduce the bank’s exposure to factor 2, $\phi_2$, increases. As it can be seen in the first panel of the figure, the weight of factor 2 in the optimal scenario increases and then decreases with $\phi_2$. At first when intervening to reduce the exposure to factor 2 is slightly higher than the cost of intervening along the dimension of factor 1, the regulator finds it optimal to learn more about factor 2 at the expense of decreasing her learning about the exposure to factor 1 to minimize her aggregate intervention costs. However, as $\phi_2$ continues to increase, the sensitivity of $a^*_2$ to the information in the stress test decreases making information about the exposure to factor 2 less effective in decreasing the aggregate intervention cost than information about the exposure to factor 1. This leads to a decrease in the weight of factor 2 and an increase in the weight of factor 1 in the stress test. In the limit, when $\phi_2$ goes to infinity and becomes prohibitive, the regulator chooses not to learn about the exposure to factor 2 at all, since her intervention policy will be independent of any information about this exposure.

Figure (5) shows the expected optimal intervention as a function of the intervention cost to reduce the exposure to factor 2. The expected intervention along the dimension of factor 2 is decreasing and convex in the intervention cost $\phi_2$. When the intervention cost $\phi_2$ is small, the expected intervention to reduce the exposure to factor 2 decreases for two reasons. First, the regulator chooses to intervene less because intervening is more expensive. Second, more precise information about the bank’s exposure to factor 2 allows the regulator to intervene more accurately and less wastefully. This second effect is not present when the intervention cost $\phi_2$ is high, and therefore, the decline in the expected intervention cost is lower.
Figure 4: Optimal scenario and information choice as a function the intervention cost to reduce the exposure to factor $2$, $\phi_2$.

Note: Figure 4 illustrates the regulator’s optimal choice of scenario and the implied posterior variance as a function of the intervention cost along the dimension of factor $2$. The parameters used are $N = 1$, $J = 2$, $\gamma = 0.3$, $\phi_1 = 1$, $\bar{x} = [1, 1]$, $\Sigma_x = I_J$, $\alpha_1 (M) = 0.8M$, $\beta_1 = 1$, $M = 1$, and $\theta = 2$, $\mathbb{E} [s^2_k] = 1$, $\mathbb{E} [\epsilon^2_{1,0}] = 1$ and $\mathbb{E} [\epsilon^2_{1,1}] = 2$.

**Prior mean risk exposure**

The prior mean of the exposure of the bank to each factor affects how much the regulator wants to learn about each risk exposure. Similarly to the comparative statics exercise with respect to intervention costs, the relation between the prior mean exposure and the optimal scenario is non-monotonic. There are two opposing effects of a higher expected risk exposure on the optimal stress scenario. First, the regulator’s intervention policy is higher when the expected risk exposure to a factor is higher. This makes intervening accurately along this dimension more valuable and increases the value of learning about the exposures to factors with high expected exposures. Moreover, the regulator’s beliefs, and hence her intervention policy, are less sensitive to the new information in the stress test when the expected risk exposure is higher. For example, if the prior mean exposure to factor $1$ is high, the regulator’s posterior mean is anchored around this value, i.e., the posterior mean is likely to be large regardless of the information revealed in the stress test.
Figure 5: Optimal expected ex-post interventions as a function of the intervention cost to reduce the exposure to factor 2, $\phi_2$.

Note: Figure 5 illustrates the regulator’s optimal expected ex-post interventions to reduce the bank’s exposure to factors 1 and 2 as a function of the intervention cost along the dimension of factor 2. The parameters used are $N = 1, J = 2, \gamma = 0.3, \phi_1 = 1, \bar{x} = [1, 1], \Sigma_x = I_J, \alpha_1(M) = 0.8M, \beta_1 = 1, M = 1$, and $\theta = 2, E[s_k^2] = 1, E[\epsilon_{1,0}^2] = 1$ and $E[\epsilon_{1,1}^2] = 2$.

results. In this case, the regulator expects to intervene substantially along dimension 1 for most stress test results. Therefore, new information about the exposure to factor 1 is not very valuable, as it does not change the regulator’s ex-post behavior substantially. In this case, the weight of factor 1 in the stress test scenario decreases with the expected prior mean exposure to factor 1.

