Online Linear Programming: Dual Convergence, New Algorithms, and Regret Bounds

Xiaocheng Li† Yinyu Ye†

†Department of Management Science and Engineering, Stanford University
{chengli1, yyye}@stanford.edu

Abstract

We study an online linear programming (OLP) problem under a random input model in which the columns of the constraint matrix along with the corresponding coefficients in the objective function are generated i.i.d. from an unknown distribution and revealed sequentially over time. Virtually all current online algorithms were based on learning the dual optimal solutions/prices of the linear programs (LP), and their analyses were focused on the aggregate objective value and solving the packing LP where all coefficients in the constraint matrix and objective are nonnegative. However, two major open questions are: (i) Does the set of LP optimal dual prices in OLP converge to those of the “offline” LP, and (ii) Could the results be extended to general LP problems where the coefficients can be either positive or negative. We resolve these two questions by establishing convergence results for the dual prices under moderate regularity conditions for general LP problems. Then we propose a new type of OLP algorithm, Action-History-Dependent Learning Algorithm, which improves the previous algorithm performances by taking into account the past input data as well as and decisions/actions already made. We derive an $O(\log n \log \log n)$ regret bound for the proposed algorithm, against the $O(\sqrt{n})$ bound for typical dual-price learning algorithms, and show that no dual-based thresholding algorithm achieves a worst-case regret smaller than $O(\log n)$, where $n$ is the number of decision variables. Numerical experiments demonstrate the superior performance of the proposed algorithms and the effectiveness of our action-history-dependent design. Our results also indicate that, for solving online optimization problems with constraints, it’s better to utilize a non-stationary policy rather than the stationary one.

1 Introduction

Sequential decision making has been an increasingly attractive research topic with the advancement of information technology and the emergence of new online marketplaces. As a key concept appearing widely in the fields of operations research, management science, and artificial intelligence, sequential decision making concerns the problem of finding the optimal decision/policy in a dynamic environment where the knowledge of the system, in the form of data and samples, amasses and evolves over time. In
this paper, we study the problem of solving linear programs in a sequential setting, usually referred to as online linear programming (OLP) (See e.g., (Agrawal et al., 2014)). The formulation of OLP has been widely applied in the context of online Adwords/advertising (Mehta et al., 2005), online auction market (Buchbinder et al., 2007), resource allocation (Asadpour et al., 2019), packing and routing (Buchbinder and Naor, 2009), and revenue management (Talluri and Van Ryzin, 2006). One common feature in these application contexts is that the customers, orders, or queries arrive in a forward sequential manner, and the decisions need to be made on the fly with no future data/information available at the decision/action point.

The OLP problem takes a standard linear program as its underlying form (with \( n \) decision variables and \( m \) constraints), while the constraint matrix is revealed column by column with the corresponding coefficient in the linear objective function. In this paper, we consider the standard random input model (See (Goel and Mehta, 2008; Devanur et al., 2019)) where the orders, represented by the columns of constraint matrix together with the corresponding objective coefficient, are sampled independently and identically from an unknown distribution \( P \). At each timestamp, the value of the decision variable needs to be determined based on the past observations and cannot be changed afterwards. The goal is to minimize the gap (formally defined as regret) between the objective value solved in this online fashion and the “offline” optimal objective value where one has the full knowledge of the linear program data.

There were many algorithms and research results on OLP in the past decade due to its importance and wide applications. Virtually all current online algorithms were based on learning the LP dual optimal solutions/prices, and their analyses of OLP were focused on the aggregate objective value and solving the packing LP where all coefficients in the constraint matrix and objective are nonnegative. Two major open questions in the literature are: (1) Does the set of LP optimal dual prices of OLP converge to those of the offline LP, and (2) Could the results be extended to the true general LP problems where the coefficients can be either positive or negative. We resolve these two questions in this paper as part of our results. Moreover, we propose a new type of OLP algorithm and develop tools for the regret analysis of OLP algorithms. Our key results and main contributions are summarized as follows.

### 1.1 Key Results and Main Contributions

**Dual convergence of online linear programs.** We establish self-contained results on the convergence of online dual optimal prices of linear programs in Section 3. We first point out how the dual linear program, under random input model, can be viewed as a Sample Average Approximation (SAA) (Kleywegt et al., 2002; Shapiro et al., 2009) of a constrained stochastic programming problem. The stochastic program is defined by the LP right-hand-side and the distribution \( P \). Our key result states that, under moderate regularity conditions, the optimal solution of the sampled dual program will converge to the optimal solution of the stochastic program. The literature on stochastic programming and statistics
provides either finite-sample convergence results for a single function value or asymptotic convergence results for the componentwise estimation error (the distance between the optimal solution of SAA problem and that of the stochastic program). But our result here is a finite-sample result on the estimation error which explicitly characterizes the convergence rate of the dual optimal solution in terms of $n$ and $m$. This result complements to the typical results of SAA in stochastic programming, and the study of asymptotic properties of M-estimators in statistics. We emphasize that our results on dual convergence are not only pertaining to the online packing LP, but also hold for general LPs where the input data coefficients can be either positive or negative.

**Action-history-dependent learning algorithm.** We develop a new type of OLP algorithm — Action-history-dependent Learning Algorithm in Section 4.4. This new algorithm is a dual-based algorithm (as (Devanur et al., 2011; Agrawal et al., 2014; Gupta and Molinaro, 2014), etc.) and it utilizes our results on the convergence of the sampled dual optimal solution. One common pattern in the designing of most previous algorithms is that the choice of the decision variable at time $t$ depends on only the past input data, i.e., the coefficients in the constraints and the objective function revealed, but not the decisions were already made, until time $t - 1$. However, our new action-history-dependent algorithm considers both the past input data and the past choice of decision variables. This new algorithm is more conscious of the constraints/resources consumed by the past actions, and thus the decisions can be made in a more dynamic, close-loop, and nonstationary way. We demonstrate in both theory and numerical experiments that this actions-history-dependent mechanism significantly improves the online performance.

**Regret bounds for OLP.** We analyze the worst-case gap (regret) between the expected online objective value and the “offline” optimal objective value. Specifically, we study the regret in an asymptotic regime where the number of constraints $m$ is fixed and the LP right-hand-side input scales linearly with the number of decision variables $n$. As far as we know, this is the first regret analysis result in the general OLP formulation. We show that there is no dual-based learning algorithm that achieves a worse-case regret smaller than $O(\log n)$ (Theorem 7). The proof of lower bound relates the OLP problem to a statistical estimation problem and adopts the same methodology as the proof of $O(\log n)$ lower bound in the dynamic pricing problem (Keskin and Zeevi, 2014) and the data-driven newsvendor problem (Besbes and Muharremoglu, 2013). We are able to derive an $O(\log n \log \log n)$ regret upper bound for the proposed action-history-dependent learning algorithm; while the algorithm, with exact knowledge of the optimal dual prices of the corresponding stochastic program, has the regret bound $O(\sqrt{n})$. Our regret analysis provides an algorithmic insight for a constrained online optimization problem: a successful algorithm should have a good control of the binding constraints (resources) consumption – not exhausting the resources too early or having too much leftover at the end, an aspect usually overlooked by typical dual-price learning algorithms. Moreover, the results and methodologies used here (such as Theorem 2
and Theorem 6) are potentially applicable to other contexts of online learning and online decision making problems.

1.2 Literature Review

Online optimization/computation problems have been long studied in the community of operations research and theoretical computer science. We refer readers to the papers, e.g., (Borodin and El-Yaniv, 2005; Buchbinder et al., 2009) for a general overview of the topic and the recent developments. We discuss the scope of this paper and compare against the literature from the following four aspects.

<table>
<thead>
<tr>
<th></th>
<th>Literature</th>
<th>Ours</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) Input</td>
<td>Permutation</td>
<td>i.i.d. columns</td>
</tr>
<tr>
<td>(ii) Market</td>
<td>Single-sided (a_{ij} ≥ 0)</td>
<td>Double-sided (a_{ij} ∈ ℝ)</td>
</tr>
<tr>
<td>(iii) Constraints b</td>
<td>Scarce</td>
<td>Linear growth</td>
</tr>
<tr>
<td>(iv) Measure</td>
<td>Competitiveness ratio</td>
<td>Regret</td>
</tr>
</tbody>
</table>

Table 1: Comparison of our setting with literature

(i) Input model: We consider the standard random input model (also known as random-order model) where the columns of constraint matrix are generated i.i.d. from an unknown distribution \( P \), whereas many papers on this topic adopt the random permutation model which assumes the columns are arriving at a random order and the arrival order is uniformly distributed over all permutations (e.g., (Molinaro and Ravi, 2013; Agrawal et al., 2014; Gupta and Molinaro, 2014) and the references therein). In many application contexts, such as the online advertising problem, revenue management in flight and hotel booking, or the online auction market, each column in the constraint matrix together with the corresponding coefficient in the objective function represents an individual order/bid/query. In this sense, our random input model can be interpreted as an independence assumption across different customers.

(ii) Double-sided market: A typical assumption in the OLP literature requires the data entries in the constraint matrix and objective to be non-negative. We do not make this assumption in our model. Therefore our model can capture the double-sided market with both buying and selling orders.

(iii) Right-hand-side assumption: An important distinction between our work and the OLP literature is the assumption on the right-hand-side of the LP, also referred to as resource capacity in the literature of online network revenue management. A stream of important papers (Devanur and Hayes, 2009; Molinaro and Ravi, 2013; Agrawal et al., 2014; Kesselheim et al., 2014; Gupta and Molinaro, 2014) explore the trade-off between the algorithm competitiveness and the resource capacity. It culminates in a necessary and sufficient condition for the existence of \( \epsilon \)-competitive algorithm, that the resource capacity should scale as \( O(\log m/\epsilon^2) \). We argue that this result is only effective when \( \epsilon \) is constant or larger than \( O(1/\sqrt{n}) \). Because if \( \epsilon \leq O(1/\sqrt{n}) \), this condition
entails resource scales at least as $O(n \log m)$ (usually with a large constant), then we can simply accept all the orders. On the contrary, if $1/\epsilon = o(\sqrt{n})$, then the resource capacity is $o(n \log m)$. In this case, we can only accept a smaller proportion of orders; in other words, the service level - proportion of the orders being served, goes to zero as $n$ grows. In this paper, we study the situation when the resource capacity scales linearly with $n$, i.e., keeping the service level at a constant level, whether we can further improve the algorithm competitiveness. Therefore, our model is more suitable in a decision environment where the decision problem is continuous and repeated, and the resources/inventories can be refilled or replenished.

(iv) Performance Measure: In this paper, we consider regret instead of competitiveness as our research objective. In our regime of constant service level, the LP’s offline optimal objective value may also scale linearly with $n$. In this scenario, an $\epsilon$-competitiveness result will potentially incur an $O(n)$ gap between the online objective value and the offline optimal objective. Alternatively, the regret directly measures this gap and better compares the performance among different algorithms/mechanisms. Moreover, our result of $O(\log n \log \log n)$ regret is equivalent to a competitiveness ratio of $O\left(\frac{\log n \log \log n}{n}\right)$, notably better than the constant or $O\left(\frac{1}{\sqrt{n}}\right)$ competitiveness ratio given by the previous algorithms.

Another line of research originates from the revenue management literature and studies a parameterized type of the OLP problem. Specifically, it assumes heterogeneous customers arrive according to a Poisson process and the customers can be divided into finitely many classes based on their requests and prices. In this way, the columns of the constraint matrix with the corresponding coefficient in the objective follow a well-parameterized distribution and have a finite and known support. Both the situations that the parameters in the Poisson process and the price distribution are known (Reiman and Wang, 2008; Jasin and Kumar, 2013; Bumpensanti and Wang, 2018) and unknown (Jasin, 2015; Ferreira et al., 2018) are studied. The objective of regret in our current paper aligns with this line of literature, and our action-history-dependent learning algorithm can be viewed as a non-parametric generalization of deterministic LP based algorithms (Jasin, 2015) (for network revenue management problem) and the adaptive algorithm (Arlotto and Gurvich, 2019) (for multi-secretary problem).
2 Problem Formulation

In this section, we formulate the OLP problem and define the objective. Consider a generic LP problem

\[
\text{max } \sum_{j=1}^{n} r_j x_j \tag{1}
\]

\[
\text{s.t. } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m
\]

\[
0 \leq x_j \leq 1, \quad j = 1, \ldots, n
\]

where \( r_j \in \mathbb{R} \), \( a_j = (a_{1j}, \ldots, a_{mj}) \in \mathbb{R}^m \), and \( b = (b_1, \ldots, b_m) \in \mathbb{R}_+^m \). Throughout this paper, we use bold symbols to denote vectors/matrices and normal symbols for scalars.

In the online setting, the parameters of the linear program are revealed in an online fashion and one needs to determine the value of decision variables sequentially. Specifically, at each time \( t \), the coefficients \((r_t, a_t)\) are revealed, and we need to decide the value of \( x_t \) instantaneously. Different from the offline setting, at time \( t \), we do not have the information of the following coefficients to be revealed. Given the history \( H_{t-1} = \{r_j, a_j, x_j\}_{j=1}^{t-1} \), the decision of \( x_t \) can be expressed as a policy function of the history and the coefficients observed at the current time period. That is,

\[
x_t = \pi_t(r_t, a_t, H_{t-1}). \tag{2}
\]

The policy function \( \pi_t \) can be time-dependent and we denote policy \( \pi = (\pi_1, \ldots, \pi_n) \). The \( x_t \)'s must conform to the constraints that

\[
\sum_{j=1}^{t} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m,
\]

\[
0 \leq x_t \leq 1.
\]

The objective is to maximize the objective \( \sum_{j=1}^{n} r_j x_j \).

We illustrate the problem setting through the following practical example. Consider a market making company receiving both buying and selling orders, and the orders arrive sequentially. At each time \( t \), we observe a new order, and we need to decide whether to accept or reject the order. The order is a buying/selling request for the resources, or it could be a mixed request, e.g., selling the first resource for 1 unit and buying the second resource for 2 units with a total order price of $1. Once our decision is made, the order will leave the system, either fulfilled or rejected. In this example, the term \( b_i \) can be interpreted as the total available inventory for the resource \( j \), and the decision variables \( x_t \)'s can be interpreted as the acceptance and rejection of an order. Particularly, we do not allow shorting of the resource along the process. Our goal is to maximize the total revenue.

We assume the LP parameters \((r_j, a_j)\) are generated i.i.d. from an unknown distribution \( P \in \Xi \).
\( \Xi \) denotes a family of distributions satisfying some regularity conditions (to be discussed in Section 3). We denote the offline optimal solution of linear program (1) as \( x^* = (x^*_1, ..., x^*_n) \), and the offline (online) objective value as \( R_n^* (R_n) \). Specifically,

\[
R_n^* = \sum_{j=1}^{n} r_j x^*_j \\
R_n(\pi) = \sum_{j=1}^{n} r_j x_j.
\]

in which online objective value depends on the policy \( \pi \). In this paper, we consider a fixed \( m \) and large \( n \) regime, and focus on the worst-case gap between the online and offline objective. This regime aligns with the OLP literature of the revenue management (See Jasin and Kumar (2013); Jasin (2015) and the references therein). Specifically, we define the regret

\[
\Delta_{P}^n(\pi) = E_{P} [R_n^* - R_n(\pi)]
\]

and the worst-case regret

\[
\Delta_n(\pi) = \sup_{P \in \Xi} \Delta_{P}^n(\pi) = \sup_{P \in \Xi} E_{P} [R_n^* - R_n(\pi)].
\]

Throughout this paper, we omit the subscript \( P \) in the expectation notation when there is no ambiguity. The worst-case regret takes the supremum regret over a family of distributions so that it is suitable as a performance guarantee when the distribution \( P \) is unknown.

### 3 Dual Convergence

OLP algorithms rely on solving the dual problem of linear program (1). However, there is still a lack of theoretical understanding of the properties of the dual optimal solutions. In this section, we establish convergence results on the OLP dual solutions and lay foundations for the analyses of the OLP algorithms.

To begin with, the dual of linear program (1) is

\[
\begin{align*}
\min & \quad \sum_{i=1}^{m} b_i p_i + \sum_{j=1}^{n} y_j \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} p_i + y_j \geq r_j, \quad j = 1, ..., n. \\
& \quad p_i, y_j \geq 0 \text{ for all } i, j.
\end{align*}
\]

Here the decision variables are \( p = (p_1, ..., p_m) \) and \( y = (y_1, ..., y_n) \).

From the LP duality, if \( (p_n^*, y_n^*) \) is an optimal solution for the dual LP (3), then the primal optimal
solution must satisfy

\[ x_j^* = \begin{cases} 
1, & r_j > a_j^T p_n^* \\
0, & r_j < a_j^T p_n^*.
\end{cases} \]  

(4)

And the degeneration case can be ignored when \( n \) is large. One observation is that the primal optimal solution largely depends on the dual optimal solution \( p_n^* \). This underscores the importance of studying the optimal dual solutions. Following this observation, we present an equivalent form of the dual program (3):

\[
\min \sum_{i=1}^m b_i p_i + \sum_{j=1}^n \left( r_j - \sum_{i=1}^m a_{ij} p_i \right)^+ \\
p_i \geq 0, \ i=1,...,m.
\]

(5)

where \((\cdot)^+\) is the positive part function, also known as the ReLu function. The optimization problem (5), despite not being a linear program, has a convex objective function. It has the advantage of only involving \( p \) which is closely related to the optimal primal solution. More importantly, the summand in the second part of the objective function (5) are independent of each other, and therefore the sum (after a normalization) will converge to certain deterministic function. To better make this point, let \( d_i = b_i/n \) and divide the objective function in (5) by \( n \). Then the optimization problem can be rewritten as

\[
\min f_n(p) := \sum_{i=1}^m d_i p_i + \frac{1}{n} \sum_{j=1}^n \left( r_j - \sum_{i=1}^m a_{ij} p_i \right)^+ \\
p_i \geq 0, \ i=1,...,m.
\]

(6)

The second term in the objective function (6) is a summation of \( n \) i.i.d. random functions.

