The Price and Variety Effects of Vertical Mergers

Vertical mergers within a multi-echelon market result in equilibrium price changes, for wholesalers and retailers, alike. They may also impact the product variety that is available to the consumer, i.e., the equilibrium product assortment sold in the market. In this paper, we consider the simultaneous price and variety effects of vertical mergers in a general two-echelon base model, in which an arbitrary number of firms compete at each echelon; each of the upstream suppliers offers one or multiple products to some or all of the retailers or directly to the end consumer. We assume linear price contracts, with prices selected sequentially, in a sequence of oligopolistic price competitions: the process starts with the firms at the upper echelon, followed by simultaneous price selections by the retailers in the downstream echelon. To assess the impact on product variety, we employ a demand model with the characteristic that, depending on the selected retail prices, a smaller or larger subset of all potential products is sold in the market.

For an arbitrary merger of a supplier with a group of retailers, we characterize the post-merger equilibrium behavior and show how the changes of all performance measures of interest, i.e., wholesale and retail price equilibria, product assortment, sales volumes, the firms’ profit levels and consumer welfare, can be computed efficiently. When the merger is strictly vertical, i.e., involves a single retailer (organization) with whom the supplier deals on an exclusive basis, we prove that the merger results in a (weak) reduction of all equilibrium prices, along with a (weak) shrinkage of the product assortment.

1. Introduction and summary

Since the nineteen-fifties, the competitive effects of vertical integration have been the subject of continuous debate among economists, antitrust lawyers, public policy makers, and last but not least, in many corporate board rooms. Vertical integration occurs when a supplier acquires or merges with one or more of its client firms. Alternatively, vertical integration may result from internal growth: a firm in an intermediate echelon of a supply chain may integrate forward by creating its own direct distribution channel to final consumers or buyers at an echelon further downstream from the very next downstream echelon. Similarly, a firm may integrate backward by deciding to manufacture the products it used to buy from an upstream supplier, or to purchase them directly from firms further upstream than their traditional supplier(s).

It is important to establish robust price effects that pertain to markets with an arbitrary number

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of general, non-identical suppliers and retailers and any one of the retailers procuring any number of differentiated products from any given subset of the suppliers. Beyond characterizing whether vertical mergers result in price increases or decreases, it is important to assess the actual resulting magnitude in price and consumer welfare changes. For example, in the high-profile 2018 litigation of the Department of Justice against AT&T and Time Warner’s vertical merger, the government’s expert witness predicted that the merger would result in an increase of 45 cents per month in the subscription fee for DIRECT TV, which is owned by AT&T and is the largest US pay-TV provider. This minor increase, itself contested on methodological grounds, was further viewed as insignificant and contributed to the Judge’s dismissal of significant harmful consumer effects. Last but not least, it is important to understand what impact vertical mergers have on the product variety offered in the market. For example, even if prices decline, consumers may be disenfranchised if the vertical merger results in reduced product variety.

Product variety is one of the principal “non-price effects” used in antitrust and merger investigations in the US (along with “quality,” “service” and “innovation”). For example, the 2010, most recent merger guidelines by the Department of Justice (DOJ) and the Federal Trade Commission (FTC) state that, in addition to price effects, “[e]nhanced market power can also be manifested in non-price terms and conditions that adversely affect customers, including reduced product quality, reduced product variety, reduced service, or diminished innovation,” see DOJ and FTC (2010). While the latter are focused on horizontal mergers, the concerns are no less prevalent for vertical mergers, and for good reasons, as shown in our paper. Moreover, legal scholars such as Averitt and Lande (1997, 1998), economists Carlton and Israel (2010) and former FTC Commissioners Leary (2000, 2001) and Rosch (2014) have argued that the above 2010 FTC and DOJ guidelines “do not go far in their recognition of non-price effects.” Finally, the American Antitrust Institute (AAI) devoted its 2016 annual symposium to the topic of “Non-Price Effects of Mergers,” see also Gundlach (2016).

To address the simultaneous price and variety effects of vertical mergers, we consider a general two-echelon base model, in which an arbitrary number of firms compete at each echelon; each of the upstream suppliers offers one or multiple products to some or all of the retailers or directly to the end consumer. We stipulate a general, albeit parsimonious demand model for all the products (potentially) offered to the end consumer. We assume linear price contracts, with prices selected sequentially, in a sequence of oligopolistic price competitions: the process starts with the firms at the upper echelon, followed by simultaneous price selections by the retailers in the downstream
echelon. The suppliers, when selecting their wholesale prices, of course, consider the equilibrium price responses by the retailers.