When the expected risk exposure is low (high), the first (second) effect dominates and the weight of a factor in the stress test is increasing (decreasing) in the prior mean exposure to it. When the prior risk exposure is high enough, the regulator finds it optimal to set factor 1 to zero in the stress test and learn only about factor 2. This does not imply that the regulator stops caring about the bank’s exposure to risk factor 1. However, the regulator will choose to reduce the risk generated by the bank’s exposure to factor 1 by intervening more heavily ex-post and not learning about it at all.

Panel (a) in Figure 6 shows the weights of risk factors 1 and 2 in the optimal scenario as $\bar{x}_1$ changes. Panel (b) in the same figure shows the posterior precisions for the bank’s risk exposures as a function of the prior risk exposure to factor 1. Figure 7 shows the optimal expected ex-post interventions as a function of the prior mean exposure to factor 1.

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Figure 6: Optimal scenario and information choice as a function of the prior mean \( \bar{x}_1 \).

Note: Figure 6 illustrates the regulator’s optimal choice of scenario and the implied posterior variance as a function of the prior mean of the risk exposure to factor 1. The parameters used are \( N = 1 \), \( J = 2 \), \( \gamma = 1 \), \( \phi_1 = 1 \), \( \bar{x}_2 = 1 \), \( \Sigma_x = I_J \), \( \alpha_1 (M) = 1.3M \), \( \beta_1 = 1 \), \( M = 1 \), and \( \theta = 3 \), \( E [s^2_k] = 1 \), \( E [\epsilon_{1,0}^2] = 1.2 \) and \( E [\epsilon_{1,1}^2] = 1 \).

Figure 7: Optimal expected ex-post interventions as a function of the prior mean \( \bar{x}_1 \).

Note: Figure 6 illustrates the regulator’s optimal expected ex-post interventions to reduce the bank’s exposure to factors 1 and 2 as a function of the prior mean exposure to factor 1. The parameters used are

\[ N = 1, \quad J = 2, \quad \gamma = 1, \quad \phi_1 = 1, \quad \bar{x}_2 = 1, \quad \Sigma_x = I_J, \quad \alpha_1 (M) = 1.3M, \quad \beta_1 = 1, \quad M = 1, \quad \text{and} \quad \theta = 3, \quad E [s^2_k] = 1, \quad E [\epsilon_{1,0}^2] = 1.2 \quad \text{and} \quad E [\epsilon_{1,1}^2] = 1. \]
Prior uncertainty

Figure 8: Optimal scenario and information choice as a function of the regulator’s prior variance of the exposure to factor 2, $\Sigma_{x,22}$.

Note: Figure 8 illustrates the regulator’s optimal choice of scenario and the implied posterior variance as a function of the regulator’s prior variance of the exposure to factor 2, $\Sigma_{x,22}$. The parameters used are $N = 1$, $J = 2$, $\gamma = 1$, $\phi_j = 1$, $\bar{x} = [1, 1]$, $\Sigma_x = I_J$, $\alpha_1(M) = 0.6M$, $M = 1$, $B_1 = I_{2 \times 2}$, and $C_1 = I_{2 \times 2}$.

Another important component of the regulator’s beliefs is the prior uncertainty about the risk exposure to the risk factors. As one would expect, the regulator finds it more valuable to learn about an exposure to a risk factor the larger her prior uncertainty about it. Therefore, the weight of factor 2 (factor 1) is increasing (decreasing) in the regulator’s prior uncertainty about the bank’s exposure to factor 2. When the regulator’s prior uncertainty about the exposure to factor 2 is large enough, the regulator chooses to learn only about this exposure and only puts weight on factor 2 in the optimal scenario. Panels (a) and (b) in Figure 8 show respectively the weights of risk factors 1 and 2 in the optimal scenario and the posterior precisions for the bank’s risk exposures as a function of the prior uncertainty about the bank’s exposure to factor 2, $\Sigma_{x,22}$.