Consider a stochastic programming problem defined as follows.

\[
\min f(p) := a^T p + \mathbb{E} [(r - a^T p)^+] \\
\text{subject to } p \geq 0,
\]

(7)

where the expectation is taken with respect to \((r,a)\). In the rest of the paper, unless otherwise stated, the expectation is always taken with respect to \((r,a)\). Apparently, we have

\[ \mathbb{E} f_n(p) = f(p) \]

for all \( p \). This observation casts the dual convergence problem in a form of stochastic programming problem. The function \( f_n(p) \) in (6) can be viewed as a sample average approximation (SAA) (See
(Kleywegt et al., 2002; Shapiro et al., 2009)) of the function \( f(p) \). Specifically, the dual program associated with a primal linear program with \( n \) decision variables is then an \( n \)-sample approximation of the stochastic program (7). We denote the optimal solutions to the \( n \)-sample approximation problem (6) and the stochastic program (7) with \( p^*_n \) and \( p^* \), respectively. Under some regularity conditions, the sequence of random function \( f_n(p) \)'s will (weakly)\(^1\) converge to some deterministic function \( f(p) \),

\[
f_n(p) \Rightarrow f(p),
\]

and the optimal solution \( p^*_n \) of \( f_n(p) \) will converge to that of \( f(p) \)

\[
p^*_n \Rightarrow p^*.
\]

In this section, we provide a rigorous treatment to the above two convergences. Naturally, the deterministic function \( f(p) \) and its corresponding optimal solution \( p^* \) are dependent on the distribution \( P \) that governs the generation of the parameter \((r_j, a_j)\)'s, but we will establish universal convergence rates that do not depend on \( P \). This is important because in the OLP setting, we do not assume the knowledge of distribution when implementing the online algorithms.

3.1 Assumptions

The first group of assumptions concerns the boundedness and the linear growth of the constraints.

**Assumption 1 (Boundedness and Constraints’ Linear Growth).** We assume

(a) \( \{(r_j, a_j)\}_{j=1}^n \) are generated i.i.d. from distribution \( P \).

(b) There exist constants \( a, \bar{a} \in \mathbb{R}^+ \) such that \( a \leq \|a_j\|_2 \leq \bar{a} \).

(c) There exists a constant \( \bar{r} \in \mathbb{R}^+ \) such that \( |r_j| \leq \bar{r} \).

(d) \( d_i = b_i/n \in (d, \bar{d}) \) for \( d, \bar{d} > 0 \), \( i = 1, ..., m \). We know \( d_i \)'s a priori.

(e) \( n > m \).

Throughout this paper, \( \| \cdot \|_2 \) denotes the \( l_2 \)-norm of a vector.

Assumption 1 (a) states that parameters (coefficients in the objective function and columns in the constraint matrix) of linear program (1) is generated i.i.d. from some distribution \( P \). The vectors \( \{(r_j, a_j), j = 1, ..., n\} \) are independent of each other, but their components may be dependent. Assumption 1 (b) and (c) requires the parameters are bounded. The bound parameters \( a, \bar{a} \) and \( \bar{r} \) are introduced only for analysis purpose and will not be used for algorithm implementation. Assumption 1 (d) requires

\(^1\)The convergence refers to the weak convergence in the probability space of the continuous functions.
the right-hand-side of the LP constraints grows linearly with \( n \). This guarantees that for the (optimal) solutions, \( O(n) \) of the \( x_j \)'s could be 1. As explained earlier, it means \( O(n) \) of the orders/requests can be fulfilled and ensures a constant service level - percentage of orders satisfied. It is a legitimate assumption in that if this is not true, the service level will go to zero when the business running periods \( n \) goes to infinity. \( d_i \) has the interpretation of available resource per period. The parameters \( d \) and \( \bar{d} \) are only for analysis purpose and it can be arbitrarily positive numbers. Also, we require that the number of decision variables \( n \) is larger than number of constraints \( m \). While discussing the dual convergence, the dimension of the dual variable \( p \) is equal to the number of constraints \( m \) and the number of primal decision variables \( n \) can be viewed as the number of samples used to estimate \( p^* \). The assumption of \( n > m \) restricts our attention to a low-dimensional setting.

We introduce a second group of assumptions on the distributional properties of \((r_j, a_j)\).

**Assumption 2 (Distribution).** We assume

(a) The expected second-order moment matrix \( M := \mathbb{E}[aa^\top] \) is positive-definite. Denote the minimum eigenvalue with \( \lambda_{\min} \).

(b) Assume the conditional distribution \( r|a \) has a density function \( f_a(r) \). There exist constants \( \nu_1, \nu_2 \in \mathbb{R}_+ \) such that \( \nu_1 \leq f_a(r) \leq \nu_2 \) holds for any \( r \in [-\bar{r}, \bar{r}] \) and \( a \).

(c) The optimal solution \( p^* \) to the stochastic optimization problem (7) satisfies \( p^*_i = 0 \) if and only if \( d_i - \mathbb{E}_{(r,a) \sim P}[a_i I(r > a^\top p^*)] > 0 \).

Assumption 2 is essentially to enforce a strong convexity assumption for the function \( f(p) \). This is the key for an exponential convergence rate of \( p^*_n \) to \( p^* \). Part (a) is mild in that the matrix \( aa^\top \) is positive semi-definite; the positive definiteness holds as long as there is no vector \( \beta \) such that \( \beta^\top a = 0 \) with probability 1. The intuition for part (b) is that the reward \( r \) needs to have enough dispersion. When we set the decision \( x_j = I(r_j > a^\top p) \), this assumption ensures that different choices of dual price will lead to different decisions. Part (c) comes from the LP's complimentary slackness and it states that the binding and non-binding constraints can be clearly distinguished.

We use the notation \( \Xi \) to denote the family of distributions that satisfy Assumption 1 and 2. In the rest of the paper, all the theoretical results, including the dual convergence and the analyses of OLP algorithms, are established under Assumption 1 and 2.

### 3.2 Dual Convergence

Proposition 1 summarizes several basic properties related to the dual LP (3) and the stochastic program (7). Part (a) states that the optimal solution to the dual problem is also the optimal solution of \( f_n \). Part (b) and (c) are easily implied from the assumptions. Part (d) of the proposition comes from the KKT
optimality conditions. The optimal solution will have either \( p_i^* = 0 \) or \( d_i = E(r,a) \sim P [a_i I(r > a^\top p^*)] \). In the first case, the \( i \)-th constraint is nonbinding, while in the second case, the \( i \)-th constraint is binding and the resource \( d_i \) will be exhausted in expectation, if we set the acceptance-rejection decision rule with \( p^* \). This result provides a characterization of function \( f(p) \) at the point \( p^* \).

**Proposition 1.** Under Assumption 1, we have the following results on \( f_n \) and \( f \).

(a) The optimal solution of problem (5) is identical to the optimal solution of problem (3).

(b) Both \( f_n(p) \) and \( f(p) \) are convex.

(c) The optimal solutions \( p^*_n \) and \( p^* \) satisfy

\[
\begin{align*}
  d^\top p^*_n & \leq \bar{r}, \\
  d^\top p^* & \leq \bar{r}.
\end{align*}
\]

(d) The optimal solution \( p^* \) satisfies

\[
p^*_i (d_i - E(r,a) \sim P [a_i I(r > a^\top p^*)]) = 0
\]

for \( i = 1, \ldots, m \). Here and hereafter, \( I(\cdot) \) denotes the indicator function. Specifically, if the condition in its argument is satisfied, it will output 1; otherwise it will output 0.

Define

\[
\Omega_p := \left\{ p \geq 0 : e^\top p \leq \frac{\bar{r}}{2} \right\}
\]

where \( e \in \mathbb{R}_m \) is all-one vector. From Proposition 1, we know \( \Omega_p \) covers all possible optimal solutions to (6) and (7). Define two index sets

\[
I_B = \left\{ i : d_i - E(r,a) \sim P [a_i I(r > a^\top p^*)] = 0 \right\} \\
I_N = \left\{ i : p^*_i = 0 \right\},
\]

where the subscripts “B” and “N” denote binding and non-binding, respectively. From Assumption 2 (c) and Proposition 1 (d), we know \( I_B \cap I_N = \emptyset \) and \( I_B \cup I_N = \{1, \ldots, m\} \). With \( I_B \) and \( I_N \), we define the binding and non-binding constraints of the OLP problem according to the stochastic program (7).

Figure 1 illustrates the relationship between \( f(p) \) and \( f_n(p) \). Intuitively, with a large \( n \), the function \( f_n(p) \) is close to the function \( f(p) \) and so are their optimal solutions. Most of the literature in the fields of stochastic programming and the machine learning theory study the quantity \( f(p^*_n) - f(p^*) \) or
$f_n(p^*_n) - f(p^*)$. Under certain regularity conditions, researchers provide asymptotic results and finite-sample bounds for the uniform convergence of $f_n$ to $f$; for example, $f(p^*_n) - f(p^*) = O(1/\sqrt{n})$ with high probability. However, in the context of OLP, we will use the learned dual price $p^*_n$ as the decision rule. So, we care more about the convergence of $p^*_n$ rather than the convergence of $f(p^*_n)$. Additionally, as we will see in the later sections, to conduct the regret analysis, the convergence of $p^*_n$ should be characterized in a finite-sample (but not asymptotic) way.

To establish the convergence of $p^*_n$, we take an approach of doing Taylor expansion at the point $p^*_n$ for the function $f_n$. The approach is inspired by the study of M-estimator in statistics (See Chapter 3 of (Van der Vaart, 2000) for definitions and examples). In fact, $p^*_n$ can be viewed as an M-estimator for some true statistical parameters $p^*$. Different from the statistics literature, we derive a finite-sample convergence rate for $E\left[\|p^*_n - p^*\|_2\right]$ with the help of Bernstein/Hoeffding concentration arguments. Also, we apply the techniques used in the asymptotic analysis for quantile regression (See Chapter 3 of (Koenker, 2005)) to resolve the non-differentiability of the function $f_n$.

Define $h : \mathbb{R}_m \times \mathbb{R}_{m+1} \to \mathbb{R}$ as

$$h(p, u) \triangleq \sum_{i=1}^m d_i p_i + \left(u_0 - \sum_{i=1}^m u_i p_i\right)^+$$

and $\phi : \mathbb{R}_m \times \mathbb{R}_{m+1} \to \mathbb{R}_m$ as

$$\phi(p, u) \triangleq \frac{\partial h(p, u)}{\partial p} = (d_1, \ldots, d_m)^\top - (u_1, \ldots, u_m)^\top \cdot I\left(u_0 > \sum_{j=1}^m u_j p_j\right)$$

where $u = (u_0, u_1, \ldots, u_m)$, $(\cdot)^+$ is the positive part function and $I(\cdot)$ is the indicator function as defined in Proposition 1. The function $\phi$ has the interpretation of partial sub-gradient of the function $h$ with respect to $p$. The function $f_n(p)$ and $f(p)$ can be represented as

$$f_n(p) = \frac{1}{n} \sum_{j=1}^n h(p, u_j)$$

$$f(p) = \mathbb{E}_{u \sim \mathcal{P}}[h(p, u)].$$
where \( u_j = (r_j, a_j) \). For any \( p \in \mathbb{R}_m \), we have the following identity,

\[
f_n(p) - f_n(p^*) = \frac{1}{n} \sum_{j=1}^{n} \phi(p^*, u_j)^\top (p - p^*) + \frac{1}{n} \sum_{j=1}^{n} a_j^\top p \left( I(r_j > v) - I(r_j > a_j^\top p^*) \right) dv.
\]  
(8)

First-order

Second-order

Note that this identity holds regardless of the distribution \( P \). Its derivation shares the same idea with the Knight’s identity in (Knight, 1998). Indeed, the first and second term on the right-hand side of the identity can be interpreted as the first- and second- order term in Taylor expansion. They are disguised in this special form due to the non-differentiability of \( h \). The following two propositions analyze these two terms respectively. Proposition 2 tells that the sample average sub-gradient \( \frac{1}{n} \sum_{j=1}^{n} \phi(p^*, u_j) \) stays close to the gradient \( \nabla f(p^*) \) evaluated at \( p^* \) with high probability. Its proof is similar to the proof of convergence \( f_n \) to \( f \) in literature of SAA but just applies the same machinery for the gradient function. Proposition 3 states that the second-order term – the integral on the right hand of (8), is uniformly lower bounded by a strongly convex quadratic function with high probability.

**Proposition 2.** We have

\[
P \left( \left\| \frac{1}{n} \sum_{j=1}^{n} \phi(p^*, u_j) - \nabla f(p^*) \right\|_2 \leq \epsilon \right) \geq 1 - 2m \exp \left( - \frac{ne^2}{2\bar{a}^2m} \right)
\]

hold for any \( \epsilon > 0 \), all \( n > m \) and \( P \in \Xi \). Additionally, \( (\nabla f(p^*))_i = 0 \) for \( i \in B \) and \( (\nabla f(p^*))_i > 0 \) for \( i \in N \).

Proposition 2 is proved by componentwise concentration analysis of the sample gradient and then applying the union bound to obtain the vector result. Detailed proof can be found in Section A2. The probability bound on right-hand-side is not dependent on the distribution \( P \) and the inequality holds for all \( \epsilon > 0 \).

**Proposition 3.** We have

\[
P \left( \frac{1}{n} \sum_{j=1}^{n} a_j^\top p \left( I(r_j > v) - I(r_j > a_j^\top p^*) \right) dv \geq -e^2 - 2\epsilon \|p^* - p\|_2 + \frac{\nu_1 \lambda_{\min}}{16} \|p^* - p\|_2 \right)
\]

for all \( p \in \Omega_p \) \( \geq 1 - m \exp \left( - \frac{ne^2}{4\bar{a}^2} \right) - 2 (2N)^m \exp(-2ne^2) \)

holds for any \( \epsilon > 0 \), \( n > m \) and \( P \in \Xi \). Here

\[
N = \left\lfloor \log_q \left( \frac{e^2}{v^2 \bar{a}^{2/(\sqrt{m})}} \right) \right\rfloor + 1, \quad q = \max \left\{ \frac{1}{1 + \sqrt{m}}, \frac{1}{1 + \sqrt{m}} \left( \frac{\nu_1 \lambda_{\min}}{4\bar{a}^2} \right)^\frac{1}{2} \right\}
\]

where \( \lfloor \cdot \rfloor \) is the floor function.
Proposition 3 discusses the second-order term in (8). The challenging part of the proof is to show that the inequality (9) holds uniformly for all \( p \in \Omega_p \). Proving the inequality for a fix \( p \) can be easily done by a concentration argument like in Proposition 2. To prove the uniform bound on \( p \), the idea is to find a collection of sets that covers \( \Omega_p \) and then analyze each covering set separately. We utilize the same covering scheme as in (Huber et al., 1967) which is originally used to develop consistency results for non-standard maximum likelihood estimators. The advantage of this covering scheme is that it provides a tighter probability bound than the traditional \( \epsilon \)-covering scheme. Detailed proof can be found in Section A3. We point out that both probability bounds in Proposition 2 and 3 have a exponential rate in \( n \), and that all the other parameters involved are not dependent on the distribution \( P \).

With Proposition 2 and 3, the identity (8) can be written heuristically as

\[
f_n(p) - f_n(p^*) \geq \nabla f(p^*)^\top (p - p^*) - \epsilon \|p^* - p\|_2 - \epsilon^2 - 2\epsilon \alpha^\top |p^* - p|_2 + \frac{\nu_1 \lambda_{\min}}{16} \|p^* - p\|_2^2 \geq -\epsilon^2 - (2\alpha + 1)\epsilon \|p^* - p\|_2 + \frac{\nu_1 \lambda_{\min}}{16} \|p^* - p\|_2^2 \text{ uniformly for all } p \in \Omega_p
\]

with high probability. Note that the right side in above is a quadratic function of \( p \) and from the optimality of \( p_n \)

\[
f_n(p_n) \leq f_n(p^*).
\]

If we put together (10) with (11) and then do integral for the tail probability, we can obtain the dual convergence result in Theorem 1. Its detailed proof can be found in Section A4.

**Theorem 1.** There exists a constant \( C \) such that

\[
\mathbb{E}[\|p_n^* - p^*\|_2^2] \leq \frac{Cm \log m \log \log n}{n}
\]

holds for all \( n > m \) and distribution \( P \in \Xi \). Also, it implies

\[
\mathbb{E}[\|p_n^* - p^*\|_2] \leq \sqrt{\frac{Cm \log m \log \log n}{n}}.
\]

Theorem 1 states the convergence rate in terms of the \( L_2 \) error. The uniform convergence rate in \( P \) creates convenience for the analyses of OLP algorithms in that we do not assume the knowledge of \( P \) for the OLP problem. Huber et al. (1967) and Shapiro (1993) also proved asymptotic convergence results on the MLE estimators and the SAA optimal solutions. Our proof utilizes the structure of the problem and allows a discrete distribution on the coefficient \( a_j \), while their results require all the random variables to have density functions. More importantly, our convergence in terms of \( L_2 \) error is stronger than their results (convergence in probability). Moreover, the \( L_2 \) convergence is necessary in the regret analysis for
the OLP-type problem. For example, Keskin and Zeevi (2014) develop $L_2$ error bound for the ordinary least square estimator to analyze the regret of dynamic pricing algorithms.

Figure 2: An illustrative visualization of the dual convergence

We discuss some implications of the dual convergence result on the OLP problem. OLP algorithms usually involve computing dual optimal solution and form primal decision rules based on the dual solution. In the literature, due to the lack of convergence knowledge of the dual optimal solutions, papers derived other approaches to directly analyze the objective value. Here, dual convergence is explicitly characterized and it could provide us a powerful and natural instrument for the theoretical analyses of the online algorithms.

Figure 2 illustrates the idea of dual-based OLP algorithms. We know that the sequence of $p_n^*$’s will converge to $p^*$. The offline solution employs $p_n^*$ as the decision rule. When $n$ is large, $p_n^*$ stays rather close to $p^*$. The dual convergence result tells us that we do not actually need as many as $n$ samples to get a good estimate of $p^*$. At each time $t$, we can form an estimate for $p^*$ based on the observed $t$ inputs. If $t$ is sufficiently large, the estimate $p_t^*$ could be very close to both $p^*$ and the offline optimal $p_n^*$. More importantly, the results on convergence rate quantifies the distance between $p_t^*$ and $p^*$ (also the distance between $p_t^*$ and $p_n^*$). This quantification makes the first step in analyzing the difference between online objective and the offline optimal objective value.