To assess the impact on product variety, we need a demand model with the characteristic that, depending on the selected retail prices, a smaller or larger subset of all potential products is sold in the market. Our demand model has this characteristic, while in more traditional consumer choice models, for example those with MultiNomial Logit (MNL) demand functions, or its various generalizations (e.g., nested or mixed MNL)—all products always attain some market share, even if offered at highly uncompetitive prices. Indeed, existing workhorse models for merger simulations are based on mixed MNL consumer choice models and are therefore ill-suited to assess the impact of mergers on product variety.

In our model, as shown in Federgruen and Hu (2016) and Federgruen and Hu (2018), there exists a pure Nash equilibrium of wholesale and retail prices for the sequential oligopoly model. These equilibria do not need to be unique; in fact, there may be infinitely many such equilibria. However, even if the sequential Nash equilibrium fails to be unique, there is one equilibrium which stands out in the sense that it is the unique equilibrium to be globally stable under robust best responses. For the second-stage competition model among the retailers, under given wholesale prices, this means that, irrespective of the initial price choices of the retailers, the market is guaranteed to converge to this equilibrium when firms, iteratively, adjust their prices as “robust” best responses to the competitors’ prices. A response is “robustly best,” if it is best among all price choices whose impact can be assessed based on local information only, see Section 3 for a precise definition. Similarly, in the first-stage competition model among the suppliers, there is an equilibrium which is either categorically unique, or stands out as the only equilibrium that is globally stable under robustly best responses. Given their unique structural properties, we, henceforth refer to them as “the” equilibrium in the retailer and supplier competition model, respectively.

Consider, now, a vertical merger between one of the suppliers and a group of retailers who this supplier sells to, possibly along with other suppliers. (The supplier may, likewise, sell to other retailers, outside of the group she merges with.) After the merger, any outside supplier continues to offer the merged company all products she used to offer one of the merging retailers; similarly, the merging supplier continues to offer outside retailers any of the products she offered them before the merger. Our first objective is to demonstrate how all post-merger performance measures of interest can be determined and efficiently computed. These measures include the wholesale and retail price equilibria, the product assortment, the sales volumes of the different products, the firms’ profit values, and a consumer and a social welfare measure. When characterizing the post-merger
equilibrium behavior, it is reasonable to assume that the retail prices of the products sold by the merged company, continue to be determined along with all other retail prices, i.e., as part of the second stage retailer competition game. However, in some settings, these prices may be determined by the merged company during the first-stage competition game. The sequence in which price decisions are made, is often of critical importance. We therefore cover both settings, in a dedicated section, for each (Sections 4 and 6, respectively). When the merged company discloses its retail prices, upfront, during the first competition stage, this is equivalent to a setting, where those products are sold directly from the (merging) supplier to the end consumer. This gives rise to a new type of sequential oligopoly, with “direct sales,” in contrast to the pre-merger, strictly hierarchical network structure. The model with direct sales for some of the products is developed in Section 5.

Of critical significance in the comparison of the pre-merger and post-merger equilibria, is a comparison of the pre- and post-merger, so-called effective cost rate polyhedra. The interior of this polyhedron describes the region of cost rate vectors under which, all products capture a share of the market, i.e., under which consumers enjoy the maximum product variety.

Our post-merger equilibrium characterizations apply to the above, fully general, merger setting. Numerical examples in Section 7 show that reductions in product variety may arise, both when the merged firm’s retail prices are determined and disclosed in the first-stage game, and when this occurs in the second stage (along with the remainder of the retail prices). However, the merger may also increase the product variety. In this general setting, it is impossible to prove categoric directional changes for either the prices or the product variety: On the one hand, the vertical merger eliminates some of the double marginalization, hence, tends, in and of itself, to reduce prices. However, when a group of retailers participates in the merger, the merger also has a horizontal component. Horizontal mergers tend to result in increased prices, since reducing the number of competing retail organizations. The net impact of these two conflicting forces depends on the specifics of the network structure, the demand functions, and last but not least, the exogenously specified cost rates incurred by the suppliers.

To isolate the impact of a vertical merger, we therefore analyze the merger of a supplier with a single retailer with whom she has an exclusive relationship: the supplier sells exclusively via the retailer and the retailer sources all of his products from this supplier. In the more common setting where all retail prices continue to be determined during the second-stage competition game, we can show that the wholesale price equilibrium decreases, as does the retail price equilibrium. Moreover, we show that the post-merger equilibrium product assortment is a subset of its pre-merger counterpart. In other words, any product that failed to be competitive in the pre-merger
market, continues to be uncompetitive after the merger; moreover, the merger may “elbow” some of the products out of the market. Put differently, foreclosure of some of the products may arise as the market equilibrium outcome in the post-merger world. The intuition is as follows: because a vertical merger eliminates double marginalization for the integrated firm, it leads to lower equilibrium prices. Some products that have high marginal costs with positive demand levels in the pre-merger world, may be priced out of the market after the merger, as the merger drives down the equilibrium prices.