Prior correlation in risk exposures within banks

The prior correlation among risk exposures plays a crucial role in determining how much the regulator can learn from a stress test, as it can be seen from the feasible set of posterior precisions,
Figure 9: Optimal scenario and information choice as a function of the prior correlation in risk exposures, $\Sigma_{x,12}$.

Note: Figure 9 illustrates the regulator’s optimal choice of scenario and the implied posterior variance as a function of prior correlation in risk exposures. The parameters used are $N = 1, J = 3, \gamma = 0.3, \phi_1 = 1, \bar{x} = [1, 1, 1], \Sigma_x = I_J, \alpha_1(M) = 0.8M, \beta_1 = 1, M = 1, \theta = 2, E[s_k^2] = 1, E[\epsilon_{1,0}^2] = 1$ and $E[\epsilon_{1,1}^2] = 1$

When the prior correlation among risk exposures is zero, the regulator faces a trade-off between learning about one risk exposure or the other. Scenarios that provide a lot of information about the bank’s risk exposure to factor 1 contain very little information about the bank’s risk exposure to factors 2 and 3. For example, an extreme scenario that puts all the weight on risk factor 1 and no weight on the other risk factors contains no information about the bank’s risk exposure to factors 2 and 3 at all. When the correlation between risk exposures is non-zero, this trade-off is attenuated as signals about one risk exposure always contain some information about the other.

Panel (a) in Figure 9 plots the weight on risk factor 1 in an optimal scenario as a function of the regulator’s prior correlation among risk exposures. As one can see in this figure, the correlation in risk exposures does not affect the scenario choice. However, as panel (b) shows, the amount of information that the regulator can get from the same scenario changes as the prior correlation in risk exposures varies. As bank’s risk exposures become more correlated in the bank’s prior, the more informative the stress test.

When there are more than two factors, the scenario choice will be affected by the prior
correlation among the bank’s risk exposures to factors 1 and 2. As this correlation increases, the regulator will put more weight on the factors to which the bank’s exposures are correlated and decrease the weight of the remaining factor. When the exposures of the bank to factors 1 and 2 are very correlated, the regulator finds it optimal to learn only about these two risk exposures and chooses not to deviate from the benchmark for factor 3.

**Uncertainty about risk factors**

![Figure 10: Optimal scenario and information choice as a function of the uncertainty about risk factor 2, $E[s^2_2]$.](image)

Note: Figure 10 illustrates the regulator’s optimal choice of scenario and the implied posterior variance as a function of the uncertainty about risk factor 2, $E[s^2_2]$. The parameters used are $N = 1$, $J = 2$, $\gamma = 0.3$, $\phi_i = 1$, $\bar{x} = [1,1]$, $\Sigma_x = I_J$, $\alpha_1 (M) = 0.8M$, $\beta_1 = 1$, $M = 1$, and $\theta = 1$, $E[s^2_k] = 1$, $E[\epsilon_{1,0}^2] = 1$ and $E[\epsilon_{1,1}^2] = 1$.

The optimal stress scenario depends also on the uncertainty about the aggregate macroeconomic risk factors. If one risk factor has a very low variance and will stay close to the the baseline, then it is less valuable to learn about the exposures to it and to intervene to reduce them. In this case, the factor’s weight on the expected losses will be small and uncertainty about the exposure to it is less costly. However, if the variance of a risk factor is large it has the potential to be an important driver of bank losses depending on the risk exposures to it. In this case, the regulator has more incentives to learn and intervene along the dimension of this factor.
to curve its potential impact on losses. Therefore, the regulator will stress a risk factor more in
the optimal scenario the highest the uncertainty about it. Figures 10 show the optimal scenario
choice as a function of the uncertainty about risk factor 2, $E[s^2_2]$.