Besides, the dual convergence result also contributes to the literature of approximate algorithms for large-scale LPs. Specifically, we can perform a one-time learning with the first $t$ inputs and then we use $p_t^*$ as an approximation for $p_n^*$. In this way, we obtain an approximate algorithm for solving the original LP problem by only accessing to the first $t$ columns $\{(r_j, a_j)\}_{j=1}^t$. This complements the recent work (Vu et al., 2018) in which approximate algorithms for large-scale LP are developed under certain statistical assumptions on the coefficients of the LP problem.
4 Learning Algorithms for OLP

4.1 Online Policy, Constraint Process, and Stopping Time

In this section, we present several online algorithms based on the dual convergence results derived in Section 3. We first revisit the definition of online policies and narrow down our scope to a special class of policies that rely on the dual-based thresholding. Specifically, at each time $t$, a vector $p_t$ is computed based on historical data

$$p_t = h_t(H_{t-1})$$

and we attempt to set

$$\tilde{x}_t = \begin{cases} 
1, & \text{if } r_t > a_t^\top p_t, \\
0, & \text{if } r_t \leq a_t^\top p_t.
\end{cases}$$

In other words, a threshold is set by the dual price vector $p_t$ and if the reward $r_t$ is larger than the threshold then we desire to accept the order. Then, we check the constraints satisfaction and assign

$$x_t = \begin{cases} 
\tilde{x}_t, & \text{if } \sum_{j=1}^{t-1} a_{ij}x_j + a_{it}\tilde{x}_t \leq b_i, \text{ for } i = 1, \ldots, m, \\
0, & \text{otherwise}.
\end{cases}$$

We emphasize that in this policy class, $p_t$ is first computed based on history (up to time $t-1$), and then $(r_t, a_t)$ is observed. This creates a natural conditional independence

$$(\tilde{x}_t, r_t, a_t) \perp H_{t-1}|p_t.$$ 

This matches the setting in online convex optimization where at each time $t$, the online player makes her decision before we observe the function $f_t$ (See (Hazan et al., 2016)). We will frequently resort to this conditional independence in the regret analysis. In this policy class, an online policy $\pi$ could be fully specified by the sequence $h_t$’s, i.e., $\pi = (h_1, \ldots, h_n)$. To facilitate our analysis, we introduce the constraint process $\{B_t\}_{t=1}^n$:

$$B_0 = b$$

$$B_t = B_{t-1} - a_t x_t.$$ 

In this way, $B_t = (b_1^{(t)}, \ldots, b_m^{(t)})^\top$ represents the vector of remaining resources at the end of the $t$-th period. In particular, $B_n = (b_1^{(n)}, \ldots, b_m^{(n)})^\top$ represents the remaining resources at the end of horizon. By the definition of OLP, $B_t \geq 0$ for $t = 1, \ldots, n$. Also, the process of $\{B_t\}_{t=0}^n$ is pertaining to the policy
π. Based on the constraint process, we define
\[
\tau_s := \min\{n\} \cup \left\{ t \geq 1 : \min_i b_i(t) < s \right\}
\]
for \( s > 0 \). In this way, \( \tau_s \) denotes the first time that there are less than \( s \) units for some type of resources. Precisely, \( \tau_s \) is a stopping time adapted to the process \( \{B_t\}_{t=1}^n \). Similar to the process \( B_t \), the stopping time \( \tau_s \) is also pertaining to the policy \( \pi \). In executing an online policy, we do not close the business at the first time that some constraints are violated. This is because we are considering a double-sided problems including both buying and selling orders. If certain type of resource is exhausted, we may accept selling orders containing that resource as a way of replenishment. However, we emphasize that with a careful design of the algorithm, the constraints violation will happen only at the very end of the procedure. So, the decision afterwards actually does not affect much the cumulative revenue.

In the rest of this section, we first derive an upper bound for the regret of OLP algorithms and then present three OLP algorithms. These three algorithms all belong to the above dual-based policy class, and all of them rely on the dual convergence and are applicable to the double-sided setting. All the results in this section are subject to Assumption 1 and 2. We restrict our attention to large-\( n \) and small-\( m \) setting, and the regret bounds will be presented with big-O notation which treats the parameters in Assumption 1 and 2 and \( m \) as constants.

### 4.2 Upper Bound for OLP Regret

Now, we derive a generic regret bound for dual-based online policies. First, consider the optimization problem
\[
\begin{align*}
\max_{\mathbf{p} \geq 0} & \quad \mathbb{E} \left[ rI(r > \mathbf{a}^\top \mathbf{p}) \right] \\
\text{subject to} & \quad \mathbb{E} \left[ \mathbf{a}I(r > \mathbf{a}^\top \mathbf{p}) \right] \leq d.
\end{align*}
\]
(12)

There are two ways to interpret this optimization problem. On one hand, we can interpret this problem as a “deterministic” relaxation of the primal LP (1). We substitute both the objective and constraints of (1) with an expectation form expressed in dual variable \( \mathbf{p} \). On the other hand, we can view this optimization problem as the primal problem of the stochastic program (7). The consideration of a deterministic form for an online decision making problem has appeared widely in the literature of network revenue management (Talluri and Van Ryzin, 1998; Jasin and Kumar, 2013; Bumpensanti and Wang, 2018) and dynamic pricing (Besbes and Zeevi, 2009; Wang et al., 2014; Lei et al., 2014; Chen and Gallego, 2018), and bandit problem (Wu et al., 2015). The idea behind is that when analyzing the regret of an online algorithm in such problems, the offline optimal value usually does not have a tractable form (such as the
primal LP problem (1)). The deterministic formulation serves as a tractable upper bound for the offline optimal, and then the gap between the deterministic optimal and the online objective values is an upper bound for the regret of the online algorithm. Different from the literature, we proceed one step forward and consider the Lagrangian of the deterministic formulation. Define

\[ g(p) = \mathbb{E}\left[ r I(r > a^\top p) + (d - a I(r > a^\top p))^\top p^* \right]. \]

We can view \( g(p) \) as the Lagrangian of the optimization problem (12) with a specification of the dual variable by \( p^* \). Lemma 1 establishes that the deterministic formulation does provide an upper bound for the offline optimal and that the optimization problems (7) and (12) share the same optimal solution.

**Lemma 1.** Under Assumption 1 and 2, we have

\[ \mathbb{E} R_n^* \leq n g(p^*) \]

and

\[ g(p^*) \geq g(p) \]

for any \( p \geq 0 \). Here \( p^* \) is the optimal solution to the stochastic program (7). Additionally,

\[ \frac{1}{2} \nu_1 \lambda_{\min} \| p^* - p \|_2^2 \leq g(p^*) - g(p) \leq \nu_2 \bar{a}^2 \| p^* - p \|_2^2 \]

holds for all \( p \in \Omega_p \) and all the distribution \( P \in \Xi \).

The existence of constraints makes the regret analysis challenging, because the way how the constraints affect the objective value in an online setting is elusive and very problem-specific. The Lagrangian form \( g(p) \) provides a solution by incorporating the constraints into the objective. Intuitively, it assigns a cost/reward to the constraint consumption and thus unifies the two seemingly conflicting sides – revenue maximization and constraint satisfaction.

Theorem 2 compares the online objective value \( \mathbb{E} R_n(\pi) \) with \( ng(p^*) \) and develops a generic upper bound for dual-based online policies. Specifically, the upper bounds highlights three aspects of the underlying policy: (i) the cumulative “estimation” error, (ii) the remaining periods after the resource is almost exhausted, and (iii) the remaining resources. The first term (i) justifies why all OLP algorithms are essentially dual-based, and it relates the regret with \( \mathbb{E} \left[ \| p_t - p^* \|_2^2 \right] \) studied in the last section. The second term (ii) concerns \( \tau_{\bar{a}} \), where \( \bar{a} \) is the maximal possible resource consumption per period. The intuition for \( \tau_{\bar{a}} \) is that the order \( (r_t, a_t) \), if necessary, can always be fulfilled before time \( \tau_{\bar{a}} \). The third term (iii) considers the resource leftovers for binding constraints. The intuition is that the binding constraints are the bottleneck for generating revenue, so a leftover of those resources will induce a cost.
Theorem 2. There exists a constant $K$ such that the worst-case regret under policy $\pi$, 

$$
\Delta_n(\pi) \leq K \cdot E \left[ \sum_{t=1}^{\tau_n} \| p_t - p^* \|_2^2 + (n - \tau_n) + \sum_{i \in I_B} b_i^{(n)} \right]
$$

holds for all $n \in \mathbb{N}^+$. Here $I_B$ is the set of bidding constraints, $p_t$ is specified by the policy $\pi$, and $p^*$ is the optimal solution of the stochastic program (7).

Theorem 2 provides us important insights in the design of an online policy. At the first place, the policy should conduct learning of $p^*$. Meanwhile, the online policy should have a stable control of the resource/constraint consumption. Exhausting the resources too early may result in a large value of the term $n - \tau_a$ while the remaining resources at the end of the horizon may also induce regret through the term $\sum_{i \in I_B} b_i^{(n)}$. Essentially, both terms are stemmed from the fluctuation of the constraint consumption. The ideal case, although not possible due to the randomness, is that the $i$-th resource is consumed exactly of $d_i$ units in each period. The following corollary states that the result in Theorem 2 holds for a general class of stopping times.

Corollary 1. For any given $B_t$-adapted stopping time $\tau$, if $\mathbb{P}(\tau \leq \tau_a) = 1$, 

$$
\Delta_n(\pi) \leq K \cdot E \left[ \sum_{t=1}^{\tau-1} \| p_t - p^* \|_2^2 + (n - \tau) + \sum_{i \in I_B} b_i^{(n)} \right]
$$

holds for all $n \in \mathbb{N}^+$ with the same constant $K$ as in Theorem 2. Here $I_B$ is the set of bidding constraints, $p_t$ is specified by the policy $\pi$, and $p^*$ is the optimal solution of the stochastic program (7).

In the remainder of this section, we present three different algorithms and derive corresponding regret bounds with the help of Theorem 2 and Corollary 1.

4.3 Simplified Dynamic Learning Algorithm

We first present a simplified version of the dynamic learning algorithm in the paper (Agrawal et al., 2014). In Algorithm 1, the dual price vector $p_t$ is updated only at geometric time intervals and it is computed based on solving the $t$-sample approximation - minimizing $f_t(p)$. The key difference between this simplified algorithm and the dynamic learning algorithm in (Agrawal et al., 2014) is that we get rid of the shrinkage term $\left(1 - \epsilon \sqrt{\frac{n}{t}}\right)$ in the constraints. Specifically, in that paper, they have $\left(1 - \epsilon \sqrt{\frac{n}{t}}\right) t_k d_i$ on the right-hand side of the constraints in Algorithm 1 Step 6. This term resulted in an over-estimated dual price $p_t$ and hence will be more conservative in accepting orders. However, we argue that this term may be unnecessary because of our dual convergence theorem proved earlier. With this term removed, we no longer need to have the knowledge of $n$ but only need to know the average available resource $d_i$'s per period.
Algorithm 1 Simplified Dynamic Learning Algorithm

1: Input: \( d_1, \ldots, d_m \) where \( d_i = b_i/n \)
2: Initialize: Find \( \delta \in (1, 2] \) and \( L \in \mathbb{N}^+ \) s.t. \[\delta^L \equiv n.\]
3: Let \( t_k = \lfloor \delta^k \rfloor, k = 1, 2, \ldots, L - 1 \) and \( t_L = n + 1 \)
4: Set \( x_1 = \ldots = x_{t_1} = 0 \)
5: for \( k = 1, 2, \ldots, L - 1 \) do
6: Specify an optimization problem
   \[
   \max \sum_{j=1}^{t_k} r_j x_j \\
   \text{s.t.} \sum_{j=1}^{t_k} a_{ij} x_j \leq t_k d_i, \ i = 1, \ldots, m \\
   0 \leq x_j \leq 1, \ j = 1, \ldots, t_k
   \]
7: Solve its dual problem and obtain the optimal dual variable \( p_k^* \)
   \[
   p_k^* = \arg \min \sum_{i=1}^m d_i p_i + \frac{1}{t_k} \sum_{j=1}^{t_k} \left( r_j - \sum_{i=1}^m a_{ij} p_i \right) \\
   p_i \geq 0, \ i = 1, \ldots, m.
   \]
8: for \( t = t_k + 1, \ldots, t_{k+1} \) do
9: If constraints permit, set
   \[
   x_t = \begin{cases} 
   1, & \text{if } r_t > a_{i}^T p_k^* \\
   0, & \text{if } r_t \leq a_{i}^T p_k^*
   \end{cases}
   \]
10: Otherwise, set \( x_t = 0 \)
11: if \( t = n \), stop the whole procedure.
12: end for
13: end for

Theorem 3. With the online policy \( \pi_1 \) specified by Algorithm 1,

\[
\Delta_n(\pi_1) \leq O(\sqrt{n} \log n).
\]

Theorem 3 tells that the policy incurs a worst-case regret of \( O(\sqrt{n} \log n) \). The proof relies on an analysis of the three components in the regret bound in Theorem 2. Specifically, the term \( \sum_{t=1}^n \| p_t - p^* \|_2^2 \) is dealt with the dual convergence result. The two terms related to the stopping time and remaining resources are studied based on a careful analysis of the process

\[
Y_t = \sum_{j=1}^t a_j I(r_j > a_j^T p_j).
\]

Since the dual price \( p_j \) only depends on the history \( \{(r_j, a_j)\}_{j=1}^{t-1} \), the stochastic process \( \{Y_t\}_{t=1}^n \) can be viewed as an adapted and integrable stochastic process. The expectation and variance of \( Y_t \) can be
analyzed based on a treatment same as the Doob’s decomposition. The connection between

\[ E[n - \tau_0], \ E\left[ \sum_{i \in I_n} b_i^{(n)} \right] \]

and \( \{Y_t\}_{t=1}^n \) is then established and the proof is completed. The detailed proof can be found in Section A8.

### 4.4 Action-History-Dependent Learning Algorithm

Now we present our second algorithm — action-history-dependent learning algorithm. In Algorithm 1, \( p_t \) is a function of the past inputs \( \{(r_j, a_j)\}_{j=1}^{t-1} \) but it does not consider the past actions \( (x_1, ..., x_{t-1}) \), whereas Algorithm 2 integrates the past actions into the constraints of the optimization problem of \( p_t \).

Specifically, at the beginning of period \( t + 1 \), we observe the first \( t \) inputs \( \{(x_j, r_j, a_j)\}_{j=1}^t \). Algorithm 1 normalizes \( b_1 \) to \( \frac{t}{n} b_1 = td_i \) for the right-hand-side of the LP; in Algorithm 2, we normalize the remaining resources \( b_i^{(t)} \) for the right-hand-side of the LP (Step 6 of Algorithm 2). The intuition is that if we happen to consume too much resources in the past periods, the remaining resource \( b_i^{(t)} \) will shrink, and Algorithm 2 will accordingly push up the dual price and be more inclined to reject an order. On the contrary, if we happen to reject a lot of orders at the beginning and it results in too much remaining resources, the algorithm will lower down the dual price so as to accept more orders in the future. This pendulum-like design in Algorithm 2 incorporates the past actions in computing dual prices indirectly through the remaining resources.

**Theorem 4.** With the online policy \( \pi_2 \) specified by Algorithm 2,

\[ \Delta_n(\pi_2) \leq O(\log n \log \log n). \]

Theorem 4 states that Algorithm 2 incurs a worst-case regret of \( O(\log n \log \log n) \). The proof also relies on an analysis the three components of the upper bound in Theorem 2. A different methodology than the proof of Theorem 3 is adopted, but the analysis is still focused on the constraint process \( \{B_t\}_{t=1}^n \). The proof can be found in Section A9.

### 4.5 No-Need-to-Learn Algorithm

To complete our discussion, we present another algorithm in the situation when we know the distribution \( \mathcal{P} \) that generates the LP coefficients. With the knowledge of \( \mathcal{P} \), the stochastic programming problem (7) is well-specified. The study of this algorithm is more for theoretical purpose, so we do not discuss the practicability of knowing the distribution \( \mathcal{P} \). Moreover, we assume the stochastic programming
Algorithm 2  Action-history-dependent Learning Algorithm

1: Input: Input: $n, d_1, \ldots, d_m$
2: Initialize the constraint $b_i^{(0)} = nd_i$ for $i = 1, \ldots, m$
3: Initialize the dual price $p_1 = 0$.
4: for $t = 1, \ldots, n$ do
5: Observe $(r_t, a_t)$ and set
   \[ x_t = \begin{cases} 
   1, & \text{if } r_t > a_t^p p_t \\
   0, & \text{if } r_t \leq a_t^p p_t 
   \end{cases} \]
   if the constraints are not violated
6: Update the constraint vector
   \[ b_i^{(t)} = b_i^{(t-1)} - a_i x_t \text{ for } i = 1, \ldots, m \]
7: Specify an optimization problem
   \[
   \begin{align*}
   & \max \sum_{j=1}^t r_j x_j \\
   \text{s.t.} \quad & \sum_{j=1}^t a_{ij} x_j \leq \frac{tb_i^{(t)}}{n-t}, \quad i = 1, \ldots, m \\
   & 0 \leq x_j \leq 1, \quad j = 1, \ldots, t-1
   \end{align*}
   \]
8: Solve its dual problem and obtain the dual price $p_{t+1}$
   \[
   p_{t+1} = \arg \min_p \sum_{i=1}^m \frac{b_i^{(t)}}{n-t} p_i + \frac{1}{t} \sum_{j=1}^t \left( r_j - \sum_{i=1}^m a_{ij} p_i \right)^+ \\
   p_i \geq 0, \quad i = 1, \ldots, m.
   \]
9: end for

Theorem 5. With the online policy $\pi_3$ specified by Algorithm 3,
   \[
   \Delta_n(\pi_3) \leq O(\sqrt{n}).
   \]

Theorem 5 tells that the worst-case regret under an online policy with the knowledge of full distri-
Algorithm 3 No-Need-to-Learn
1: Input: Input: \( n, d_1, \ldots, d_m \), Distribution \( P \)
2: Compute the optimal solution for the stochastic programming problem
\[
p^* = \arg \min d^T p + \mathbb{E}_{(r,a) \sim P} [(r - a^T p)^+] \quad \text{subject to} \quad p \geq 0.
\]
3: for \( t = 1, \ldots, n \) do
4: If constraints are not violated, choose
\[
x_t = \begin{cases} 1, & \text{if } r_t > a_t^T p^* \\ 0, & \text{if } r_t \leq a_t^T p^* \end{cases}
\]
5: end for

bution is \( O(\sqrt{n}) \), which is comparable to Algorithm 1. The proof follows a similar idea as the proof of Theorem 3 and can be found in Section A10.

5 One-Constraint Case and Lower Bound

5.1 One-Constraint Case

In this section, we discuss the OLP regret lower bound by relating the OLP problem to a statistical estimation problem. Specifically, we consider a one-constraint linear program problem where \( m = 1 \) in LP (1). We set \( a_{1j} = 1 \) for \( j = 1, \ldots, n \). Then the optimization problem becomes

\[
\max \sum_{j=1}^{n} r_j x_j \quad \text{s.t.} \sum_{j=1}^{n} x_j \leq nd, \quad x_j \in [0,1].
\]

This one-constraint case of OLP is also known as the multi-secretary problem (Kleinberg, 2005; Arlotto and Gurvich, 2019). To simplify our discussion, we assume \( dn \) to be an integer. Then there exists an integer-valued optimal solution of (13), given by

\[
x_t^* = \begin{cases} 1, & r_t \geq \hat{Q}_n(1-d) \\ 0, & r_t < \hat{Q}_n(1-d) \end{cases}
\]

where \( \hat{Q}_n(\eta) \) defines the sample \( \eta \)-quantile of \( \{r_j\}_{j=1}^{n} \), i.e. \( \hat{Q}_n(\eta) = \inf \left\{ v \in \mathbb{R} : \frac{1}{n} \sum_{j=1}^{n} I(r_j > v) \geq \eta \right\} \). The sample quantile \( \hat{Q}_n(1-d) \) is indeed the dual optimal solution \( p^*_n \) in the general setting. Intuitively, this optimal solution allocates resources to the proportion of orders with highest returns. We restate Assumption 1 and 2 in this one-constraint case as follows.