In the alternative setting where the (retail) prices of the merged firm’s products are determined and disclosed in the first stage game, we obtain somewhat more restricted comparison results: for any given vector of wholesale prices, the corresponding retail price equilibrium is smaller than before the merger. When the cost-rate vector belongs to the above mentioned pre-merger effective cost rate polyhedron $C$, all equilibrium wholesale and retail prices weakly decline, as does the product assortment. However, when the cost rate vector is outside of $C$, price increases or expansions of the product assortment may arise, as evidenced by several examples in Section 7.

There are very few workhorse models for two-echelon industries with an arbitrary number of firms in both echelons and a general set of products offered by the upper-echelon suppliers to the lower-echelon retailers. In the economics literature, the state-of-the-art models employ a framework, commonly referred to as “Nash-in-Nash.” There, the retailers are assumed to engage in a standard multi-product non-cooperative price competition game based on a mixed MNL consumer choice model (or, analogously an MNL model with random coefficients in the utility measures). Assuming linear price contracts, as in our paper, the unit wholesale prices for the various products are assumed to be determined by the solution of a set of parallel independent cooperative Nash bargaining games, one for each product, between its supplier and its retailer. The model also assumes that bargaining over these wholesale prices happens simultaneously with the retailers making their non-cooperative pricing decisions.

Several reservations have been raised vis-à-vis this framework. First, it adopts a hybrid approach, with one set of price decisions, i.e., retail prices, being made non-cooperatively, and the others, i.e., wholesale prices, cooperatively as the solutions of independent Nash bargaining games. Even for the collection of products sold by a given supplier to a specific retailer, the wholesale prices are assumed to be “negotiated” independently, via independent Nash bargaining games, rather than comprehensively.

Most importantly, by assuming that the bargaining over the wholesale prices occurs simultaneously with the retailers making their retail price decisions, Crawford et al. (2018) state, for example:
“[t]his assumption simplifies the estimation and computation of [the] model.” In particular, the assumed *simultaneity* of wholesale cooperative bargaining and retail competitive pricing implies that, for any given product, there is no anticipated change in the retail price if the corresponding wholesale price changes. Crawford *et al.* (2018) admit that “[a]n alternative timing assumption would be to assume that [wholesale prices] are first negotiated, and then [retail] prices,” as in our framework. The authors conclude that “the [Nash-in-Nash] approach taken here [is] a reasonable approximation.”

Finally, even when the wholesale prices are exogenously given, rather than endogenously determined, the equilibrium behavior in the retail price competition game may be complex. As shown in Aksoy-Pierson *et al.* (2013), the game may have no Nash equilibrium, one or multiple equilibria, without any one of the equilibria necessarily standing out in terms of predictability; in particular, the equilibria may not necessarily be characterized by the First Order Conditions for an interior solution.

It is for these various reasons that we offer an alternative workhorse model, to be used to assess the effects of changes in the market structure, for example due to vertical mergers.

The remainder of this paper is organized as follows. In Section 2, we provide a brief literature review. Section 3 describes our general sequential oligopoly model and summarizes its equilibrium behavior. In Section 4 we characterize the price and product assortment implications of a vertical merger, assuming that all of the retail prices of these retailers’ products continue to be determined in the *second* stage of the two-stage competition model. Section 5 extends our hierarchical two-stage price competition model to one in which some of the suppliers’ products are sold directly to the consumers without the intermediation of a retailer. We refer to this as the “direct sales model.” As mentioned, this extension is of interest, by itself; moreover, it enables the vertical merger analysis in Section 6, where post-merger retail prices are assumed to be determined by the merging supplier as part of the *first* stage price competition model. Both Sections 4 and 6 pay special attention to the special case where the supplier merges with a single dedicated retailer, to prove the various above discussed directional changes for prices and product assortment. Section 7 discusses numerical examples, and Section 8 ends the paper with some concluding remarks.

2. Literature review

The economics literature on vertical mergers goes back to the 50s and 60s. Those early papers advanced arguments that vertical mergers could have adversarial effects, in particular by imposing exclusionary practices that *foreclose* competitors, see, e.g., Bain (1965). Their arguments failed
to be supported by formal competition models or even solid micro-economic foundations. In the next two decades, i.e., the 60s and 70s, economists, mostly part of the so-called Chicago School, explained that, to the contrary, vertical integration advances economic efficiency, for example by eliminating double marginalization.

Nobel Prize winner Williamson (1975, 1985) was the leading contributor to the area of transaction cost economics, in the 70s and 80s. This stream of literature identified both positive and negative welfare effects resulting from vertical mergers.