**Correlated risk exposures across banks**

When there are multiple (types of) banks in the economy, the regulator’s beliefs about the risk
exposures within and across banks determine the optimal choice of stress scenario. Consider the
case in which there are two banks and two risk factors. In this case, the banking sector’s wealth
is given by

$$W = \bar{\omega} - \bar{\eta} - \sum_{i=1}^{2} \left( \sum_{j=1}^{2} \left( 1 - a_{i,j}^\star (\hat{y}) \right) x_{i,j}s_j \right).$$

The system that characterizes optimal action choices in Equation (16) becomes

$$a_{i,j}^\star (\hat{y}) = \frac{\gamma E[s^2_j] \left( \phi_j \sum_{n=1}^{2} E[x_{i,j}x_{n,j} | \hat{y}] + \gamma E[s^2_j] \left( E[x_{i,j}^2 | \hat{y}] E[x_{z,j}^2 | \hat{y}] - (E[x_{i,j}x_{z,j} | \hat{y}])^2 \right) \right)}{\phi_j^\star + \gamma E[s^2_j] \left( \phi_j \sum_{n=1}^{2} E[x_{n,j}^2 | \hat{y}] + \gamma E[s^2_j] \left( E[x_{i,j}^2 | \hat{y}] E[x_{z,j}^2 | \hat{y}] - (E[x_{i,j}x_{z,j} | \hat{y}])^2 \right) \right)} \quad \forall i, j = 1, 2. \quad (19)$$

As in the one bank case, the optimal ex-post intervention for bank $i$ along the dimension of
factor $j$ depends on the intervention cost $\phi_j$ and on the uncertainty about bank $i$’s exposure to
factor $j$. However, when there are multiple banks, the optimal intervention $a_{i,j}^\star$ also depends on
how the posterior covariance of exposures to factor $j$ across banks, as it can be seen from Equation
(19). This posterior covariance in turn depends on the regulator’s prior beliefs about the banks’
risk exposures and, in turn, on the amount of scenario choice. In the remainder of this section,
we provide numerical examples to understand how the regulator’s beliefs affect the optimal stress
scenario design when there are multiple banks.

When there are multiple banks, the exposures to the risk factors are likely to be correlated
among banks, even when the factors are orthogonal. Figure 11 shows the optimal scenario weights
as a function of the correlation between the exposure to risk factor 1 for banks 1 and 2, when
there are two banks, two risk factors and one scenario.

When the exposures to factor 1 are correlated across banks, the stress test results of bank $i$
contain information about bank $i$’s exposures to both risk factors and about bank $j$’s exposure
to factor 1, as long as the weight of factor 1 in the stress test’s scenario is non-zero. Figure 11
shows that, in this case, learning about factor 1 becomes more valuable for the regulator to the
point that the stress scenario only puts weight on factor 1 when the risk exposures of the banks to
factor 1 are very correlated. In this case, the regulator does not learn about the risk exposures to
factor 2 and only chooses to intervene ex-post to reduce the losses associated with this factor. The
Figure 11: Optimal scenario and information choice as a function of the prior correlation between the bank’s risk exposures to factor 1, Σ_{x,12}.

Note: Figure 11 illustrates the regulator’s optimal choice of scenario and the implied posterior variance as a function of prior correlation in risk exposures. The parameters used are N = 2, J = 2, γ = 0.3, φ_j = 1, \bar{x}_2=1, Σ_x = I_J, \alpha_1 (M) = 0.1M, \beta_i = 1, M = 1, and θ = 3, E[s^2_k] = 1, E[\epsilon^2_{j,0}] = E[\epsilon^2_{j,1}] = 1 and

\[ E[\epsilon_{1,0}\epsilon_{2,0}] = E[\epsilon_{1,1}\epsilon_{2,1}] = 1. \]

posterior variance of the risk exposures to factor 2 remain constant and equal to the prior. On the other hand, the posterior variance of the exposure of the banks to factor 1 decreases mechanically with Σ_{x,11} since the the stress test of bank j becomes more informative of bank i’s exposure to factor 1 as this correlation increases.

6 Conclusion

Despite the growing importance of stress testing for financial regulation and risk management, economists still lack a theory of the design of stress scenarios. We model stress testing as a learning mechanism and show how to map the scenario choice problem into an information acquisition choice problem. In this framework, we derive optimal scenarios and characterize how the design of these optimal scenarios depends on the cost of interventions, the prior beliefs of the regulator, the precision of regulatory information, the uncertainty about the risk factors, and the presence of systemic risk factors.
Our approach is consistent with the general principles of current policies implement in various jurisdiction, but it has the advantage that our optimal scenarios are not arbitrary. For example, the current policy on stress scenario design in the U.S. allows for the stress scenarios to “follow either a recession approach, a probabilistic approach, or an approach based on historical experiences.” These concepts are somewhat vague and have generated much discussion among banks and regulators. Some commentators argue that scenarios should be predictable while others advocate a flexible design to accommodate emerging risks and changing exposures. Our learning approach shows how to incorporate this goals in the design of the stress scenarios.