Assumption 3. Assume \( d \in (0,1) \) and \( \{r_j\}_{j=1}^{n} \) is a sequence of i.i.d. random variables supported in
[0, 1]. Assume it has a density function \( f_r(x) \) s.t. \( \nu_3 \leq f_r(x) \leq \nu_4 \) for \( \nu_3, \nu_4 \in \mathbb{R}^+ \). Denote the set of all distributions \( \mathcal{P}_r \) satisfying the above assumptions as \( \Xi_r \).

We restrict our attention to a class of thresholding policies as discussed in Section 4.1. At each time \( t \), we compute a dual price from the history inputs, \( p_t = h_t(r_1, x_1, ..., r_{t-1}, x_{t-1}) \) and if the constraint permits, set

\[
x_t = \begin{cases} 
1, & r_t > p_t \\
0, & r_t \leq p_t 
\end{cases}
\]

If \( \sum_{j=1}^t x_t = d n \) for some \( t \), we require \( p_s = h_s(r_1, x_1, ..., r_{s-1}, x_{s-1}) = 1 \) for all \( s > t \). In this way, all the future orders will be automatically rejected. Similar to the general setting, an online algorithm/policy in this one-constraint problem can be specified by the sequence of functions \( f_t \)'s, \( \pi = (h_1, ..., h_n) \).

### 5.2 Lower Bound

Theorem 6 states an inequality on the lower bound regret for the one-constraint problem. The inequality tells that if we view the threshold \( p_j \) at each step as a statistical estimator, the regret of this one-constraint problem is no less than the cumulative estimation error of a certain quantile of the distribution \( p^* = Q_r(1-d) \). The significance of this inequality lies in the fact that the term on its right-hand side does not involve the constraint. In an online optimization/learning setting, a violation of the binding constraint will potentially improve the reward and reduce the regret; however, it does not necessarily help decrease the term on the right-hand side. Therefore, while studying the lower bound, we can focus on the right-hand side and view it as the estimation error from an unconstrained problem.

**Theorem 6.** There exists a constant \( K' \) such that

\[
\mathbb{E}_{\mathcal{P}_r} \left[ R_n^* - R_n(\pi) \right] \geq K' \cdot \mathbb{E}_{\mathcal{P}_r} \left[ \sum_{j=1}^n (p_j - p^*)^2 \right] + \mathbb{E}_{\mathcal{P}_r} \left[ R_n^* \right] - ng(p^*)
\]  

(14)

holds for any the thresholding policy \( \pi \) and any distribution \( \mathcal{P}_r \in \Xi_r \). Here \( p_j = h_t(r_1, x_1, ..., r_{j-1}, x_{j-1}) \) is specified by the policy \( \pi \), and \( p^* = Q_r(1-d) \) is the \((1-d)\)-quantile of the random variable \( r_j \) with \( Q_r(\eta) := \inf \{ v \in \mathbb{R} : P(r \leq v) \geq \eta \} \).

The proof of Theorem 6 follows the same approach as the derivation of the upper bound in Theorem 2. We establish \( ng(p^*) \) as an upper bound for \( \mathbb{E} [R_n^*] \) and compare \( \mathbb{E} [R_n(\pi)] \) against \( ng(p^*) \). Therefore, the gap between \( \mathbb{E}_{\mathcal{P}_r} [R_n^*] \) and \( ng(p^*) \) will affect the tightness of this lower bound. Fortunately, this gap should be small in many cases; for example, in the proof of Theorem 7, we use the dual convergence result and show that

\[
\mathbb{E}_{\mathcal{P}_r} [R_n^*] - ng(p^*) \geq -O(\log \log n)
\]
for this one-constraint problem. The following corollary presents the lower bound in a more similar form as the upper bound. The right hand side of the lower bound in Corollary 2 also involves \( E[n - \tau_0] \). Note that the terms \( E[n - \tau_0] \) and \( E \left[ \sum_{i \in I} b^{(i)}_t \right] \) symmetrically capture the overuse and underuse of the resources, they are often on the same order. Thus Corollary 2 implies that the upper bound given in Theorem 2 is rather tight, and that a stable control of the resource consumption is indispensable because the term \( E[n - \tau_0] \) also appears in the lower bound of the regret.

**Corollary 2.** With the same constant \( K' \) in Theorem 6,

\[
E_{\mathcal{P}_r} [R_n^* - R_n(\pi)] \geq K' \cdot E_{\mathcal{P}_r} \left[ \sum_{j=1}^{n_0} (p_j - p^*)^2 \right] + K'(1 - p^*)^2 E[n - \tau_0] + E_{\mathcal{P}_r} [R_n^*] - ng(p^*)
\]

holds for any the thresholding policy \( \pi \) and any distribution \( \mathcal{P}_r \in \Xi_r \). Here \( p_j = h_t(r_1, x_1, ..., r_{j-1}, x_{j-1}) \) is specified by the policy \( \pi \), and \( p^* = Q_r(1 - d) \). The stopping time

\[
\tau_0 = \min \{ n \} \cup \left\{ t : \sum_{i=1}^t x_i = nd \right\}
\]

represents the first time that the resource is exhausted.

Based on Theorem 6, we can derive an lower bound for the OLP problem by analyzing the right-hand-side of (14). The idea is to find a parametric family of distributions and we relate \( p^* \) with the parameter that specifies the distribution \( \mathcal{P}_r \). The \( j \)-th term on the right-hand side then can be viewed as the estimation error of the parameter \( p^* \) with \( j - 1 \) observations. Theorem 7 states the lower bound for the OLP problem. Its proof considers a family of truncated exponential distributions that satisfies Assumption 3, and then employs the same idea as the derivation of lower bounds in (Keskin and Zeevi, 2014; Besbes and Muharremoglu, 2013). The core part is the usage of van Trees inequality (Gill and Levit, 1995) – a Bayesian version of the Cramer-Rao bound. The reason for using such a Bayesian version is that we do not know the bias of the estimator \( p_j \), which makes the traditional Cramer-Rao bound ineffective.

**Theorem 7.** There exist constants \( C \) and \( n_0 \in \mathbb{N}^+ \) such that

\[
\Delta_n(\pi) \geq C \log n
\]

holds for all \( n \geq n_0 \) and any dual-based policy \( \pi \).

Theorem 7 indicates that Algorithm 2 is an asymptotically near-optimal algorithm for the OLP problem under the fixed-\( m \) and large-\( n \) regime. The lower bound \( O(\log n) \) is also consistent with the lower bound of the unconstrained online convex optimization problem (Abernethy et al., 2008).
6 Experiments, Remarks, and Future Directions

6.1 Numerical Experiments

We implement the three proposed algorithms on three different models, with model details given in Table 2. In the first model (Random Input I), the constraint coefficients $a_j$'s and objective coefficient $r_j$'s are i.i.d. generated bounded random variables. This setting satisfies both Assumption 1 and 2. All $d_i$'s are set to be 0.25. In the second model (Random Input II), the constraint coefficient $a_{ij}$ is generated from normal distribution, which violates the boundedness assumption. The assignment of $r_j$ is deterministic conditional of $a_j$ and thus violates Assumption 2 (b). Both $a_{ij}$ and $r_j$ take negative values with a positive probability. In Random Input II, we set $d_i$'s alternatively to be 0.2 and 0.3. In the third model, we consider a random permutation model, same as the worst-case example in (Agrawal et al., 2014). Note that the number of decision variables $n$ is a random variable itself in this permutation model, so we are specifying its expectation to be 100 and 300 in the experiment.

<table>
<thead>
<tr>
<th>Model</th>
<th>$a_j$</th>
<th>$r_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random Input I</td>
<td>$a_{ij} \sim \text{Uniform}[0,1]$</td>
<td>$r_j \perp a_j$ and $r_j \sim \text{Uniform}[0,10]$</td>
</tr>
<tr>
<td>Random Input II</td>
<td>$a_{ij} \sim \text{Normal}(0.5,1)$</td>
<td>$r_j = \sum_{i=1}^{m} a_{ij}$</td>
</tr>
<tr>
<td>Permutation</td>
<td></td>
<td>(Agrawal et al., 2014)</td>
</tr>
</tbody>
</table>

Table 2: Models used in the experiments

Table 3 reports the estimated regrets of the three algorithms under different combinations of $m$ and $n$. The estimation is based on 200 simulation trials and in each simulation trial, a problem instance ($a_{ij}$'s and $r_j$'s) is generated from the corresponding model. While solving the stochastic program in Algorithm 3, we use an SAA scheme with $10^6$ samples. We have the following observations based on the experiment results. First, Table 3 shows that Algorithm 2 performs uniformly better than Algorithm 1 and Algorithm 3. Importantly, Algorithm 2 also excels in the models of Random Input II and Permutation where the assumptions for theoretical analysis are violated. As far as we know, Algorithm 2 is the first algorithm that employs the action-history-dependent mechanism in a generic OLP setting. In this regard, Algorithm 1 and Algorithm 3 well represent the conventional algorithms and the experiments so as to demonstrate the effectiveness of the action-history-dependent design. Besides, the lead of Algorithm 2 becomes smaller as the ratio $m/n$ goes up. This can be explained by the fact that the dual convergence rate is of order $\sqrt{m/n}$ and therefore a dual-based algorithm like Algorithm 2 would be more effective in a large-$n$ regime.

To illustrate how the regret scales with $n$, we fix $m = 4$ and run the experiments for different $n (= 25, 50, 100, 250, 500, 1000, 2000)$. The results are presented in Figure 3, where the curves are plotted by connecting the sample points. For the left panel, the model of Random Input I satisfies both Assumption 1 and 2; the curves verify the regret results in Theorem 3, 4, and 5. Meanwhile, the right panel looks
Table 3: Regret performance: A1, A2, and A3 stand for Algorithm 1 (Simplified Dynamic Learning), Algorithm 2 (Action-history-dependent), and Algorithm 3 (No-need-to-learn), respectively.

interesting in that the regrets of Algorithm 1 and 3 scale linearly with $n$ while the regret of Algorithm 2 is $O(1)$ (a horizontal line can be fitted). This phenomenon is potentially caused by the deterministic assignment of $r_j$’s in the Random Input II model.

Figure 3: Regret curves with $m = 4$

6.2 Remarks on the Online Algorithms

Table 4 and Table 5 provide a summary of the three algorithms. Table 4 presents the regret upper bounds of the algorithms separately in terms of the three components in Theorem 2. First, we notice that the bottleneck for Algorithm 1 and 3 lies in the control of resource consumption. Intuitively, these two algorithms do not employ the action-history-dependent design and thus the resource consumption in each period is “independent” of the current resource level. This causes a fluctuation of $O(\sqrt{n})$ after $n$ periods, and that is essentially the result why the last two terms are $O(\sqrt{n})$ for these two algorithms. Moreover, Algorithm 2, in contrast with the geometrically updating scheme in Algorithm 1, updates the dual price after every period. Its goal is not to further reduce the estimation error of $p^*$, but to stabilize the resource consumption and reduce the other two terms.

The computational cost is measured by the number of LPs that need to be solved throughout the process. Algorithm 1 is notably more computationally efficient than Algorithm 2. However, the $O(n)$ computational cost of Algorithm 2 can be significantly curtailed in practice by using $p_t$ as the initial
Algorithm 1

\[ \mathbb{E} \left[ \sum_{t=1}^{T} \| p_t - p^* \|^2_2 \right] \]
\[ \mathbb{E} [n - \tau] \]
\[ \mathbb{E} \left[ \sum_{i \in I} b_i^{(n)} \right] \]

Regret

Algorithm 2

Algorithm 3

Table 4: Three components of the upper bound in Theorem 2/Corollary 1.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Prior Knowledge</th>
<th>Computational Cost</th>
<th>Regret</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algorithm 1</td>
<td>((d_1, ..., d_m))</td>
<td>(O(\log n))</td>
<td>(O(\sqrt{n} \log n))</td>
</tr>
<tr>
<td>Algorithm 2</td>
<td>((d_1, ..., d_m), n)</td>
<td>(O(n))</td>
<td>(O(\log n \log \log n))</td>
</tr>
<tr>
<td>Algorithm 3</td>
<td>((d_1, ..., d_m), P, p^*)</td>
<td>(O(1))</td>
<td>(O(\sqrt{n}))</td>
</tr>
</tbody>
</table>

Table 5: Algorithm summary and comparison

6.3 Summary, Future Direction, and Open Questions

In this paper, we study the OLP problem under a random input model. Our model considers a double-sided market with both buying and selling orders. We view the dual LP problem as a sample average approximation for a stochastic program and establish convergence results on the dual optimal solutions. Based on the dual convergence results, we develop a new type of OLP algorithm - the action-history-dependent learning algorithm, which integrates both the past inputs and actions in making the online decisions. Regret bounds for general OLP algorithms are developed and they reveal an important aspect of the OLP problem – a stable control of the resource/constraint consumption. We derive an \(O(\log n \log \log n)\) regret upper bound for the action-history-dependent learning algorithm and show that no dual-based algorithms achieves better regret than \(O(\log n)\). Also, we derive an \(O(\sqrt{n})\) regret upper bound for the traditional dual-based learning algorithms. Empirical and experimental evidences highlight the effectiveness of the non-stationary control in our new algorithm. The theoretical results and algorithms developed in this paper can be potentially extended to other online learning problems. We conclude the paper with several open questions motivated from the numerical experiments.
Based on the experiments and the theoretical results developed in this paper, we raise several open questions for future study. First, how does the regret depend on $m$? In the experiments of Random Input I, we observe that the regret does not scale up with $m$, but this is apparently not the case in Random Input II. A possible explanation is that the generation of $r_j$'s in Random Input II causes that $r_j$ scales with $m$ and consequently the offline optimal objective values scales linearly with $mn$. A natural question then is what are the conditions that render the regret dependent on $m$ and in what way the regret depends on $m$. Second, is it possible to relax the assumptions and extend the theoretical results for more general random input models and permutation models? We observe a good performance of Algorithm 2 when the assumptions are violated and even in the permutation model. For example, we observe an $O(1)$ regret of Algorithm 2 in the experiment under Random Input II (Figure 3). Since the assumptions are violated, the lower bound does not hold in Random Input II. Also, it is an interesting question to ask that whether the dual convergence and the regret results still hold in a permutation model. This question entails a proper definition of the stochastic program 7 in the permutation context. Third, in Assumption 1 (d), we require the constraints $b$ scales linearly with $n$. We did not answer the question whether this linear growth rate is necessary for establishing the dual convergence results. In other words, how the dual convergence and regret bounds can be extended to the limited-resource regimes? Besides, Algorithm 2 updates the dual price in every periods. This raises the question that if it is possible to have a less frequent updating/learning scheme but still achieve the same order of regret.

Acknowledgement

We thank Simai He, Peter W. Glynn, Daniel Russo, Zizhuo Wang, Zeyu Zheng, and participants at Stanford AFTLab seminar and ADSI summer school for helpful discussions and comments. We thank Chunlin Sun and Guanting Chen for proofreading the proof and Yufeng Zheng for assistance in the simulation experiments.

References


Asadpour, Arash, Xuan Wang, Jiawei Zhang. 2019. Online resource allocation with limited flexibility. *Management Science*.


Appendix

A1 Proof of Proposition 1

Proof. (a) The original dual problem 3 can be recovered by substituting \( y_j = (r_j - \sum_{i=1}^{m} a_{ij}p_i)^+ \) in the objective function (5). Then the feasible solutions and objective are matched. Therefore these two problems share the same optimal solution.

(b) We know that each component in the first summation is linear and in the second summation is convex. Also, the summation operation preserves convexity (See Chapter 3.2.1 (Rockafellar, 1970)). So, both \( f_n \) and \( f \) are convex functions.

(c) If \( d^\top p > \bar{r} \), then

\[
f(p) = d^\top p > \bar{r} \geq E[r] = f(0).
\]

Hence \( p \) cannot be the optimal solution. In the same way, we can show the result for \( p_n^* \).

(d) This can be derived from the KKT conditions (See Chapter 5.5.3 (Rockafellar, 1970)).

\[ \Box \]

A2 Proof of Proposition 2

First, we introduce the Hoeffding’s inequality for scalar random variables.

**Lemma 2** (Hoeffding’s inequality). Let \( X_1, ..., X_n \) be independent random variables such that \( X_i \) takes its values in \([u_i, v_i]\) almost surely for all \( i \leq n \). Let

\[
S = \sum_{i=1}^{n} (X_i - \mathbb{E}X_i).
\]

Then for every \( t > 0 \),

\[
\mathbb{P}(S \geq t) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(u_i - v_i)^2}\right)
\]

Proof. We refer to Chapter 2 of the book (Boucheron et al., 2013).

Now, we present the proof of Proposition 2.

Proof. We use \( \phi(p^*, u_j) \) to denote the \( i \)-th coordinate of the gradient vector \( \phi(p^*, u_j) \). By the definition of \( \phi \), we know that

\[
\mathbb{E}\phi(p^*, u_j)_i = (\nabla f(p^*)_i).
\]
From the boundedness \(a_j\)'s, we know that
\[
|\phi(p^*, u_j)| \in [d_i - \bar{a}, d_i + \bar{a}].
\]

Then by applying the Hoeffding’s inequality, we obtain
\[
P \left( \left| \frac{1}{n} \sum_{j=1}^{n} \phi(p^*, u_j)_i - \left( \nabla f(p^*) \right)_i \right| \geq \epsilon \right) \leq 2 \exp \left( \frac{-n\epsilon^2}{2\bar{a}^2} \right).
\]

In fact,
\[
\left\{ \left\| \frac{1}{n} \sum_{j=1}^{n} \phi(p^*, u_j) - \left( \nabla f(p^*) \right)_i \right\|_2 \geq \epsilon \right\} \subset \bigcup_{i=1}^{m} \left\{ \left| \frac{1}{n} \sum_{j=1}^{n} \phi(p^*, u_j)_i - \left( \nabla f(p^*) \right)_i \right| \geq \frac{\epsilon}{\sqrt{m}} \right\}.
\]

If we apply the union bound,
\[
P \left( \left\| \frac{1}{n} \sum_{j=1}^{n} \phi(p^*, u_j) - \left( \nabla f(p^*) \right)_i \right\|_2 \geq \epsilon \right) \leq mP \left( \left| \frac{1}{n} \sum_{j=1}^{n} \phi(p^*, u_j)_i - \left( \nabla f(p^*) \right)_i \right| \geq \frac{\epsilon}{\sqrt{m}} \right) \leq 2m \exp \left( \frac{-n\epsilon^2}{2\bar{a}^2m} \right).
\]

Thus we obtain Proposition 2. \(\square\)

### A3 Proof of Proposition 3

We first introduce a matrix version for the Hoeffding’s inequality.