Formal game-theoretical models were published in the subsequent two decades. These papers typically consider models with at most two suppliers and at most two retailers and at most one product transacted between any given supplier-retailer pair. A few papers deal with an arbitrary number of suppliers or retailers, however assumed to be with identical characteristics. Here are two examples: Ordover et al. (1990) consider a model with two upstream firms and two downstream retailers. The retailers sell a differentiated item produced from a common homogeneous input by the two identical suppliers with the (same) constant input cost rate. They face a general pair of demand functions and engage in price competition. The authors show that equilibrium retail prices may increase and that welfare may decline due to a vertical merger, depending upon how input prices are set at the upper echelon after such a merger. One such setting is when the merged firm chooses not to deliver to the unmerged retailer, leaving him with the unmerged supplier as the only possible supply source.

Salinger (1988) analyzes a market with a general number of identical suppliers and retailers; the market faces sequential Cournot competition for a homogenous product, first at the suppliers’ level and then at the retailers’ level. The (inverse) demand function at both echelons is assumed to be linear. The author shows that vertical mergers result in an increase of the price in the retail market, if more than half of the suppliers merge, each with one of the retailers, and the number of retailers is sufficiently larger than the number of suppliers. See Riordan (2008) for a comprehensive survey of this literature. This survey paper also reviews antitrust regulation and legal challenges of proposed vertical integrations by the Department of Justice (DOJ) and the Federal Communications Commission (FCC).

The “Nash-in-Nash” approach, reviewed in the Introduction, represents the state-of-the-art workhorse model for two-echelon competition models and associated vertical merger simulations. The approach was initiated by the seminal paper of Horn and Wolinsky (1988). Crawford and Yurukoglu (2012) and Crawford et al. (2018) applied this framework to model the two-echelon industry of multi-channel video programming distributors and the channels. Other empirical papers
that employ this framework include Draganska et al. (2010), Grennan (2013), Gowrisankaran et al. (2015) and Ho and Lee (2017).

Some papers deal with two-echelon oligopoly models in which some of the products are sold directly by a supplier to the end consumer, either in parallel or as alternatives to traditional sales of the same or related products via independent retailers; these models thus allow for “direct sales” channels as in our model. Recent examples include Abhishek et al. (2016) and Hotkar and Gilbert (2018), the former (latter) with one (two) supplier(s) and two (one) retailers (retailer), see also the references therein. Both papers assume affine demand functions. The existence of a direct sales channel in parallel to a traditional one for identical or closely related products is referred to as “supplier encroachment.”

A few competition models have endogenized the determination of the retailers’ assortments. Besbes and Sauré (2016) was the first paper to address a joint price and assortment competition model. However, in their model, firms start out by each making its own price and assortment decisions. The specific sales volumes for all selected products are then determined by an underlying pure or mixed MNL model. In other words, each firm is assumed to control its own product assortment, irrespective of the price-assortment decisions made by its competitors. The authors show that a unique equilibrium always exists, with the property that every firm selects an identical profit margin for all of its products. In our model, assortment choices are implied by price selections allowing for general firm- and product-dependent price sensitivities and explaining general profit margins. Heese and Martínez-de Albéniz (2018) consider a model with a single retailer and any number of suppliers, each offering a single product. The retailer selects a subset of these products of a given pre-announced cardinality, the so-called assortment breadth. After the retailer selects her assortments, sales volumes are determined by an MNL model with a no-purchase option. As in our paper, competition is modeled as a two-stage Stackelberg game: in the first stage, the suppliers select their wholesale prices; in the second stage, the retailer determines the subset of products of the given cardinality which maximizes expected profits.

A few recent papers in the operations management literature analyze the impacts of horizontal mergers. Federgruen and Pierson (2011) show that in a single echelon price competition model, with general non-linear demand and cost functions, horizontal mergers result in higher equilibrium prices and profit levels but reduced consumer welfare, when the price competition model is supermodular, each firm’s profit function is quasi-concave in its own price and markets are competitive, i.e., in the pre-merger industry, each firm’s profits increase when any of the competitors increases his price. Their results generalize those obtained in Deneckere and Davidson (1985) for the case of
linear demand and cost functions. Cho (2014) analyzes the impact of horizontal mergers in the multi-echelon Cournot competition model of Corbett and Karmarkar (2001). In this model, a single homogeneous final good is sold to the end consumer. At the most downstream echelon, firms engage in Cournot competition for the single homogeneous good, with an affine demand function. At each echelon, all competing firms have identical characteristics and engage in Cournot competition as well.

Korpeoglu et al. (2017) consider a variant of the Corbett and Karmarkar model, with two echelons. The suppliers face an identical cost rate and charge an identical wholesale price for a single homogeneous item in the market. The retailers share the same demand function, which depends on the retailer’s own retail price, only. The common wholesale price is determined assuming the suppliers engage in Cournot competition or the Shapley and Shubik (1977) generalization thereof. The authors characterize how the equilibrium varies with the number of retailers and suppliers. The paper is also related to Adida and DeMiguel (2011).