Recently, the 2020 supervisory stress tests in the U.S. have been scrutinized for not being, in hindsight, stressful enough, and featuring a severely adverse scenario that was a milder economic catastrophe than the one experienced during the COVID-19 pandemic. These criticisms seem to be grounded in the belief that stress scenarios are predictors of the next crisis. Should this be the case? Should the severely adverse scenario be the worst-case scenario? Or are there other considerations that guide the stress scenario choice? Without pondering the goal of the stress tests and the role of the stress scenarios it is impossible to have a serious discussion of what the different scenarios are meant to capture and how they should be chosen. Our paper is a first step in this direction.

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6See 12 CFR Part 252 Appendix A.
7See, for example, https://www.bloomberg.com/opinion/articles/2020-06-29/banks-are-managing-their-stress.
References


Appendix

The Appendix contains some auxiliary calculation for formulas in the text. It needs to be completed.

A Optimal intervention policy

Under linear quadratic preferences, the first order condition that characterizes the regulator’s optimal intervention policy is

\[ \phi_j a_{i,j}^* (\hat{y}) = E \left[ x_{i,j} s_j (1 - \gamma W) | \hat{y} \right] \quad \forall i = 1, \ldots, N, \quad \forall j = 1, \ldots, J, \]

where \( W = \sum_{i=1}^N \left( \sum_{j=1}^J \left( 1 - a_{i,j}^* (\hat{y}) \right) x_{i,j} s_j + \eta_i - d_i \right) \). This is the same as

\[ \phi_j a_{i,j}^* (\hat{y}) = E \left[ x_{i,j} s_j \left( 1 + \gamma \sum_{n=1}^N \sum_{h=1}^J \left( 1 - a_{n,h}^* (\hat{y}) \right) x_{n,h} s_h + \gamma \sum_{i=1}^N d_i \right) | \hat{y} \right]. \]

Since \( E [ s_j ] = 0 \) for all \( j \), we have

\[ \phi_j a_{i,j}^* (\hat{y}) = \gamma \sum_{n=1}^N \left( 1 - a_{n,j}^* (\hat{y}) \right) E \left[ x_{i,j} x_{n,j} | \hat{y} \right] E \left[ s_j^2 \right]. \]

Let \( a_j^* = (a_{1,j}^*, \ldots, a_{N,j}^*)' \). Then, the FOCs can be written as

\[ \phi_j a_j^* = -\gamma \left[ \begin{array}{c} E \left[ x_{1,j} x_{1,j} | \hat{y} \right] & E \left[ x_{1,j} x_{2,j} | \hat{y} \right] & \cdots & E \left[ x_{1,j} x_{N,j} | \hat{y} \right] \\ E \left[ x_{2,j} x_{1,j} | \hat{y} \right] & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ E \left[ x_{N,j} x_{1,j} | \hat{y} \right] & \cdots & \cdots & E \left[ x_{N,j} x_{N,j} | \hat{y} \right] \end{array} \right] E \left[ s_j^2 \right] a_j^* + \gamma \sum_{n=1}^N \left[ E \left[ x_{1,j} x_{n,j} | \hat{y} \right] E \left[ s_j^2 \right] \right] + \gamma \sum_{n=1}^N \left[ E \left[ x_{2,j} x_{n,j} | \hat{y} \right] E \left[ s_j^2 \right] \right] + \cdots + \gamma \sum_{n=1}^N \left[ E \left[ x_{N,j} x_{n,j} | \hat{y} \right] E \left[ s_j^2 \right] \right], \]