**Lemma 3 (Matrix Hoeffding’s Inequality).** Let \(X_1, \ldots, X_n \in \mathbb{R}_d\) be i.i.d. random vectors with \(\mathbb{E}(X_iX_i^\top) = M\). Also, we assume \(\|X_i\|_2 \leq B\) almost surely. Let
\[
Z = \sum_{k=1}^{n} X_iX_i^\top.
\]

Then
\[
P \left( \|M - Z\|_S \geq t \right) \leq d \cdot \exp \left( -\frac{t^2}{Bn} \right)
\]
for all \(t > 0\). Here \(\|\cdot\|_S\) refers to the spectral norm of a matrix.

**Proof.** We refer to the Corollary 4.2 (Matrix Hoeffding Inequality) of the paper (Mackey et al., 2014).

The proof of this lemma simply reduces the matrix in Corollary 4.2 to a vector setting. \(\square\)

Now we prove Proposition 3.
Proof of Proposition 3. Consider

\[ M_n = \frac{1}{n} \sum_{j=1}^{n} a_j a_j^\top \]

\[ M = \mathbb{E} [a_j a_j^\top] . \]

We know from Assumption 2 that the minimum eigenvalue of \( M \) is \( \lambda_{\text{min}} \). Also,

\[ \lambda_{\text{min}} - \lambda_{\text{min}}(M_n) \leq \lambda_{\text{max}}(M - M_n) \leq \|M - M_n\|_2 \]

where \( \lambda_{\text{min}}(\cdot) \) and \( \lambda_{\text{max}}(\cdot) \) refer to the smallest and largest eigenvalue of a matrix, respectively. Denote event

\[ E_0 = \left\{ \lambda_{\text{min}}(M_n) \leq \frac{\lambda_{\text{min}}}{2} \right\} . \]

Applying Lemma 3,

\[ \mathbb{P}(E_0) = \mathbb{P} \left( \lambda_{\text{min}}(M_n) \leq \frac{\lambda_{\text{min}}}{2} \right) \leq \mathbb{P} \left( \|M - M_n\|_2 \geq \frac{\lambda_{\text{min}}}{2} \right) \]

\[ \leq m \cdot \exp \left( \frac{-n\lambda_{\text{min}}^2}{4\lambda_{\text{min}}^2} \right) . \quad (15) \]

From Proposition 1 (c), we know that the optimal solution \( p_n^* \) and \( p^* \) is bounded. Define set \( \Omega = \left\{ p \in \mathbb{R}^m \left\| p - p^* \right\|_\infty \leq \frac{\epsilon}{2} \right\} \) and \( p_n^* \in \Omega \) almost surely. Next, we split \( \Omega \) into a union of disjoint sets

\[ \Omega = \bigcup_{k=1}^{N} \bigcup_{l=1}^{l_k} \Omega_{kl} . \]

The splitting scheme is inspired by (Huber et al., 1967). Specifically, these sets are divided layer by layer. The set \( \Omega_k = \left\{ p \in \mathbb{R}^m \left\| p - p^* \right\|_\infty \leq q^k \frac{\epsilon}{2} \right\} \) for \( k = 0, ..., N \). Here \( N \) and \( q \in (0, 1) \) will be determined later. The \( k \)-th layer \( \Omega_{k-1} \setminus \Omega_k \) is further divided into disjoint cubes \( \{\Omega_{kl}\}_{l=1}^{l_k} \) with edges of length \( (1 - q)q^{k-1} \frac{\epsilon}{2} \) for \( k = 1, ..., N - 1 \) and \( l = 1, ..., l_k \). The center cube is simply \( \Omega_N = \Omega_{N1} \) with edge of length \( q^N \frac{\epsilon}{2} \) and \( l_N = 1 \). Also, the length \( q \) is adjusted in a way that the splitting scheme cut the whole space into integer number of cubes.

Figure 4 gives a visualization of the splitting scheme. In total, there are no more than \( (2N)^m \) cubes (See Lemma 3 in (Huber et al., 1967)). Let \( p_{kl} \) be the center of the cube \( \Omega_{kl} \), \( p_{kl} \) and \( p_{kl} \) be the points in \( \Omega_{kl} \) that are closest and furthest from \( p^* \), respectively. That is,

\[ p_{kl} = \arg \min_{p \in \Omega_{kl}} \|p - p^*\|_2 , \]

\[ \bar{p}_{kl} = \arg \max_{p \in \Omega_{kl}} \|p - p^*\|_2 . \]

35
Define
\[ \Gamma_{kl}(r_j, a_j) = \max_{p \in \Omega_{kl}} \int_{a_j^T p}^{a_j^T p_{kl}} I(r_j > v) - I(r_j > a_j^T p^*) dv. \]

We know that
\[
\mathbb{E}[\Gamma_{kl}(r_j, a_j)] = \mathbb{E} \left[ \max_{p \in \Omega_{kl}} \int_{a_j^T p}^{a_j^T p_{kl}} I(r_j > v) - I(r_j > a_j^T p^*) dv \right]
\leq \mathbb{E} \left[ \max_{p \in \Omega_{kl}} \int_{a_j^T p}^{a_j^T p_{kl}} I(v < r_j \leq a_j^T p^*) dv \right]
\leq \mathbb{E} \left[ \max_{p \in \Omega_{kl}} |a_j^T p - a_j^T p_{kl}| \cdot \max_{p \in \Omega_{kl}} (|I(v < r_j \leq a_j^T p^*)| + |I(v \geq r_j > a_j^T p^*)|) \right]
\leq \bar{a} \max_{p \in \Omega_{kl}} ||p - p_{kl}||_2 \cdot \nu_2 \bar{a} ||p^* - \bar{p}_{kl}||_2
\]
\[= \nu_2 \bar{a} ||p^* - \bar{p}_{kl}||_2 \max_{p \in \Omega_{kl}} ||p - p_{kl}||_2 \]

and
\[ \Gamma_{kl}(r_j, a_j) \leq \max_{p \in \Omega_{kl}} |a_j^T p - a_j^T p_{kl}| \leq \bar{a} \max_{p \in \Omega_{kl}} ||p - p_{kl}||_2 \]
hold for all \( k = 1, ..., N - 1 \) and \( l = 1, ..., l_k \). Let
\[ E_{kl,1} = \left\{ \frac{1}{n} \sum_{j=1}^{n} \Gamma_{kl}(r_j, a_j) \geq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}[\Gamma_{kl}(r_j, a_j)] + \epsilon \bar{a} \max_{p \in \Omega_{kl}} ||p - p_{kl}||_2 \right\} \]
By applying Hoeffing’s Inequality with the independence of \( \Gamma_{kl}(r_j, a_j) \)'s,

\[
\mathbb{P}(E_{kl,1}) \leq \exp(-2n\epsilon^2)
\]  

(16)

for all \( k = 1, \ldots, N - 1 \) and \( l = 1, \ldots, l_k \).

On the other hand, we know,

\[
\int_{a_j^T P_k}^{a_j^T \hat{p}^*} (I(r_j > v) - I(r_j > a_j^T \hat{p}^*)) \, dv \leq |a_j^T \hat{p}_{kl} - a_j^T \hat{p}^*| \leq \bar{a}\|\hat{p}^* - \hat{p}_{kl}\|_2.
\]

Let

\[
E_{kl,2} = \left\{ \frac{1}{n} \sum_{j=1}^{n} \int_{a_j^T \hat{p}_{kl}}^{a_j^T \hat{p}^*} (I(r_j > v) - I(r_j > a_j^T \hat{p}^*)) \, dv \leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ \int_{a_j^T \hat{p}_{kl}}^{a_j^T \hat{p}^*} (I(r_j > v) - I(r_j > a_j^T \hat{p}^*)) \, dv \right] - \epsilon \bar{a}\|\hat{p}^* - \hat{p}_{kl}\|_2 \right\}
\]

By applying Hoeffing’s Inequality with the independence of \( (r_j, a_j) \)'s,

\[
\mathbb{P}(E_{kl,2}) \leq \exp(-2n\epsilon^2)
\]  

(17)

for \( k = 1, \ldots, N - 1 \) and \( l = 1, \ldots, l_k \).

Conditional on event \( E_0 \),

\[
E \left[ \frac{1}{n} \sum_{j=1}^{n} \int_{a_j^T \hat{p}_{kl}}^{a_j^T \hat{p}^*} (I(r_j > v) - I(r_j > a_j^T \hat{p}^*)) \, dv \bigg| a_1, \ldots, a_n \right] 
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \int_{a_j^T \hat{p}_{kl}}^{a_j^T \hat{p}^*} \hat{F}_{a_j}(v) - \hat{F}_{a_j}(a_j^T \hat{p}^*) \, dv 
\]

\[
\geq \frac{\nu_1}{n} \sum_{j=1}^{n} (a_j^T \hat{p}^* - a_j^T \hat{p}_{kl})^2 
\]

\[
= \nu_1 (\hat{p}^* - \hat{p}_{kl})^\top \left( \frac{1}{n} \sum_{j=1}^{n} a_j a_j^\top \right) (\hat{p}^* - \hat{p}_{kl}) 
\]

\[
\geq \frac{\nu_1 \lambda_{\min}}{2} \|\hat{p}^* - \hat{p}_{kl}\|_2^2
\]

(18)

Given

\[
\max_{p \in \Omega_{kl}} \|p - \hat{p}_{kl}\|_2 = \sqrt{m}(1 - q)q^{k-1}\frac{\bar{p}}{d}
\]

and

\[
\|\hat{p}^* - \hat{p}_{kl}\|_2 \geq q^{k}\frac{\bar{p}}{2}.
\]
for all \( k = 1, \ldots, N - 1 \) and \( l = 1, \ldots, l_k \), we have

\[
\|p^* - \bar{p}_{kl}\|_2 \leq \|p^* - p_{kl}\|_2 + \max_{p \in \Omega_{kl}} \|p - p_{kl}\|_2 \\
\leq \left( 1 + \frac{\sqrt{m}(1 - q)}{q} \right) \|p^* - p_{kl}\|_2 \tag{19}
\]

and

\[
\max_{p \in \Omega_{kl}} \|p - p_{kl}\|_2 \leq \frac{\sqrt{m}(1 - q)}{q} \|p^* - p_{kl}\|_2 \leq \frac{\sqrt{m}(1 - q)}{q} \|p^* - \bar{p}_{kl}\|_2 \tag{20}
\]

With (18), (19) and (20),

\[
E \left[ \frac{1}{n} \sum_{j=1}^{n} \int a_j^\top p^* (I(r_j > v) - I(r_j > a_j^\top p^*)) \, dv \right] - E \left[ \frac{1}{n} \sum_{j=1}^{n} \Gamma_{kl}(r_j, a_j) \right] \\
\geq \nu_1 \lambda_{\min} \frac{1}{2} \|p^* - p_{kl}\|_2^2 - \nu_2 \bar{a}^2 \|p^* - \bar{p}_{kl}\|_2 \max_{p \in \Omega_{kl}} \|p - p_{kl}\|_2 \\
\geq \nu_1 \lambda_{\min} \frac{1}{2} \left( \frac{1}{1 + \frac{\sqrt{m}(1 - q)}{q}} \right)^2 \|p^* - p_{kl}\|_2^2 - \nu_2 \bar{a}^2 \frac{\sqrt{m}(1 - q)}{q} \|p^* - \bar{p}_{kl}\|_2^2.
\]

By choosing

\[
q = \max \left\{ \frac{1}{1 + \frac{1}{\sqrt{m}}}, \frac{1}{1 + \frac{1}{\sqrt{m}} \left( \frac{\nu_1 \lambda_{\min}}{4 \nu_2 \bar{a}} \right)^\frac{1}{2}} \right\}
\]

we have

\[
E \left[ \frac{1}{n} \sum_{j=1}^{n} \int a_j^\top p^* (I(r_j > v) - I(r_j > a_j^\top p^*)) \, dv \right] - E \left[ \frac{1}{n} \sum_{j=1}^{n} \Gamma_{kl}(r_j, a_j) \right] \geq \frac{\nu_1 \lambda_{\min}}{16} \|p^* - \bar{p}_{kl}\|_2^2. \tag{21}
\]
We use $E^c$ to denote the complement of $E$. on event $E_0 \cap E_{kl,1}^c \cap E_{kl,2}^c$.

\[
\frac{1}{n} \sum_{j=1}^{n} \int_{a_j^* p} (I(r_j > v) - I(r_j > a_j^* p^*)) \, dv
\]

\[= \frac{1}{n} \sum_{j=1}^{n} \int_{a_j^* p} (I(r_j > v) - I(r_j > a_j^* p^*)) \, dv + \frac{1}{n} \sum_{j=1}^{n} \int_{a_j^* p} (I(r_j > v) - I(r_j > a_j^* p^*)) \, dv
\]

\[(a) \frac{1}{n} \sum_{j=1}^{n} \int_{a_j^* p} (I(r_j > v) - I(r_j > a_j^* p^*)) \, dv - \frac{1}{n} \sum_{j=1}^{n} \Gamma_{kl}(r_j, a_j)
\]

\[= \frac{1}{n} \sum_{j=1}^{n} \int_{a_j^* p} (I(r_j > v) - I(r_j > a_j^* p^*)) \, dv - \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} \Gamma_{kl}(r_j, a_j) \right]
\]

\[+ \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} \int_{a_j^* p} (I(r_j > v) - I(r_j > a_j^* p^*)) \, dv \right]
\]

\[\geq - 2\bar{a} ||p^* - \bar{p}_{kl}||_2 + \frac{\nu_1 \lambda_{\text{min}}}{16} ||p^* - \bar{p}_{kl}||_2^2
\]

\[(b) \geq - 2\bar{a} ||p^* - \bar{p}_{kl}||_2 + \frac{\nu_1 \lambda_{\text{min}}}{16} ||p^* - p||_2^2 \tag{22}
\]

\[(c) \geq - 2\bar{a} ||p^* - p||_2^2 + \frac{\nu_1 \lambda_{\text{min}}}{16} ||p^* - p||_2^2 \tag{23}
\]

holds for any $p \in \Omega_{kl}$, $k = 1, \ldots, N - 1$ and $l = 1, \ldots, l_k$. Here (a) is from the definition of $\Gamma_{kl}$; (b) comes from a combination of (16), (17) and (21); (c) comes from the definition of $\bar{p}_{kl}$.

For $\Omega_{N1}$, its center is $p^*$, and

\[
\frac{1}{n} \sum_{j=1}^{n} \int_{a_j^* p} (I(r_j > v) - I(r_j > a_j^* p^*)) \, dv \geq - \frac{1}{n} \sum_{j=1}^{n} \Gamma_{N1}(r_j, a_j)
\]

\[\geq - \nu_2 \bar{a}^2 ||p^* - \bar{p}_{N1}||_2^2 = - \nu_2 \bar{a}^2 \sqrt{m} q N \frac{\bar{p}^2}{2}.
\]

We choose

\[N = \left| \log_2 \left( \frac{d^2}{\nu_2 \bar{a}^2 \sqrt{m}} \right) \right| + 1
\]

and obtain

\[
\frac{1}{n} \sum_{j=1}^{n} \int_{a_j^* p} (I(r_j > v) - I(r_j > a_j^* p^*)) \, dv \geq - \epsilon^2 \tag{23}
\]

for $p \in \Omega_{N1}$.

Therefore, we obtain from (22) and (23) that, conditional on $\cap_{k=1}^{N_k} \cap_{l=1}^{l_k} (E_{kl,1}^c \cap E_{kl,2}^c) \cap E_0$,

\[
\frac{1}{n} \sum_{j=1}^{n} \int_{a_j^* p} (I(r_j > v) - I(r_j > a_j^* p^*)) \, dv \geq - \epsilon^2 - 2\bar{a} ||p^* - p||_2^2 + \frac{\nu_1 \lambda_{\text{min}}}{16} ||p^* - p||_2^2
\]
for all \( p \in \Omega \). Also,

\[
1 - P \left( \bigcap_{k=1}^{N} \bigcap_{l=1}^{l_k} \left( E_{kl,1}^c \cap E_{kl,2}^c \right) \cap E_0 \right) = P \left( \bigcup_{k=1}^{N} \bigcup_{l=1}^{l_k} \left( E_{kl,1} \cup E_{kl,2} \right) \cap E_0^c \right)
\]

\[
\leq P(E_0^c) + \sum_{k=1}^{N} \sum_{l=1}^{l_k} (P(E_{kl,1}) + P(E_{kl,2}))
\]

\[
\leq m \exp \left( -\frac{n \lambda_{\min}^2}{4 \bar{a}^2} \right) + 2 \exp(-2n\epsilon^2) \cdot (2N)^m.
\]

\( \square \)

**A4 Proof of Theorem 1**

**Proof.** Let

\[
E_1 = \left\{ \left\| \frac{1}{n} \sum_{j=1}^{n} \phi(p^*, u_j) - \nabla f(p^*) \right\|_2 \leq \epsilon \right\}.
\]

From Proposition 2,

\[
P(E_1^c) \leq 2m \exp \left( -\frac{n \epsilon^2}{2 \bar{a}^2 m} \right).
\]

Let

\[
E_2 = \left\{ \frac{1}{n} \sum_{j=1}^{n} \int_{a_j^* p}^{a_j p} \left( I(r_j > v) - I(r_j > a_j^* p^*) \right) dv \geq -\epsilon^2 - 2\epsilon \bar{a} \|p^* - p\|_2 + \frac{\nu_1 \lambda_{\min}}{16} \|p^* - p\|_2^2 \right\}
\]

From Proposition 3,

\[
P(E_2^c) \leq m \exp \left( -\frac{n \lambda_{\min}^2}{4 \bar{a}^2} \right) + 2 \exp(-2n\epsilon^2) \cdot (2N)^m.
\]

On the event \( E_1 \cap E_2 \), we have

\[
f_n(p) - f_n(p^*) \geq -\epsilon^2 - \epsilon (2\bar{a} + 1) \|p^* - p\|_2 + \frac{\nu_1 \lambda_{\min}}{16} \|p^* - p\|_2^2
\]

for all \( p \in \Omega_p \). By the definition of \( p_n^* \),

\[
-\epsilon^2 - \epsilon (2\bar{a} + 1) \|p^* - p_n^*\|_2 + \frac{\nu_1 \lambda_{\min}}{16} \|p^* - p_n^*\|_2^2 \leq f_n(p_n^*) - f_n(p^*) \leq 0
\]

and this implies,

\[
\|p_n^* - p^*\|_2 \leq \kappa \epsilon
\]

with

\[
\kappa = \frac{2\bar{a} + 1 + \sqrt{(2\bar{a} + 1)^2 + \frac{\nu_1 \lambda_{\min}}{4}}}{\nu_1 \lambda_{\min}/8}.
\]
On the other hand,

\[ P(E_1 \cap E_2) = 1 - P(E_1^c) - P(E_2^c) \]

\[ \geq 1 - 2m \exp \left( -\frac{\kappa \epsilon^2}{2\bar{a}^2 m} \right) - m \exp \left( -\frac{n \lambda_{\min}}{4\bar{a}^2} \right) - 2 \exp(-2n\epsilon^2) \cdot (2N)^m. \]

Given that \( \|p_n^* - p^*\|_2 \leq \frac{\bar{r}}{2}\),

\[ \frac{1}{K} E\|p_n^* - p^*\|_2 = \int_0^\frac{\bar{r}}{2} e dP(\|p_n^* - p^*\|_2 \leq \kappa \epsilon) \]

\[ = \int_0^\frac{\bar{r}}{2} P(\|p_n^* - p^*\|_2 \geq \kappa \epsilon) d\epsilon \]

\[ \leq \int_0^\frac{\bar{r}}{2} (1 - P(E_1 \cap E_2)) d\epsilon \]

\[ \leq \int_0^\frac{\bar{r}}{2} \left( \left( 2m \exp \left( -\frac{\kappa \epsilon^2}{2\bar{a}^2 m} \right) + m \exp \left( -\frac{n \lambda_{\min}}{4\bar{a}^2} \right) + 2 \exp(-2n\epsilon^2) \cdot (2N)^m \right) \wedge 1 \right) d\epsilon \]

(24)

where \( y \wedge z = \min\{y, z\} \). We point out that because \( r_j \) has a density function, \( \|p_n^* - p^*\|_2 \) has a density function as well, which makes the above equation valid. Next, we analyze the integral (24) term by term.