Bimpikis et al. (2016) analyze a model with $N$ suppliers and a number of separate markets; in each market, a homogenous good is offered by a specific subset of the $N$ suppliers who engage in Cournot competition. Each supplier’s costs depend on her aggregate production quantity across all markets. The authors investigate the impact of a horizontal merger among suppliers. Cho and Wang (2017) analyze horizontal mergers among newsvendors.

Our work is also related to the literature on finite population non-cooperative games with linear-quadratic utilities, with many applications in the study of peer effects, see, e.g., Ballester et al. (2006). Since the demand functions in our model are piecewise affine, the firm’s profit functions are piecewise linear-quadratic, which is the reason why our characterization of the equilibrium behavior is more challenging and more complex. Ballester et al. (2006) focus on the implications of changes in the network structure, similar to our focus where such network changes arise due to mergers. Candogan et al. (2012) incorporate pricing decisions into the framework of Ballester et al. (2006). Most recently, Zhou and Chen (2015) extended the results in Ballester et al. (2006) to settings where the players are partitioned into leaders and followers engaging in a two-stage competition game, similar to our settings with suppliers as leaders and retailers as followers. Other papers emphasizing the impact of the network structure on market outcomes include Borenstein et al. (1999), Wu et al. (2013) and Bose et al. (2014).

To our knowledge, only Federgruen and Hu (2016) address how changes in the structure of the supply chain network may impact the product variety sold in the market. More specifically, the paper considers a market with a given number of manufacturers each selling a group of products to
a manufacturer-associated chain of independent retailers. The demand model is of the same type as considered in this paper: demands are generated by a base system of affine equations, as long as the price vector belongs to a given polyhedron; they are extended to the full Euclidean price space to satisfy the following intuitive regularity condition, see Shubik and Levitan (1980): under any given price vector, when some product is priced out of the market, i.e., has zero demand, any increase of its price has no impact on the demand volumes of any of the products. Initially, each manufacturer sells its products through an exclusive wholesaler. Federgruen and Hu (2016) investigate how the equilibrium changes under disintermediation, i.e., when the manufacturers decide to sell directly to their retailers, eliminating the wholesalers as middlemen. They prove that while equilibrium prices decline, the set of products sold in the market stays the same or declines as well, similar to the results on vertical mergers obtained in Section 4 of our paper.

In the US, guidelines for the evaluation of vertical mergers were established in 1984, see DOJ (1984). Unlike the guidelines for horizontal mergers which have been updated respectively, these 1984 guidelines are still in place. Salop and Culley (2014), in their guide for practitioners write: “Those Guidelines are now out of date. They do not reflect current economic thinking about vertical mergers or current agency practice. Nor do they reflect the approach taken in the 2010 Horizontal Merger Guidelines.” In Europe, the most recent guidelines are contained in EEC (2008).

From a methodological point of view, our analysis makes extensive usage of so-called ZP-matrices, otherwise referred to as M-matrices, and their properties. This linear algebra area has, recently, proven to be of great value in the analysis of other operational models, for example, Li and Chen (2018) dealing with replenishment policies for substitutable products.

3. The model; characterization of equilibrium behavior

For a vector \( a \) and an index set \( S \), \( a_{S} \) denotes the subvector with entries specified by \( S \). Similarly, for a matrix \( M \) and index sets \( S, T \in \mathcal{N} \), \( M_{ST} \) denotes the submatrix of \( M \) with rows specified by the set \( S \) and columns by the set \( T \). The transpose of a matrix \( M \) (vector \( a \)) is denoted by \( M^{\top} \) (\( a^{\top} \)). For notational simplicity, 0 may denote a scalar, a vector or a matrix of any dimensions with all entries being zeros, and \( I \) is an identity matrix of appropriate dimensions. The matrix inequality \( X = (x_{ij}) \geq 0 \) means that \( x_{ij} \geq 0 \) for all \( i, j \). For any polyhedral subset \( \Pi \subseteq \mathbb{R}^{N} \), \( int(\Pi) \) denotes its interior and \( \partial \Pi \) its boundary. \( \mathbb{R}_{+} = \{ x \in \mathbb{R} \mid x \geq 0 \} \) and \( \mathbb{R}_{++} = \{ x \in \mathbb{R} \mid x > 0 \} \).

We employ some properties of square matrices of special structure.

Definition 1 (Z-matrix). A square matrix whose off-diagonal entries are non-positive is called a Z-matrix.
Definition 2 (P-matrix). A square matrix whose principal minors are all positive is called a P-matrix.

Definition 3 (ZP-matrix). A matrix which is a Z- and a P-matrix is called a ZP-matrix.

We consider a market with a set \( J = \{1, \ldots, J\} \) of suppliers who compete by selling any number of grossly substitutable products, via the same pool \( I = \{1, \ldots, I\} \) of competing retailers. In the ex-ante state of the market, all \( J \) suppliers and all \( I \) retailers are independent firms, each attempting to maximize her own profit. Extensions to supply networks with any number of echelons, larger than two, are possible, following the methodology presented in Section 4 of Federgruen and Hu (2016).