which is the same as

\[ \phi_j a_j^* = - \left( \hat{\Sigma}_{x,j} + \hat{x}_j \hat{x}_j' \right) \gamma E \left[ s_j^2 \right] a_j^* + \left( \hat{\Sigma}_{x,j} + \hat{x}_j \hat{x}_j' \right) 1_{N \times 1} \gamma E \left[ s_j^2 \right], \]

where \( \hat{x}_j = (\hat{x}_{i,j}, \ldots, \hat{x}_{N,j})' \). Then,

\[ a_j^* = \left( \phi_j \mathbb{1}_N + \gamma \left( \hat{\Sigma}_{x,j} + \hat{x}_j \hat{x}_j' \right) E \left[ s_j^2 \right] \right)^{-1} \gamma \left( \hat{\Sigma}_{x,j} + \hat{x}_j \hat{x}_j' \right) 1_{N \times 1} E \left[ s_j^2 \right] \quad \forall j = 1, \ldots, J \quad (A.1) \]

A.1 Proof of Proposition 2

It follows directly from the system in Equation (A.1).
B  Optimal scenario choice

Proof of Proposition 3

When utility is linear quadratic and intervention costs are linear, the objective function is

\[
O = \mathbb{E}_x \left[ \mathbb{E}_{s, \eta, x} \left[ U \left( W (a^*, \Delta^*) \mid \hat{x}, \hat{x}_x \right) \right] - \Phi (a^*) \right] = \mathbb{E}_x \left[ \mathbb{E}_{s, \eta, x} \left[ W (a^*, \Delta^*) + \frac{\gamma}{2} (W (a^*, \Delta^*))^2 \mid \hat{x}, \hat{x}_x \right] - \Phi (a^*) \right]
\]

\[
O = W - \frac{\gamma}{2} \hat{W}^2 + \mathbb{E}_x \left[ \Delta - \frac{\gamma}{2} \sigma^2_{\eta} - \mathbb{E}_x \left[ \frac{\gamma}{2} \left( \Delta + \sum_{i=1}^N \sum_{j=1}^J (1 - a^*_{i,j} (\hat{x})) x_{i,j} s_j \right)^2 \mid \hat{x}, \hat{x}_x \right] + \frac{1}{2} \sum_j \phi_j (a^*_{i,j} (\hat{x}))^2 \right]
\]

\[
= W - \frac{\gamma}{2} \hat{W}^2 + \mathbb{E}_x \left[ \Delta - \frac{\gamma}{2} \Delta^2 \right] - \frac{\gamma}{2} \sigma^2_{\eta} - \mathbb{E}_x \left[ \frac{1}{2} \sum_{j=1}^J \sum_{i=1}^N (1 - a^*_{i,j} (\hat{x})) \phi_{i,j} (a^*_{i,j} (\hat{x})) + \frac{1}{2} \sum_j \phi_j (a^*_{i,j} (\hat{x}))^2 \right]
\]

\[
= W - \frac{\gamma}{2} \hat{W}^2 + \mathbb{E}_x \left[ \Delta - \frac{\gamma}{2} \Delta^2 \right] - \frac{\gamma}{2} \sigma^2_{\eta} - \mathbb{E}_x \left[ \frac{1}{2} \sum_{j=1}^J \sum_{i=1}^N \phi_{i,j} a^*_{i,j} (\hat{x}) \right]
\]

where we used that

\[
\phi_j a^*_{i,j} (\hat{x}) = \gamma \sum_{k=1}^N \left( 1 - a^*_{k,j} (\hat{x}) \right) \mathbb{E} \left[ x_{i,j} x_{k,j} \mid \hat{x}, \hat{x}_x \right] \mathbb{E} \left[ s_j^2 \right] .
\]

Moreover, since \( \Delta^* \) is given by

\[
\kappa' (\Delta^*) = 1 - \gamma \left( \hat{W} + \Delta^* \right),
\]

it is independent of the information revealed by the stress test.