First, with \( \epsilon = \epsilon \sqrt{\frac{m \log m}{n}} \),

\[ \int_0^{\frac{\bar{r}}{2}} 2m \exp \left( -\frac{\kappa \epsilon^2}{2\bar{a}^2 m} \right) 1 d\epsilon = \sqrt{m \log m} \int_0^{\frac{\bar{r}}{2}} \left( 2m \exp \left( -\frac{\epsilon^2 \log m}{2\bar{a}^2} \right) \right) 1 d\epsilon \]

\[ \leq \sqrt{m \log m} \int_0^{\infty} 2 \left( \exp \left( \log m - \frac{\epsilon^2 \log m}{2\bar{a}^2} \right) \right) 1 d\epsilon \leq C_{1,1} \sqrt{\frac{m \log m}{n}}. \]

(25)

where \( C_{1,1} \) is dependent only on \( \bar{a} \).

Second,

\[ \int_0^{\frac{\bar{r}}{2}} m \exp \left( -\frac{n \lambda_{\min}}{4\bar{a}^2} \right) d\epsilon = \frac{m r}{d} \exp \left( -\frac{n \lambda_{\min}}{4\bar{a}^2} \right) \leq C_{1,2} \frac{m}{n} \]

(26)

where \( C_{1,2} \) is dependent only on \( \bar{a} \) and \( \lambda_{\min} \).

Third, we can show that there exists constant \( C_{1,3} \) such that

\[ 2N \leq C_{1,3} \sqrt{m} \log \left( \frac{\sqrt{m}}{\epsilon^2} \right). \]
Hence,
\[
\int_0^1 1 \wedge (2 \exp(-2n\epsilon^2) \cdot (2N)^m) \, d\epsilon \leq \int_0^1 1 \wedge \left( 2 \exp\left(-2n\epsilon^2 \cdot \left( C_{13}\sqrt{m \log \left( \frac{\sqrt{m}}{\epsilon^2} \right)} \right)^m \right) \right) \, d\epsilon \\
= \int_0^1 1 \wedge \left( 2 \exp\left(-2n\epsilon^2 + m \log \left( C_{13}\sqrt{m \log \left( \frac{\sqrt{m}}{\epsilon^2} \right)} \right) \right) \right) \, d\epsilon \\
\leq \int_0^\infty 1 \wedge \left( 2 \exp\left(-2n\epsilon^2 + m \log \left( C_{13}\sqrt{m \log \left( \frac{\sqrt{m}}{\epsilon^2} \right)} \right) \right) \right) \, d\epsilon
\]

Let \( \epsilon = \epsilon \sqrt{ \frac{m \log m \log \log n}{n} } \). Then,
\[
\int_0^1 1 \wedge (2 \exp(-2n\epsilon^2) \cdot (2N)^m) \, d\epsilon \\
\leq \sqrt{ \frac{m \log m \log \log n}{n} } \int_0^\infty 1 \wedge \left( 2 \exp\left(-2\epsilon^2 m \log m \log \log \frac{n}{\epsilon^2} + m \log \left( C_{13}\sqrt{m \log \left( \frac{\sqrt{m}}{\epsilon^2} \right)} \right) \right) \right) \, d\epsilon \\
\leq C_{1,4} \sqrt{ \frac{m \log m \log \log n}{n} },
\]
where \( C_{1,4} \) depends only on \( C_{1,3} \). Putting together (24), (25), (26) and (27), we conclude that
\[
\mathbb{E}\left\| \mathbf{p}_n^\ast - \mathbf{p}^\ast \right\|_2 \leq \kappa (C_{1,1} + C_{1,2} + C_{1,4}) \sqrt{ \frac{m \log m \log \log n}{n} }
\]
holds for all \( n > m \) and \( P \in \Xi \). With a similar methodology, we can show that there exists a constant \( C \) such that
\[
\mathbb{E}\left\| \mathbf{p}_n^\ast - \mathbf{p}^\ast \right\|_2^2 \leq C \frac{m \log m \log \log n}{n}.
\]

A5 Proof of Lemma 1

Proof of Lemma 1. First, \( g(\mathbf{p}^\ast) \).

\[
\mathbb{E}R_n^* = \mathbb{E}\left[ \sum_{j=1}^n r_j x_j^\ast \right] \\
= \mathbb{E}\left[ n \mathbf{a}^\top \mathbf{p}_n^\ast + \sum_{j=1}^n (r_j - \mathbf{a}_j^\top \mathbf{p}_n^\ast)^+ \right] \quad \text{(From the strong duality)} \\
\leq \mathbb{E}\left[ n \mathbf{a}^\top \mathbf{p}^\ast + \sum_{j=1}^n (r_j - \mathbf{a}_j^\top \mathbf{p}^\ast)^+ \right] \quad \text{(From the optimality of } \mathbf{p}_n^\ast \text{)} \\
= n g(\mathbf{p}^\ast).
\]
Thus, we establish \(ng(p^*)\) as an upper bound for \(\mathbb{E}R_n^*\). Now, by taking the difference between \(g(p^*)\) and \(g(p)\), we obtain,

\[
g(p^*) - g(p) = \mathbb{E} \left[ rI(r > a^\top p^*) + (d - aI(r > a^\top p^*))^\top p^* \right] - \mathbb{E} \left[ rI(r > a^\top p) + (d - aI(r > a^\top p))^\top p^* \right]
\]

\[
= \mathbb{E} \left[ (r - a^\top p^*) (I(r > a^\top p^*) - I(r > a^\top p)) \right]
\]

\[
= \mathbb{E} \left[ (a^\top p^* - r) I(a^\top p^* > r > a^\top p) \right] + \mathbb{E} \left[ (r - a^\top p^*) I(a^\top p^* < r \leq a^\top p) \right] \geq 0.
\]

This proves the maximum of \(g(p)\) is achieved at \(p^*\). Moreover, with a more careful analysis, we have

\[
g(p^*) - g(p) = \mathbb{E} \left[ (r - a^\top p^*) (I(r > a^\top p^*) - I(r > a^\top p)) \right]
\]

\[
= \mathbb{E} \left[ \int_{a^\top p} a^\top p^* (r - a^\top p^*) f_a(r) dr \right]
\]

\[
\geq (1) \mathbb{E} \left[ \int_{a^\top p} a^\top p^* \nu_1 (r - a^\top p^*) dr \right]
\]

\[
= \frac{1}{2} \nu_1 \mathbb{E} \left[ (a^\top p^* - a^\top p)^2 \right] \overset{(2)}{\geq} \frac{1}{2} \nu_1 \lambda_{\min} \|p^* - p\|^2_2.
\]

where \(\lambda_{\min}\) is defined in Assumption 2. Here inequality (1) comes from Assumption 2 (a) and inequality (2) comes from Assumption 2 (b). On the other hand,

\[
g(p^*) - g(p) = \mathbb{E} \left[ (a^\top p^* - r) I(a^\top p^* \geq r > a^\top p) \right] + \mathbb{E} \left[ (r - a^\top p^*) I(a^\top p^* < r \leq a^\top p) \right]
\]

\[
\leq \mathbb{E} \left[ (a^\top p^* - a^\top p) I(a^\top p^* \geq r > a^\top p) \right] + \mathbb{E} \left[ (a^\top p - a^\top p^*) I(a^\top p^* < r \leq a^\top p) \right]
\]

\[
= \mathbb{E} \left[ (a^\top p^* - a^\top p) (F_{r|a}(a^\top p^*) - F_{r|a}(a^\top p)) I(a^\top p^* > a^\top p) \right]
\]

\[
+ \mathbb{E} \left[ (a^\top p - a^\top p^*) (F_{r|a}(a^\top p) - F_{r|a}(a^\top p^*)) I(a^\top p^* < a^\top p) \right]
\]

\[
\leq \nu_2 \mathbb{E} \left[ (a^\top p^* - a^\top p)^2 \right]
\]

\[
\leq \nu_2 \tilde{a}^2 \|p^* - p\|^2_2.
\]

\[\square\]
A6 Proof of Theorem 2

Proof. For any dual-based online policy $\pi$, 

$$
\mathbb{E}R_n(\pi) = \mathbb{E} \left[ \sum_{j=1}^{n} r_j x_j \right] 
= \mathbb{E} \left[ \sum_{j=1}^{n} r_j x_j + \left( nd - \sum_{j=1}^{n} a_j x_j \right)^\top p^* \right] - \mathbb{E} \left[ B_n^\top p^* \right] 
= \mathbb{E} \left[ \sum_{j=1}^{n} (r_j x_j + d^\top p^* - a_j^\top p^* x_j) \right] - \mathbb{E} \left[ B_n^\top p^* \right] 
= \mathbb{E} \left[ \sum_{j=1}^{\tau_n} (r_j x_j + d^\top p^* - a_j^\top p^* x_j) \right] + \mathbb{E} \left[ \sum_{j=\tau_n+1}^{n} (r_j x_j + d^\top p^* - a_j^\top p^* x_j) \right] - \mathbb{E} \left[ B_n^\top p^* \right] \quad (29)
$$

For the first term in (29),

$$
\mathbb{E} \left[ \sum_{j=1}^{\tau_n} (r_j x_j + d^\top p^* - a_j^\top p^* x_j) \right] = \mathbb{E} \left[ \sum_{j=1}^{n} (r_j x_j + d^\top p^* - a_j^\top p^* x_j) I(\tau_n \geq t) \right] 
\overset{(a)}{=} \sum_{j=1}^{n} \mathbb{E} \left[ (r_j x_j + d^\top p^* - a_j^\top p^* x_j) I(\tau_n \geq t) \right] 
\overset{(b)}{=} \sum_{j=1}^{n} \mathbb{E} \left[ \mathbb{E} \left[ (r_j x_j + d^\top p^* - a_j^\top p^* x_j) I(\tau_n \geq t) | B_{t-1}, \mathcal{H}_{t-1} \right] \right] 
\overset{(c)}{=} \sum_{j=1}^{n} \mathbb{E} \left[ g(p_j) I(\tau_n \geq t) \right] 
\overset{(d)}{=} \mathbb{E} \left[ \sum_{j=1}^{n} g(p_j) I(\tau_n \geq t) \right] = \mathbb{E} \left[ \sum_{j=1}^{\tau_n} g(p_j) \right] 
\quad (30)
$$

where $p_j$’s are the dual price vectors specified by the policy $\pi$. Here (a) and (d) comes from the exchange of summation and expectation. (b) comes from nesting a conditional expectation. (c) is from two facts: first, on the event $\tau_n \geq t$, the remaining inventory $B_{t-1}$ is enough to satisfy the $t$-th order and second, the dual-based policy is adopting the price vector $p_j$ in deciding the value of $x_j$.

For the second term in (29), we know that $\|p^*\| \leq \frac{r}{2}$ from Proposition 1 and $\|a_j\|_2 \leq \bar{a}$ from Assumption 1, so

$$
\mathbb{E} \left[ \sum_{j=\tau_n+1}^{n} (r_j x_j + d^\top p^* - a_j^\top p^* x_j) \right] \geq -\mathbb{E}[(n - \tau_n)] \cdot \left( \bar{r} + \frac{\bar{a} \bar{d}}{d} \right). \quad (31)
$$
Plugging (30) and (31) into (29), we obtain

$$\mathbb{E} R_a(\pi) \geq \mathbb{E} \left[ \sum_{j=1}^{\tau_a} g(p_j) \right] - \mathbb{E}\left[(n - \tau_a) \cdot \left( \bar{r} + \frac{\bar{r} \bar{a}}{d} \right) - \mathbb{E}\left[ \frac{\bar{r}}{d} \cdot \sum_{i \in S_\theta} b_i(\theta) \right] \right].$$

(32)

Taking the difference between (28) and (32), and then applying Lemma 1, we have,

$$\mathbb{E} R^*_a - \mathbb{E} R_a(\pi) \leq \left[ \sum_{j=1}^{\tau_a} \nu_2 a^2 \| p_j - p^* \|_2^2 \right] + \mathbb{E}\left[ (n - \tau_a) \cdot \left( \bar{r} + \frac{\bar{r} \bar{a}}{d} \right) \right] + \mathbb{E}\left[ \frac{\bar{r}}{d} \cdot \sum_{i \in S_\theta} b_i(\theta) \right]$$

holds for all $n \in N_+$ and distribution $\mathcal{P} \in \Xi$. By choosing

$$K = \max \left\{ \nu_2 a^2, \bar{r} + \frac{\bar{r} \bar{a}}{d}, \frac{\bar{r}}{d} \right\},$$

we finish the proof.

A7 Proof of Corollary 1

Proof. From the proof of Theorem 2, the role that the stopping time $\tau_a$ plays is to guarantee the orders coming before $\tau_a$ can always be satisfied. When $\mathbb{P}(\tau \leq \tau_a) = 1$, the stopping time $\tau$ has the same property. Therefore, the derivations in the proof of Theorem 2 still hold for $\tau$.

A8 Proof of Theorem 3

Proof. Define

$$\tau_a^i = \min \{ n \} \cup \left\{ t \geq 1 : \sum_{j=1}^{t} a_{ij} l(j) \geq \bar{a}_j p_j \right\},$$

where $p_j$’s are specified by Algorithm 1. Here the stopping time $\tau_a^i$ is associated with the constraint process under policy $\pi_1$. In this way,

$$\tau_a = \min_i \tau_a^i.$$ 

From the dual convergence result, we know there exists constant $C$, such that

$$\mathbb{E} \| p_k^* - p^* \|_2^2 \leq \frac{C}{t_k} \log \log t_k$$

45
holds for all $k$ and distribution $\mathcal{P} \in \Xi$. Therefore,

$$
\mathbb{E} \left[ \sum_{j=1}^{n} \| p_j - p^* \|_2^2 \right] \leq \sum_{j=1}^{n} \mathbb{E} \| p_j - p^* \|_2^2 \\
\leq \sum_{k=1}^{L} \sum_{t=t_k+1}^{t_{k+1}} \mathbb{E} \| p_k - p^*_n \|_2^2 \\
\leq \sum_{k=1}^{L} (t_{k+1} - t_k) \frac{C}{t_k} \log \log t_k \\
= M (\delta - 1) C_1 \log \log n \leq 2C \log n \log \log n.
$$

(33)

Next, we consider the constraint process $b_i^{(t)}$. From the definition of $\tau_i^t$,

$$
\left\{ \tau_i^t \leq t \right\} = \left\{ \sum_{j=1}^{t'} a_{ij} I(r_j > a_j^\top p_j) \geq nd_i - \bar{a} \text{ for some } 1 \leq t' \leq t \right\}
$$

(34)

where $p_j$’s are specified by Algorithm 1. We then focus on analyzing the probability of the event on the right hand side. Notice that from the optimality condition of the stochastic program problem,

$$
\sum_{j=1}^{t} \mathbb{E} \left[ a_{ij} I(r_j > a_j^\top p^*) \right] \leq td_i
$$

where the expectation is taken with respect to $a_j$ and $r_j$. The equality holds for the binding constraints. Consider the function

$$
g_0(p) = \mathbb{E} \left[ a_{ij} I(r_j > a_j^\top p) \right].
$$

Given the boundedness of $\Omega_p$ and the distributional properties of $(r_j, a_j)$, there exists a universal constant $C_{2,1}$ such that

$$
|g_0(p) - g_0(p')| \leq C_{2,1} \| p - p' \|_2
$$

for all $p, p' \in \Omega_p$ and distribution $\mathcal{P} \in \Xi$. 