Any supplier may sell any number of products to each of the retailers. To distinguish among the full collection \( \mathcal{N} \) of products that may be offered in the market, we denote a product by a triple of indices \((i, j, k)\): product \((i, j, k)\) denotes the \(k\)-th product sold by retailer \(i\) and obtained from supplier \(j\). Let \( \mathcal{K}(i, j) \) denote the set of products potentially sold to retailer \(i\) by supplier \(j\) and \( \mathcal{N}(i) \) the set of all products potentially sold by retailer \(i\).

For all \(i \in I, j \in J\) and \(k \in \mathcal{K}(i, j)\), let

\[
c_{ijk} = \text{the constant marginal supply cost of supplier } j, \text{ for product } k, \text{ potentially sold to retailer } i, \\
p_{ijk} = \text{the retail price charged by retailer } i \text{ for product } (i, j, k), \\
w_{ijk} = \text{the wholesale price charged by supplier } j \text{ for product } (i, j, k), \\
d_{ijk} = \text{the consumer demand for product } k, \text{ provided by supplier } j \text{ to retailer } i.
\]

Let \(c, p, w\) and \(d\) be the corresponding vectors, with \(c\) exogenously given. Depending upon the prices \(w\) and \(p\) selected by the suppliers and retailers, respectively, only a certain subset of all potential products \(\mathcal{N}\) is sold in the market. Indeed, part of the retailers’ choices is the determination of their product assortments.

The consumer demand for each product depends, in general, on the retail prices charged for all products. As in most supply chain competition models, the dependence is, in principle, described by general affine functions, as follows:

\[
q(p) = a - Rp,
\]

where \(a \geq 0\) and \(R\) is an \(N \times N\)-matrix.

The affine structure in (1), however, can only apply on the price polyhedron \(P = \{p \geq 0 \mid q(p) = a - Rp \geq 0\}\); for price vectors \(p \notin P\), the raw demand functions \(q(\cdot)\) predict negative demand.
volumes, for some of the products. In designing an extension of the definition of the demand functions beyond $P$, Shubik and Levitan (1980) suggested that such an extension must satisfy the following regularity condition, mentioned in §2:

**Definition 4 (Regularity).** A demand function is called regular if it satisfies the following property: if for a given price vector $p \in \mathbb{R}_+^N$, a product $l \in \mathcal{N}$ is priced out of the market, i.e., has zero demand, any increase in $p_l$ leaves all demand volumes unchanged.

While this regularity condition appears very intuitive and innocuous, Soon et al. (2009) showed that it specifies the extended demand function uniquely. More specifically, for an arbitrary price vector $p$, the demand volumes are determined by applying the affine function $q(\cdot)$ to the projection $\Omega(p)$ of $p$ onto the price polyhedron $P$, i.e.,

$$d(p) = a - R\Omega(p) = q(\Omega(p)),$$

where the projection $\Omega(p)$ is defined as the vector $p' = p - t$ with $t$ the unique solution to the Linear Complementarity Problem (LCP):

$$q(p - t) = a - R(p - t) \geq 0, \quad t \geq 0 \quad \text{and} \quad t^T[a - R(p - t)] = 0. \tag{2}$$

By Theorem 3.3.7 in Cottle et al. (1992), the LCP has a unique solution, if and only if the matrix $R$ is a $P$-matrix. We therefore require the following properties for the matrix $R$.

**Assumption (P):** the matrix $R$ is positive definite (hence a $P$-matrix).

**Assumption (Z) (Competitive Market Condition):** the matrix $R$ is a $Z$-matrix, i.e., has non-positive off-diagonal elements.

The (Z)-property implies that all product pairs are substitutable, i.e., any product’s demand volume does not decrease when the price of a different product is increased. (See, however, Federgruen and Hu 2014 for a generalized model, allowing for certain types of complementarities.)

The following two properties can be verified, see Lemma 1 in Federgruen and Hu (2015):

$$\Omega(p) \in P; \quad \text{if } p \in P, \text{ then } \Omega(p) = p. \tag{3}$$

Finally, to facilitate the exposition, we will, at first, require a third property:

**Assumption (S):** The matrix $R$ is symmetric.

This assumption is somewhat restrictive: empirical studies show that price sensitivity coefficients or cross-elasticities are often asymmetric, see, e.g., Manchanda et al. (1999), Vilcassim et al. (1999), Dubé and Manchanda (2005) and Li et al. (2015). At the end of this section, we therefore explain that a weaker partial symmetry property suffices to maintain the characterization of the equilibrium
behavior. However, when the matrix $R$ is symmetric, the demand functions $d(p)$ arise from a representative consumer maximizing the quadratic utility function $U(d; p) = (R^{-1}a - p)^\top d - \frac{1}{2}d^\top R^{-1}d$, i.e., $d(p) = \arg \max_{d \geq 0} U(d; p)$. This formulation allows us to compute a consumer welfare measure, for any given vector of retail prices $p$, and a social welfare measure as the sum of consumer welfare and total firm profits. See Federgruen and Hu (2015) for a discussion of alternative foundations for the demand functions $d(p)$, and why straightforward extensions of the affine structure such as $d(p) = [a - Rp]^+$ are inappropriate.