Hence, the regulator’s scenario choice problem is equivalent to

\[
\min_{\Sigma_x \in \Sigma} \mathbb{E}_x \left[ \frac{1}{2} \sum_{j=1}^J \sum_{i=1}^N \phi_j a^*_{i,j} \left( \hat{x} + \left( \Sigma_x - \hat{\Sigma}_x \right) \frac{1}{2} z \right) \right],
\]

where

\[
a^*_j = \left( \phi_j \mathbb{I}_N + \gamma \mathbb{E} \left[ x^j x'^j \mid \hat{y} \right] \mathbb{E} \left[ s_j^2 \right] \right)^{-1} \gamma \mathbb{E} \left[ x^j x'^j \mid \hat{y} \right] 1_{N \times 1} \mathbb{E} \left[ s_j^2 \right] \quad \forall j = 1, \ldots, J
\]

and \( \hat{\Sigma}^j = [x_{1,j}, \ldots, x_{N,j}]' \).
B.1 One bank

When there is only one bank, the optimal intervention policy in Equation (16) becomes

\[
\begin{bmatrix}
\phi_1 & 0 & \cdots & 0 & 0 \\
0 & \phi_2 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \ddots & \ddots & \cdots \\
0 & \cdots & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a^* (\hat{y}) = -\gamma \\
\end{bmatrix}
= 
\begin{bmatrix}
\mathbb{E} [x_1^2 | \hat{y}] & \mathbb{E} [x_1^2 | \hat{y}] \\
0 & \mathbb{E} [x_2^2 | \hat{y}] & \mathbb{E} [s_2^2] \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \ddots & \ddots & \cdots \\
0 & \cdots & \cdots & 0 & \mathbb{E} [x_1^2 | \hat{y}] & \mathbb{E} [s_1^2]
\end{bmatrix}
\begin{bmatrix}
a^* (\hat{y}) \\
\end{bmatrix}
\]

This is the same as

\[
a^*_j (\hat{y}) = \frac{\gamma \mathbb{E} [x_j^2 | \hat{y}] \mathbb{E} [s_j^2]}{\phi_j + \gamma \mathbb{E} [x_j^2 | \hat{y}] \mathbb{E} [s_j^2]} = \frac{\gamma (\bar{x}_{x,j}^2 + \hat{x}_j^2) \mathbb{E} [s_j^2]}{\phi_j + \gamma (\bar{x}_{x,j}^2 + \hat{x}_j^2) \mathbb{E} [s_j^2]} \quad \forall j.
\]

(A.2)

B.2 Two banks

When there are two banks and two factors, the regulator’s optimal intervention policy is characterized by the following systems

\[
a^*_i,j = \left( \phi_j \bar{I}_2 + \gamma \left[ \begin{array}{c}
\mathbb{E} [x_{1,i} x_{1,j} | \hat{y}] \\
\mathbb{E} [x_{2,i} x_{1,j} | \hat{y}] \\
\mathbb{E} [x_{2,i} x_{2,j} | \hat{y}]
\end{array} \right] \mathbb{E} [s_j^2] \right)^{-1} \left[ \begin{array}{c}
\mathbb{E} \sum_{n=1}^{N} [x_{1,i} x_{n,j} | \hat{y}] \mathbb{E} [s_j^2] \\
\mathbb{E} \sum_{n=1}^{N} [x_{2,i} x_{n,j} | \hat{y}] \mathbb{E} [s_j^2]
\end{array} \right],
\]

or, equivalently,

\[
a^*_i,j = \frac{\gamma \mathbb{E} [s_j^2] \left( \phi_j \sum_{n=1}^{N} \mathbb{E} [x_{i,j} x_{n,j} | \hat{y}] + \gamma \mathbb{E} [s_j^2] \left( \mathbb{E} [x_{1,i}^2 | \hat{y}] \mathbb{E} [x_{2,i}^2 | \hat{y}] - \mathbb{E} [x_{1,i} x_{2,i} | \hat{y}] \right)^2 \right)}{\phi_j^2 + \gamma \mathbb{E} [s_j^2] \left( \phi_j \sum_{n=1}^{N} \mathbb{E} [x_{n,j}^2 | \hat{y}] + \gamma \mathbb{E} [s_j^2] \left( \mathbb{E} [x_{1,j}^2 | \hat{y}] \mathbb{E} [x_{2,j}^2 | \hat{y}] - \mathbb{E} [x_{1,j} x_{2,j} | \hat{y}] \right)^2 \right)} \quad \forall i, j.
\]