46
First, consider the expectation,

\[
\mathbb{E} \left[ \sum_{j=1}^{t} a_{ij} I(r_j > a_j^top_j) \right] = \sum_{j=1}^{t} \mathbb{E} \left[ a_{ij} I(r_j > a_j^top_j) \right] \\
\leq \sum_{j=1}^{t} (\mathbb{E} \left[ a_{ij} I(r_j > a_j^top_j) \right] - \mathbb{E} \left[ a_{ij} I(r_j > a_j^top^*) \right]) + td_i \\
\leq C_{2.1} \sum_{j=1}^{t} \mathbb{E} \left[ \|p_j - p^*\|_2 \right] + td_i \\
\leq C_{2.1} \sum_{k=1}^{M} \sum_{j=t_k+1}^{t_{k+1}} \mathbb{E} \left[ \|p_j - p^*_n\|_2 I(j \leq t) \right] + td_i \\
\leq C_{2.1} \sum_{k=1}^{M} (t_{k+1} - t_k) \frac{C}{\sqrt{t_k}} \sqrt{\log \log t_k I(j \leq t)} + td_i \\
\leq CC_{2.1} \sqrt{t} \sqrt{\log \log t} + td_i, \tag{35}
\]

for \( i = 1, \ldots, m \) and \( t = 1, \ldots, n \). Next, consider the variance

\[
\text{Var} \left[ \sum_{j=1}^{t} a_{ij} I(r_j > a_j^top_j) \right] = \mathbb{E} \left[ \sum_{j=1}^{t} a_{ij} I(r_j > a_j^top_j) - \sum_{j=1}^{t} \mathbb{E} \left[ a_{ij} I(r_j > a_j^top_j) \right] \right]^2 \\
+ \text{Var} \left[ \mathbb{E} \left[ a_{ij} I(r_j > a_j^top_j) \right] \right]. \tag{36}
\]

This is because, for two random variables \( X_1 \) and \( X_2 \), we have

\[
\text{Var}[X_1] = \mathbb{E}[X_1 - \mathbb{E}X_1]^2 \\
= \mathbb{E} [X_1 - \mathbb{E}X_1|X_2] + \mathbb{E}[X_1|X_2] - \mathbb{E}X_1 \\
= \mathbb{E} [X_1 - \mathbb{E}X_1|X_2]^2 + \text{Var}[\mathbb{E}X_1|X_2].
\]

Let \( Z_j = a_{ij} I(r_j > a_j^top_j) - \mathbb{E} \left[ a_{ij} I(r_j > a_j^top_j) \right] \). It is easy to see that \( Z_j \)'s is a martingale difference sequence adapted to \( \{\mathcal{F}_t\}_{t=1}^{n} \). Here \( \mathcal{F}_t \) is the \( \sigma \)-algebra generated by \( \{(r_j, a_j)\}_{j=1}^{t} \). Specifically, we have

\[
\mathbb{E}[Z_j] < \infty \quad \text{and} \quad \mathbb{E}[Z_j|\mathcal{F}_{j-1}] = 0.
\]

Then, for the first term in (36)

\[
\mathbb{E} \left[ \sum_{j=1}^{t} a_{ij} I(r_j > a_j^top_j) - \sum_{j=1}^{t} \mathbb{E} \left[ a_{ij} I(r_j > a_j^top_j) \right] \right]^2 \\
= \sum_{j=1}^{t} \mathbb{E} \left[ a_{ij} I(r_j > a_j^top_j) - \mathbb{E} \left[ a_{ij} I(r_j > a_j^top_j) \right] \right]^2 \leq \hat{a}^2 t. \tag{37}
\]
For the second term in (36),

\[
\text{Var} \left[ \sum_{j=1}^{t} \mathbb{E}[a_{ij}I(r_j > a_j^* p_j)|p_j] \right] \leq \mathbb{E} \left[ \sum_{j=1}^{t} \mathbb{E}[a_{ij}I(r_j > a_j^* p_j)|p_j] - \sum_{j=1}^{t} \mathbb{E}[a_{ij}I(r_j > a_j^* p^*|p_j) \right]^2
\]

\[
\leq \mathbb{E} \left[ C_{2,1} \sum_{j=1}^{t} \|p_j - p^*\|_2 \right]^2
\]

\[
\leq C^2 C_{2,1}^2 t \log \log t.
\]

Putting together (37) and (38),

\[
\text{Var} \left[ \sum_{j=1}^{t} a_{ij}I(r_j > a_j^* p_j) \right] \leq \bar{a}^2 t + C^2 C_{2,1}^2 t \log \log t,
\]

for \( i = 1, ..., m \) and \( t = 1, ..., n \). Then, we can proceed with analyzing the right-hand-side of (34),

\[
\mathbb{P} \left( \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) \geq nd_i - \bar{a} \text{ for some } 1 \leq t' \leq t \right)
\]

\[
= \mathbb{P} \left( \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) - \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) \geq nd_i - \bar{a} - \mathbb{E} \left[ \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) \right] \right)
\]

\[
\leq \mathbb{P} \left( \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) - \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) \geq nd_i - \bar{a} - \mathbb{E} \left[ \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) \right] \right)
\]

for some \( 1 \leq t' \leq t \).

We can view the process

\[
M_{t'} = \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) - \mathbb{E} \left[ \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) \right]
\]

as a martingale adapted to the filtration \( \mathcal{F}_{t'} \) generated by \( \{(r_j, a_j)\}_{j=1}^{t'} \) and apply Doob’s martingale inequality,

\[
\mathbb{P} \left( \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) \geq nd_i - \bar{a} \text{ for some } 1 \leq t' \leq t \right) \leq \frac{\text{Var} \left[ \sum_{j=1}^{t'} a_{ij}I(r_j > a_j^* p_j) \right]}{(n-t)d_i - \bar{a} - C C_{2,1} \sqrt{t\log \log t})^2}
\]

\[
\leq \frac{\bar{a}^2 t + C^2 C_{2,1}^2 t \log \log t}{(n-t)d_i - \bar{a} - C C_{2,1} \sqrt{t\log \log t})^2}.
\]
Then,

\[
\mathbb{E}[n - \tau_{\bar{a}}] = \sum_{t=1}^{n} \mathbb{P}(\tau_0 \leq t) \\
= \sum_{t=1}^{n} \mathbb{P} \left( \sum_{j=1}^{t'} a_{ij} I(r_j > \bar{a}^\top p_j) \geq nd_i - \bar{a} \text{ for some } 1 \leq t' \leq t \right) \\
\leq \sum_{t=1}^{n} \left( \frac{\bar{a}^2 t + C^2 C_{2,1}^2 t \log \log t}{((n-t)d_i - \bar{a} - CC_{2,1} \sqrt{I \log \log I})^2} \right) \wedge 1 \\
\leq C_{2,2} \sqrt{n \log n} \quad (40)
\]

for some constant \( C_{2,2} \) dependent on \( \bar{a}, d, d, C \) and \( C_{2,1} \). Then,

\[
\mathbb{E}[n - \tau_{\bar{a}}] \leq \mathbb{E}[\max_i \{n - \tau_{\bar{a}}^i\}] \\
\leq \sum_{i=1}^{m} \mathbb{E}[n - \tau_{\bar{a}}^i] \\
\leq C_{2,2} m \sqrt{n \log n}.
\]

The last thing we consider is \( \mathbb{E} \left[ b_i^{(n)} \right] \) for \( i \in I_B \). Indeed,

\[
b_i^{(n)} = \left( nd_i - \sum_{j=1}^{n} a_{ij} I(r_j > \bar{a}^\top p_j) \right)^+ \leq \left| nd_i - \sum_{j=1}^{n} a_{ij} I(r_j > \bar{a}^\top p_j) \right|.
\]

Therefore,

\[
\mathbb{E} \left[ b_i^{(n)} \right] \leq \mathbb{E} \left[ \left| nd_i - \sum_{j=1}^{n} a_{ij} I(r_j > \bar{a}^\top p_j) \right| \right] \\
\leq \mathbb{E} \left[ \left( nd_i - \sum_{j=1}^{n} a_{ij} I(r_j > \bar{a}^\top p_j) \right)^2 \right] \\
\leq \left( \mathbb{E} \left[ nd_i - \sum_{j=1}^{n} a_{ij} I(r_j > \bar{a}^\top p_j) \right]^2 \right) + \text{Var} \left[ \sum_{j=1}^{n} a_{ij} I(r_j > \bar{a}^\top p_j) \right] \\
\leq \sqrt{C^2 C_{2,1}^2 \log \log n + \bar{a}^2 n + C^2 C_{2,1}^2 \log \log n} \\
\leq (\sqrt{2}C_{2,1} + \bar{a}) \sqrt{n \log \log n} \quad (41)
\]

where the last line comes from (35) and (39). Combining the three inequalities (33), (40), and (41) with Theorem 2, we complete the proof.
Lemma 4. If a sequence \( \{z_t\}_{t=0}^{n} \) satisfies
\[
z_{t+1} = z_t + \frac{\sqrt{z_t}}{(n-t-1)^{1/2}} + \frac{1}{(n-t-1)^2},
\]
and \( z_0 = 0 \). Then, there exists a constant \( Z \) such that
\[
\sum_{t=1}^{n} z_t \leq Z \log n \log \log n
\]
holds for all \( n \in \mathbb{N}^+ \).

Proof. Specifically, without loss of generality, we assume that \( n \) is a even number and the all constant numbers in the sequence are 1, i.e.
\[
z_{t+1} = z_t + \frac{\sqrt{\log \log t}}{(n-t-1)^{1/2}} + \frac{1}{(n-t-1)^2},
\]
since if the constant is smaller than 1, we could amplify it while if the constant is larger than one we could normalize the formula. Also, we could change \( \log \log t \) to \( \log \log n \) so that we can normalize it again and now, we only need to prove that if
\[
z_{t+1} = z_t + \frac{\sqrt{z_t}}{(n-t-1)^{1/2}} + \frac{1}{(n-t-1)^2} \Rightarrow \sum_{t=1}^{n-2} z_t \leq C \log n.
\]

If \( z_t \leq \frac{4t}{(n-t-1)^2} \),
\[
z_{t+1} \leq \frac{4t}{(n-t-1)^2} + \frac{3}{(n-t-1)^2} \leq \frac{4(t+1)}{(n-t-2)^2}.
\]
Thus, for \( t \leq n/2 \), \( z_t \leq \frac{4t}{(n-t-1)^2} \) and specifically, \( z_{n/2+1} \leq 10/n \). Then, If \( z_t \leq \frac{16}{n-t-1} \) and \( t \geq n/2 \), we have
\[
z_{t+1} \leq \frac{16}{n-t-1} + \frac{1}{(n-t-1)^2} + \frac{4(t+1)}{(n-t-1)^{3/2}/2} \leq \frac{16}{n-t-2}.
\]
Thus, for \( t \geq n/2 \), \( z_t \leq \frac{16}{n-t-2} \). Now, we sum two parts together.
\[
\sum_{t=1}^{n-2} z_t \leq \sum_{t=n/2}^{n-2} \frac{16}{n-t-2} + \sum_{i=1}^{n/2} \frac{4(t+1)}{(n-t-2)^2}
\]
\[
\leq 32 \log n + \frac{n}{n-n/2-1} + \log(n-1-n/2) + \frac{1}{n-2}
\]
\[
\leq 32 \log n,
\]
where \( n > 3 \) and the second inequality is obtained by approximating the sum by integral.
Now we prove Theorem 4.

Proof. Define

\[ \mathbf{d}^* = \mathbb{E}[\mathbf{a} | \mathbf{r} \succ \mathbf{a}^\top \mathbf{p}^*] \]

where \( \mathbf{d}^* = (d_1^*, ..., d_m^*) \). Note that \( d_i = d_i^* \) for \( i \in I_B \) (binding constraints) and \( d_i < d_i^* \) for \( i \in I_N \) (non-binding constraints).

Define

\[ d_{i,t} = \frac{b_i^{(t)}}{n-t} \]

as the remaining resource per period at the end of \( t \)-th period, for \( i \in I_B \) and \( t = 1, ..., n \). For the non-binding constraints, define

\[ d_{i,t} = \frac{nd_i^* - \sum_{j=1}^{t} a_{ij} x_j}{n-t} \]

where \( x_j \)'s are the decisions specified by Algorithm 2, for \( i \in I_N \) and \( t = 1, ..., n \). Additionally, let \( d_{i,0} = d_i^* \) for \( i = 1, ..., m \) Let \( I_B \) and \( I_N \) denote the set of indices of the binding and non-binding constraints of the stochastic optimization problem (7) under \( (d_1, ..., d_n) \) and \( \mathcal{P} \). We can find \( \{d_i, \bar{d}_i\}_{i=1}^{n} \) such that \( d \leq d_i < d_i^* \leq \bar{d}_i \leq \bar{d} \), and that the binding and non-binding constraints are still \( I_B \) and \( I_N \) for the optimization problem (7) under \( \mathcal{P} \) and all \( \mathbf{d} \in \bigotimes_{i=1}^{n} [d_i, \bar{d}_i] \). In other words, the binding and non-binding constraints remain the same as long as the average resource \( \mathbf{d} \in \bigotimes_{i=1}^{n} [d_i, \bar{d}_i] \).

Define stopping time

\[ \tau = \min \{ n \} \cup \{ t \geq 0 : d_{i,t} \notin [d_i, \bar{d}_i] \text{ for some } i \} \]

The intuition for \( \tau \) is that the first time that the binding/nonbinding structure of the problem is changed, i.e., some binding constraints become non-binding or some non-binding constraints become binding. To put it in another way, it means the average remaining resource level \( d_{i,t} \) deviates from the original \( d_i \) for a constant level.

Now, we derive an upper bound for \( \mathbb{E}[n - \tau] \). Define

\[ d'_{i,t} = \begin{cases} 
  d_{i,t}, & \text{if } t \leq \tau \\
  d'_{i,t-1}, & \text{if } t > \tau 
\end{cases} \]

for \( i = 1, ..., m \) and \( t = 1, ..., n \). The process \( \{d'_{i,t}\}_{t=1}^{n} \) can be interpreted as a process associated with \( \{d_{i,t}\}_{t=1}^{n} \) and it freezes the value of \( d_{i,t} \) from the time \( \tau \).

Define stopping time

\[ \tau_i = \min \{ n \} \cup \{ t \geq 1 : d'_{i,t} \notin [d_i, \bar{d}_i] \} \]

It is easy to see that \( \tau = \min_i \tau_i \). Next, we study \( \tau_i \) and \( \mathbb{E}[n - \tau_i] \). Define \( \tilde{p}^*_{i+1} \) be the optimal solution
to the following optimization problem

$$\min \ f(p) := d_i^T p + \mathbb{E} [(r - a^T p)^+]$$
subject to \( p \geq 0, \)

(42)

where \( d_i = (d_{i,1}, ..., d_{i,m})^T \). The problem (42) is different from the original stochastic program (7) in the specification of the term \( d \). Hence, the LP solved with the first \( t \) observations in Algorithm 2 can be viewed as an SAA of the stochastic optimization problem (42). Our dual convergence results then hold for \( p_{t+1} \) and \( \hat{p}_{t+1} \).

With the execution of the algorithm, we have

$$d_{i,t+1}' = d_{i,t}' I(\tau < t) + \frac{(n-t)d_{i,t} - a_{i,t+1}r_{t+1} > a_{i,t+1}p_{t+1}}{n-t-1} I(\tau \geq t)$$

$$= d_{i,t}' I(\tau < t) + d_{i,t}I(\tau > t) + \frac{d_{i,t} - a_{i,t+1}r_{t+1} > a_{i,t+1}^T p_{t+1}}{n-t-1} I(\tau \geq t).$$

for \( t = 0, ..., n-1 \) and \( i = 1, ..., m \).

Take off \( d_i \) on both sides,

$$d_{i,t+1}' - d_{i,t}' = (d_{i,t}' - d_{i,t}^*) I(\tau < t) + (d_{i,t} - d_{i,t}^*) I(\tau \geq t) + \frac{d_{i,t} - a_{i,t+1}r_{t+1} > a_{i,t+1}^T p_{t+1}}{n-t-1} I(\tau \geq t)$$

$$= (d_{i,t}' - d_{i,t}^*) I(\tau < t) + (d_{i,t} - d_{i,t}^*) I(\tau \geq t) + \frac{d_{i,t} - a_{i,t+1}r_{t+1} > a_{i,t+1}^T \hat{p}_{t+1}}{n-t-1} I(\tau \geq t)$$

$$+ a_{i,t+1} (I(r_{t+1} > a_{i,t+1}^T p_{t+1}) - I(r_{t+1} > a_{i,t+1}^T \hat{p}_{t+1})) I(\tau \geq t).$$

Square both sides and take expectation,

$$\mathbb{E} [(d_{i,t+1}' - d_{i,t}^*)^2] = \mathbb{E} [(d_{i,t}' - d_{i,t}^*)^2 I(\tau < t)] + \mathbb{E} [(d_{i,t} - d_{i,t}^*)^2 I(\tau \geq t)]$$

$$+ \mathbb{E} \left[ \frac{(d_{i,t} - a_{i,t+1}r_{t+1} > a_{i,t+1}^T \hat{p}_{t+1})^2}{(n-t-1)^2} I(\tau \geq t) \right]$$

$$+ \mathbb{E} \left[ \frac{(a_{i,t+1}r_{t+1} > a_{i,t+1}^T \hat{p}_{t+1}) - a_{i,t+1}r_{t+1} > a_{i,t+1}^T p_{t+1})^2}{(n-t-1)^2} I(\tau \geq t) \right]$$

$$+ 2\mathbb{E} \left[ \frac{(a_{i,t+1}r_{t+1} > a_{i,t+1}^T \hat{p}_{t+1}) - a_{i,t+1}r_{t+1} > a_{i,t+1}^T p_{t+1}) (d_{i,t} - d_{i,t}^*) I(\tau \geq t)}{n-t-1} \right]$$

$$+ 2\mathbb{E} \left[ \frac{(a_{i,t+1}r_{t+1} > a_{i,t+1}^T \hat{p}_{t+1}) - a_{i,t+1}r_{t+1} > a_{i,t+1}^T p_{t+1}) (d_{i,t} - a_{i,t+1}r_{t+1} > a_{i,t+1}^T \hat{p}_{t+1}) I(\tau \geq t)}{n-t-1} \right].$$

(43)

Here the cross terms that contain both \( I(\tau < t) \) and \( I(\tau \geq t) \) will cancel out because these two events
are exclusive to each other. Also, the following cross term also disappear

\[ 2E \left[ \frac{(d_{i,t} - a_{i,t+1}I(r_{t+1} > a_{i,t+1}\tilde{p}^*_{t+1}))}{n - t - 1} (d_{i,t} - d^*_i) I(\tau \geq t) \right] \]

\[ = 2E \left[ \frac{(d_{i,t} - a_{i,t+1}I(r_{t+1} > a_{i,t+1}\tilde{p}^*_{t+1}))}{n - t - 1} (d_{i,t} - d^*_i) I(\tau \geq t) \left| d_{1,t}, ..., d_{m,t} \right. \right] \]

\[ = 2E \left[ \frac{d_{i,t} - a_{i,t+1}I(r_{t+1} > a_{i,t+1}\tilde{p}^*_{t+1})}{n - t - 1} \left| d_{1,t}, ..., d_{m,t} \right. \right] (d_{i,t} - d^*_i) I(\tau \geq t) \]

\[ = 0 \]

where the last line is due the definition of \( \tilde{p}^*_{t+1} \) and the property of the optimal solution of \((42)\) given in Proposition 1 (d).

Back to \((43)\), we analyze the right hand side term by term.