We start with a characterization of the equilibrium behavior of the retailer competition game that arises under an arbitrary, exogenously given wholesale price $w$. Note that each retailer $i \in \mathcal{I}$ attempts to maximize her profit function

$$
\pi_i(p; w) = \sum_{l \in \mathcal{N}(i)} (p_l - w_l)d_l(p).
$$

(4)

We then proceed with a characterization of the equilibrium in the full two-stage competition model, where the wholesale prices $w$ are determined in a first-stage competition model among the suppliers, who anticipate the retailers’ equilibrium response to any chosen wholesale price vector $w$. Theorems 2 and 3 in Federgruen and Hu (2015) establish that, under an arbitrary fixed wholesale price vector $w$, the retailer price competition game always has a pure Nash equilibrium.

Recall that when $p \in P$, the demand functions are affine since $d(p) = q(p) = a - Rp$, so that each firm’s profit function $\pi_i(p; w)$ is quadratic, and by Assumption (P), jointly concave in the firm’s price vector. Federgruen and Hu (2018) show that the unique solution to the First Order Conditions (FOC):

$$
\frac{\partial \pi_i(p; w)}{\partial p_{il}} = 0, \quad l \in \mathcal{N}(i), \ i = 1, \ldots, I,
$$

is given by the closed-form expression:

$$
p^*(w) = [R + T^\top(R)]^{-1}a + [R + T^\top(R)]^{-1}T^\top(R)w = w + [R + T^\top(R)]^{-1}q(w).
$$

(5)

Here,

$$
T^\top(R) = \begin{pmatrix}
R^\top_{N(1)N(1)} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & R^\top_{N(I)N(I)}
\end{pmatrix}.
$$

Indeed, $p^*(w)$ is a Nash equilibrium, as long as $p^*(w) \in P \iff w \in W$, with the “wholesale price polyhedron” $W$ defined as:

$$
W = \{w \geq 0 \mid b - Sw \geq 0\},
$$

where $b$ and $S$ are appropriately defined.
where

\[ b = \Psi^r(R)a, \quad S = \Psi^r(R)R, \]  
\[ \Psi^r(R) = T^r(R)[R + T^r(R)]^{-1}. \]  

When \( w \in \text{int}(W) \), the interior of \( W \), \( p^*(w) \in \text{int}(P) \), the interior of \( P \), and it is, in fact, the unique Nash equilibrium. For a general wholesale price vector \( w \), not necessarily in \( W \), Proposition 1 in Federgruen and Hu (2018) show that

\[ (p^*|w) \equiv p^*(w') = w' + [R + T^r(R)]^{-1}q(w') \quad (8) \]

is a Nash equilibrium, with \( w' = \Theta(w) \) and \( \Theta(\cdot) \) the projection operator with respect to the polyhedron \( W \). As shown in Federgruen and Hu (2018), under Assumptions (P), (Z) and (S), we have that \( b \geq 0 \) and \( S \) is (symmetric) positive definite, so that \( w' = \Theta(w) = w - t \) with, analogous to (2), \( t \) the unique solution to the LCP

\[ b - S(w - t) \geq 0, \quad t \geq 0 \quad \text{and} \quad t^\top[b - S(w - t)] = 0. \]

Analogous to (3), we have

\[ \Theta(w) \in W; \quad \text{if} \ w \in W, \text{then} \ w' = \Theta(w) = w. \]  

(9)

In particular, if \( w \in W \), \( p^*(w') = p^*(w) \), so that (8) reduces to (5).

While \((p^*|w)\) is the unique Nash equilibrium, where \( w \in \text{int}(W) \), alternative equilibria may exist, when \( w \notin \text{int}(W) \). A full characterization of the set of equilibria was given by Theorem 1 of Federgruen and Hu (2018), as follows:

**Theorem 1.** (a) There exists at most one equilibrium in \( \text{int}(P) \).

(b) When \( p^0 \notin P \) is an equilibrium, so is \( \Omega(p^0) \in P \). Moreover, \( \Omega(p^0) \) and \( p^0 \) are equivalent equilibria, in the sense that they generate identical product assortments, sales volumes and profit values.

(c) If \( w \in \text{int}(W) \), \( (p^*|w) = p^*(w) \in \text{int}(P) \) is the unique equilibrium, and \( N^+(p^*|w) = N \), where \( N^+(p) \equiv \{l : d(p)_l > 0\} \) denotes the assortment associated with \( p \).