\( \mathbb{E} \left[ (d_{i,t} - d^*_i)^2 I(\tau < t) \right] + \mathbb{E} \left[ (d_{i,t} - d^*_i)^2 I(\tau \geq t) \right] = \mathbb{E} \left[ (d_{i,t} - d^*_i)^2 \right] . \)

\( \mathbb{E} \left[ \frac{(d_{i,t} - a_{i,t+1}I(r_{t+1} > a_{i,t+1}\tilde{p}^*_{t+1}))^2}{(n - t - 1)^2} I(\tau \geq t) \right] \leq \frac{(\bar{d} + \hat{d})^2}{(n - t - 1)^2}. \)

\( \mathbb{E} \left[ \frac{(a_{i,t+1}I(r_{t+1} > a_{i,t+1}\tilde{p}^*_{t+1}) - a_{i,t+1}I(r_{t+1} > a_{i,t+1}\tilde{p}^*_{t+1}))}{n - t - 1} (d_{i,t} - d^*_i) I(\tau \geq t) \right] \]

\[ \leq \frac{2}{n - t - 1} \sqrt{\nu_2} \sqrt{\mathbb{E} \left[ (d_{i,t} - d^*_i)^2 \right] } \quad \text{(Applying the Cauchy–Schwarz inequality).} \]

\[ \leq \frac{2\bar{d} \sqrt{\nu_2} C \sqrt{\log \log t}}{(n - t - 1)^{1/2} \sqrt{t}} \sqrt{\mathbb{E} \left[ (d_{i,t} - d^*_i)^2 \right] } \quad \text{(Applying the dual convergence result).} \]

\( \mathbb{E} \left[ \frac{(a_{i,t+1}I(r_{t+1} > a_{i,t+1}\tilde{p}^*_{t+1}) - a_{i,t+1}I(r_{t+1} > a_{i,t+1}\tilde{p}^*_{t+1}))^2}{(n - t - 1)^2} I(\tau \geq t) \right] \]

\[ \leq \frac{\bar{a}^2}{(n - t - 1)^2}. \]

\( \mathbb{E} \left[ \frac{(a_{i,t+1}I(r_{t+1} > a_{i,t+1}\tilde{p}^*_{t+1}) - a_{i,t+1}I(r_{t+1} > a_{i,t+1}\tilde{p}^*_{t+1}))}{(n - t - 1)^2} I(\tau \geq t) \right] \)

\[ \leq \frac{2\bar{a}(\bar{a} + \hat{d})}{(n - t - 1)^2}. \]

Let \( z_{i,t} = \mathbb{E} \left[ (d_{i,t+1} - d_i)^2 \right] \) for \( i = 1, ..., m \) and \( t = 0, 1, ..., n \). Plugging the above (in)equalities back in \((43)\), we obtain

\[ z_{i,t+1} \leq z_{i,t} + \frac{4\bar{a}^2 + \bar{d}^2 + 4\bar{a}\hat{d}}{(n - t - 1)^2} + \frac{2\bar{a}^2 \sqrt{\nu_2} \sqrt{\log \log t}}{(n - t - 1)^{1/2} \sqrt{t}} \sqrt{z_{i,t}}. \]
for \( i = 1, \ldots, m \) and \( t = 0, 1, \ldots, n - 1 \). Also, the above inequality holds for all \( n \in \mathbb{N}^+ \) and distribution \( \mathcal{P} \in \Xi \). From lemma 4, we can show that there exists a constant \( C_{3,1} \) such that
\[
\sum_{i=1}^{n} z_{i,t} \leq C_{3,1} \log n \log \log n
\]
holds for all \( i = 1, \ldots, m, n \in \mathbb{N}^+ \) and distribution \( \mathcal{P} \in \Xi \).

Now, we derive the upper bound for the regret of Algorithm 2 with the help of (44). Specifically, we analyze the three components one by one on the right hand side of the upper bound in Corollary 1 with stopping time \( \tau \).

First, consider
\[
E \left[ \sum_{j=1}^{\tau} \|\mathbf{p}_j - \mathbf{p}^*\|_2^2 \right] \leq E \left[ \sum_{j=1}^{\tau} \|\mathbf{p}_j - \tilde{\mathbf{p}}_j^*\|_2^2 + \sum_{j=1}^{\tau} \|	ilde{\mathbf{p}}_j^* - \mathbf{p}^*\|_2^2 \right].
\]

We know that
\[
E \left[ \sum_{j=1}^{\tau} \|\mathbf{p}_j - \tilde{\mathbf{p}}_j^*\|_2^2 \right] \leq E \left[ \sum_{j=1}^{m} \|\mathbf{p}_j - \tilde{\mathbf{p}}_j^*\|_2^2 \right] \leq C \sum_{j=1}^{m} \frac{m}{j} \log m \log j \leq Cm \log m \log n \log \log n
\]
where \( C_{3,2} = Cm \log m \). In above, the second line comes from the dual convergence result.

Consider if we view the mapping from \( \mathbf{p} \) to \( \mathbf{d} \) specified by \( \mathbf{d} = E[\mathbf{a}(r > a^\top \mathbf{p})] \) as a function from \( \Omega_p \) to \( \otimes_{i=1}^{m} [d_i, \bar{d}_i] \). We can show that the mapping is distance-preserving with a rate \( C_{3,3} \). Then,
\[
E \left[ \sum_{j=1}^{\tau} \|\tilde{\mathbf{p}}_j^* - \mathbf{p}^*\|_2^2 \right] \leq C_{3,3} E \left[ \sum_{j=1}^{m} \sum_{i=1}^{\tau} \|d_{i,j} - d_{i,j}^*\|_2^2 \right] \leq \sum_{i=1}^{m} C_{3,3} E \left[ \sum_{j=1}^{\tau} \|d_{i,j} - d_{i,j}^*\|_2^2 \right] \leq \sum_{i=1}^{m} C_{3,3} E \left[ \sum_{j=1}^{\tau} \|d_{i,j} - d_{i,j}^*\|_2^2 I(j \leq \tau) \right] \leq \sum_{i=1}^{m} C_{3,3} \sum_{j=1}^{\tau} \|d_{i,j} - d_{i,j}^*\|_2^2 \leq \sum_{i=1}^{m} C_{3,3} \sum_{j=1}^{\tau} z_{i,j} \leq C_{3,1} C_{3,3} m \log n \log \log n.
\]
Plugging (46) and (47) into (45), we obtain

\[
\mathbb{E} \left[ \sum_{j=1}^{\tau} \|p_j - p^*\|^2 \right] \leq (C_{3,2} + C_{3,1}C_{3,3}m) \log n \log \log n
\]

(48)

for all \( n \in \mathbb{N}^+ \) and \( P \in \Xi \).

Second, let

\[
\eta = \min_i |d_i^* - d_i| \wedge |\bar{d}_i^* - \bar{d}_i|.
\]

From the choice of \( d_i \)'s and \( \bar{d}_i \)'s, we know that \( \eta > 0 \).

Then, with a similar approach as in the proof of Theorem 3,

\[
\mathbb{E}[n - \tau_i] = \sum_{t=1}^{n} \mathbb{P}(\tau_i \leq t)
\]

\[
\leq \sum_{t=1}^{n} \frac{\mathbb{E}(d_i' - d_i^*)^2}{\eta^2} = \frac{1}{\eta^2} \sum_{t=1}^{n} z_i,t \quad \text{(Applying the Chebyshev’s Inequality)}
\]

\[
\leq \frac{C_{3,1}m}{\eta^2} \log n \log \log n.
\]

So,

\[
\mathbb{E}[n - \tau] \leq \mathbb{E}[\max_i \{n - \tau_i\}]
\]

\[
\leq \sum_{i=1}^{m} \mathbb{E}[n - \tau_i]
\]

\[
\leq \frac{C_{3,1}m}{\eta^2} \log n \log \log n.
\]

(49)

Third, from the definition of \( \tau_i \), we know

\[
b_i^{(n)} \leq \bar{d}_i(n - \tau_i)
\]

for \( i \in I_B \). Consequently,

\[
\mathbb{E} \left[ b_i^{(n)} \right] \leq \mathbb{E} \left[ \bar{d}_i(n - \tau_i) \right] \leq \frac{C_{3,1} \bar{d}}{\eta^2} \log n \log \log n
\]

\[
\mathbb{E} \left[ \sum_{i \in I_B} b_i^{(n)} \right] \leq \frac{C_{3,1} \bar{d}m}{\eta^2} \log n \log \log n
\]

(50)

for all \( n \in \mathbb{N}^+ \) and \( P \in \Xi \).

Combining (48), (49), and (50) with Corollary 1, we complete the proof.
A10 Proof of Theorem 5

Proof. When we apply $p^*$ as the decision rule, $p_j = p^*$ and

$$\mathbb{E} \left[ \sum_{j=1}^{n} \| p_j - p^* \|_2^2 \right] = 0. \quad (51)$$

So, we only need to focus on $\mathbb{E}[n - \tau] \text{ and } \mathbb{E} \left[ \sum_{i \in I_B} l_i^{(n)} \right]$. In a similar way as the proof of Theorem 3, we define

$$\tau_i^d = \min \{ n \} \cup \left\{ t \geq 1 : \sum_{j=1}^{t} a_{ij} I(r_j > \bar{a}^*_j p^*) > nd_i - \bar{a} \right\}.$$ 

By the definition of $p^*$, we know

$$\mathbb{E} \left[ \sum_{j=1}^{t} a_{ij} I(r_j > \bar{a}^*_j p^*) \right] \leq td_i$$

for $i = 1, ..., m$ and $t = 1, ..., n$. Specifically, the equality holds for the binding constraints, i.e., $i \in I_B$.

$$\text{Var} \left[ \sum_{j=1}^{t} a_{ij} I(r_j > \bar{a}^*_j p^*) \right] \leq \bar{a}^2 t.$$ 

for $i = 1, ..., m$ and $t = 1, ..., n$.

Therefore,

$$\mathbb{E}[n - \tau_i^d] = \sum_{t=1}^{n} \mathbb{P}(\tau_i \leq t)$$

$$= \sum_{t=1}^{n} \mathbb{P} \left( \sum_{j=1}^{t} a_{ij} I(r_j > \bar{a}^*_j p^*) \geq nd_i - \bar{a} \right)$$

$$\leq \sum_{i=1}^{n} \left( \frac{\bar{a}^2 t}{(n - t)^2 \bar{d}^2_i} \right) \wedge 1 \quad \text{(Applying the Chebyshev’s inequality)}$$

$$\leq C_{3,1} \sqrt{n}$$

for some constant $C_{3,2}$ dependent on $\bar{a}$, $\bar{d}$ and $\bar{d}$. Then, we obtain

$$\mathbb{E}[n - \tau] \leq C_{3,1} m \sqrt{n} \quad (52)$$

with a similar derivation as in Theorem 3. Next, for $i \in I_B$,

$$\mathbb{E}[l_i^d] \leq \sqrt{\text{Var} \left[ \sum_{j=1}^{t} a_{ij} I(r_j > \bar{a}^*_j p^*) \right]} \leq \bar{a} \sqrt{n} \quad (53)$$
from (41). Combining (51), (52), and (53) with Theorem 2, we complete the proof.

\[ \square \]

A11 Proof of Theorem 6

**Proof.** First,

\[
E[R_n(\pi)] = E \left[ \sum_{j=1}^{n} r_j x_j \right] \\
\leq E \left[ \sum_{j=1}^{n} r_j x_j - \left( nd - \sum_{j=1}^{n} x_j \right) p^* \right] \\
= E \left[ \sum_{j=1}^{n} (r_j x_j + dp^* - x_j p^*) \right] \\
= \sum_{j=1}^{n} E[r_j x_j + dp^* - x_j p^*] \\
= \sum_{j=1}^{n} E[g(p_j)].
\]

where (a) comes from that the constraint must be satisfied and (b) comes from the definition of \( g(\cdot) \). We do not need to consider the stopping time because after the resource is exhausted, the setting of \( p_s = 1 \) is consistent with the enforcement of \( x_s = 0 \).

Then,

\[
ng(p^*) - E[R_n(\pi)] \geq \sum_{j=1}^{n} E[g(p^*) - g(p^*_j)] \\
\geq \frac{1}{2} \nu_3a^2 \sum_{j=1}^{n} E \|p^* - p_j\|^2.
\]

The second line comes from Lemma 1 and \( \nu_3 \) is defined in Assumption 3. By taking \( K' = \frac{1}{2} \nu_3a^2 \), we complete the proof.

\[ \square \]
A12 Proof of Corollary 2

Proof. Since the algorithm enforce \( p_t = 1 \) for \( t > \tau_0 \). The result follows by splitting the summation on the right-hand-side of (7).

\[
\mathbb{E}_{P_\pi} [R^*_n - R_n(\pi)] \geq K' \cdot \mathbb{E}_{P_\pi} \left[ \sum_{j=1}^n (p_j - p^*)^2 \right] + \mathbb{E}_{P_\pi} [R^*_n] - ng(p^*)
\]

\[
= K' \cdot \mathbb{E}_{P_\pi} \left[ \sum_{j=1}^{\tau_0} (p_j - p^*)^2 \right] + \mathbb{E}_{P_\pi} \left[ \sum_{j=\tau_0+1}^n (p_j - p^*)^2 \right] + \mathbb{E}_{P_\pi} [R^*_n] - ng(p^*)
\]

\[
= K' \cdot \mathbb{E}_{P_\pi} \left[ \sum_{j=1}^{\tau_0} (p_j - p^*)^2 \right] + K' (1 - p^*)^2 \mathbb{E}[n - \tau_0] + \mathbb{E}_{P_\pi} [R^*_n] - ng(p^*).
\]

\[\square\]

A13 Proof of Theorem 7

We first introduce the van Tree inequality and refer its proof to (Gill and Levit, 1995).

Lemma 5 (Gill and Levit (1995)). Let \( (X, F, P_\theta : \theta \in \Theta) \) be a dominated family of distributions on some sample space \( X \); denote the dominating measure by \( \mu \). The parameter space \( \Theta \) is a closed interval on the real line. Let \( f(x|\theta) \) denote the density of \( P_\theta \) with respect to \( \mu \). Let \( \lambda(\theta) \) denote the density function of \( \theta \). Suppose that \( \lambda \) and \( f(x|\cdot) \) are both absolutely continuous, and that \( \lambda \) converges to zero at the endpoints of the interval \( \Theta \). Consider \( \phi : \Theta \rightarrow \mathbb{R} \) a first-order differentiable function. Let \( \hat{\phi}(X) \) denote any estimator of \( \phi(\theta) \). Then,

\[
\mathbb{E}[(\hat{\phi}(X) - \phi(\theta))^2] \geq \frac{[\mathbb{E}\phi'(\theta)]^2}{\mathbb{E}[\mathcal{I}(\theta)] + \mathcal{I}(\lambda)}
\]

where the expectation on the left hand side is taken with respect to both \( X \) and \( \theta \), and the expectation on the right hand side is taken with respect to \( \theta \). \( \mathcal{I}(\theta) \) and \( \mathcal{I}(\lambda) \) denote the Fisher information for \( \theta \) and \( \lambda \), respectively,

\[
\mathcal{I}(\theta) := \mathbb{E} \left[ (\log f(X|\theta))^2 \bigg| \theta \right]
\]

\[
\mathcal{I}(\lambda) := \mathbb{E} \left[ (\log \lambda(\theta))^2 \right].
\]

Now we proceed to prove Theorem 7.
Proof of Theorem 7. First, we analyze the gap between $\mathbb{E}[R_n^*]$ and $ng(p^*)$.

$$ng(p^*) - \mathbb{E}[R_n^*] \leq \sum_{j=1}^{n} \left( \mathbb{E}[(p^* - p_n^*) I(p^* > r_j > p_n^*)] + \mathbb{E}[(p_n^* - p^*) I(p^* < r_j \leq p_n^*)] \right)$$

$$= \sum_{j=1}^{n} \left( \mathbb{E}[(p^* - p_n^*) \mathbb{P}(p^* > r_j > p_n^*)] + \mathbb{E}[(p_n^* - p^*) \mathbb{P}(p^* < r_j \leq p_n^*)] \right)$$

$$\leq n\nu_2 \mathbb{E}[|p^* - p_n^*|^2]$$

$$\leq \nu_4 Cm \log m \log \log n = C_{5,1} \log \log n \quad (54)$$

where $C_{5,1} = \nu_4 Cm \log m$. Here (a) comes from the proof of Lemma 1. (b) comes from the fact that

$$\mathbb{P}(p^* > r_j > p_n^*) \leq \nu_4 |p^* - p_n^*|^2 I(p^* > p_n^*).$$

(c) comes for applying the dual convergence result to this special one-constraint case. Next, we derive a lower bound for

$$\mathbb{E} \left[ \sum_{j=1}^{n} (p_j - p^*)^2 \right]$$

with the help of the van Tree’s inequality in Lemma 5. For the lower bound, we only need to derive under a specific distribution. Consider a truncated exponential distribution for $r_j$’s

$$f(r|\theta) = \frac{\theta e^{-\theta r} I(r \in [0,1])}{1 - e^{-\theta}}$$

and the Beta distribution for $\theta$

$$\lambda(\theta) = (\theta - 1)^2(2 - \theta)^2$$

with the support $\Theta = [1,2]$. Let $d = 1/2$ and then

$$p^* = \mathcal{Q}_{d/2}(r) = \phi(\theta) := \frac{1}{\theta} \log \left( \frac{1}{2} + \frac{1}{2} \theta \right).$$

Additionally,

$$\left[ \mathbb{E}\phi'(\theta) \right]^2 = \left[ \int_{1}^{2} \phi'(\theta) \lambda(\theta) d\theta \right]^2 := C_{5,2} > 0.$$
The Fisher information $\mathcal{I}(\theta)$ and $\mathcal{I}(r; \lambda)$ can be computed according to the definition.

$$
\mathbb{E}[\mathcal{I}(r; \theta)] = \mathbb{E}\left[ (\log f(r|\theta))^2 \mid \theta \right] = \int_0^1 \int_0^1 (\log f(r|\theta))^2 f(r|\theta)\lambda(d\theta) dr d\theta := C_{5,3} > 0.
$$

$$
\mathcal{I}(\lambda) = \mathbb{E}\left[ (\log \lambda(\theta))^2 \right] = \int_0^1 (\log \lambda(\theta))^2 \lambda(d\theta) := C_{5,4} > 0.
$$

In above, $C_{5,2}$, $C_{5,3}$ and $C_{5,4}$ are deterministic real numbers that can be computed from the corresponding integrals.

Then, $p_j$ can be viewed as an estimator of $p^* = \phi(\theta)$ with the first $j - 1$ samples. Consider the fact that

$$
\mathbb{E}[\mathcal{I}(r_{1:(j-1)}; \theta)] = (j - 1)\mathbb{E}[\mathcal{I}(r; \theta)].
$$

We apply Lemma 5 and obtain,

$$
\sup_{\theta} \mathbb{E}\left[ \sum_{j=1}^n (p_j - p^*)^2 \right] \geq \sum_{j=2}^n \frac{C_{5,2}}{C_{5,3}(j - 1) + C_{5,4}} \geq C_{5,5} \log n \quad (55)
$$

for all $n \in \mathbb{N}^+$ with some constant $C_{5,5}$ dependent on $C_{5,2}$, $C_{5,3}$, and $C_{5,4}$. Combining (54) and (55) with Theorem 6, there exist a $\theta$ and a distribution $P(r|\theta)$ such that

$$
\mathbb{E}R_n^* - \mathbb{E}R_n(\pi) \geq C_{5,5}K' \log n - C_{5,1} \log \log n.
$$

Set $n_0 = \min\{n \geq 0 : C_{5,5}K' \log n \leq 2C_{5,1} \log \log n\}$ and $C = \frac{C_{5,5}K'}{2}$. The lower bound result follows.

\[\square\]