(d) If \( w \in \mathbb{R}^N_{++} \setminus \text{int}(W) \), the set of all equilibria is outside \( \text{int}(P) \) and \( (p^*|w) = p^*(w') \in \partial P \) is one such equilibrium.
However, even when \((p^*|w)\) fails to be the unique equilibrium, Federgruen and Hu (2018, Theorems 2 and 3) show that this equilibrium stands out as the unique equilibrium that is globally stable, with respect to the so-called “robust best-response operator.” This means that, for any possible vector of retail prices \(p^o\), the market is guaranteed to converge to \((p^*|w)\), when each firm, iteratively, adjusts its prices as “robust best responses” to the most recent prices selected by its competitors. To this end, for each firm \(i = 1, \ldots, I\), define a best-response mapping \(RB_i(p_{-N(i)})\) as:

\[
\arg \max_{p_{N(i)} \geq 0} \left\{ (p_{N(i)} - w_{N(i)})^T (a_{N(i)} - R_{N(i), -N(i)} p_{-N(i)} - R_{N(i), N(i)} p_{N(i)}) \right. \\
\left. | a_{N(i)} - R_{N(i), -N(i)} p_{-N(i)} - R_{N(i), N(i)} p_{N(i)} \geq 0 \right\}.
\]

In other words, the \(RB_i(\cdot)\) operator selects for any vector of prices \(p_{-N(i)}\) of the products offered by firm \(i\)’s competitors, the price vector which maximizes firm \(i\)’s profits among all vectors \(p_{N(i)}\) such that

\[
p_{N(i)} \in P_i(p_{-N(i)}) \equiv \{ p_{N(i)} \geq 0 | a_{N(i)} - R_{N(i), -N(i)} p_{-N(i)} - R_{N(i), N(i)} p_{N(i)} \geq 0 \}.
\]

The robust best-response mapping \(RB_i(\cdot)\) restricts firm \(i\)’s choices to vectors \(p_{N(i)}\) such that \(p_{N(i)} \in P_i(p_{-N(i)})\). Firm \(i\), of course, has the option to select a price vector \(p_{N(i)}\) that falls outside of \(P_i(p_{-N(i)})\), and such a selection may result in additional profit enhancements. However, to assess the resulting sales volumes requires the solution of the LCP, which requires knowledge of the structure of all affine demand functions \(q_{-N(i)}(\cdot)\) for all of the competitors’ products. Such information may not be available to firm \(i\). In contrast, application of the robust best-response mapping \(RB_i(\cdot)\) merely requires local and robust information of the foundational affine functions \(q_{N(i)}(\cdot)\) pertaining to firm \(i\)’s own products, as well as its own cost vector \(w_{N(i)}\).

The fact that \((p^*|w)\) is globally stable and that it is always the unique equilibrium to possess this property, sets it apart as the likely steady state for the market. When analyzing the implication of given supplier prices, it is therefore natural to assume that the retailers respond to any such vector \(w\) by adopting the prices \((p^*|w)\), resulting in sales volumes: \(D(w) = Q(w')\), where

\[
Q(\xi) = b - S\xi.
\]

Since, as mentioned above, \(b \geq 0\) and \(S\) is a positive-definite symmetric \(Z\)-matrix, the first-stage competition game among the suppliers is of the exact same structure as the second-stage competition game among the retailers. (Note that the suppliers’ profit functions are structurally identical to those of the retailers, as well, see (4).)
To complete the full equilibrium characterization, reorder the products so that their supplier index in the triple \((i,j,k)\) comes first. Let \(\mathcal{M}(1), \ldots, \mathcal{M}(J)\) denote the set of products supplied by supplier 1, \ldots, \(J\), respectively. Define

\[
T^*(S) = \begin{pmatrix}
S_{\mathcal{M}(1)\mathcal{M}(1)}^T & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & S_{\mathcal{M}(J)\mathcal{M}(J)}^T
\end{pmatrix},
\]

\[
\Psi^*(S) = T^*(S)[S + T^*(S)]^{-1},
\]

and the effective cost rate polyhedron \(C = \{c \geq 0 \mid \Psi^*(S)Q(c) \geq 0\}\). Finally, let \(\Gamma(\cdot)\) denote the projection operator onto the polyhedron \(C\). Analogous to the definitions of \(p^*(w)\) and \((p^*|w)\), define:

\[
w^*(c) = [S + T^*(S)]^{-1}b + [S + T^*(S)]^{-1}T^*(S)c = c + [S + T^*(S)]^{-1}Q(c), \tag{12}
\]

\[
(w^*|c) \equiv w^*(\Gamma(c)). \tag{13}
\]

The following proposition follows directly, by applying Theorem 1 to the supplier competition game.

**Proposition 1** (Equilibrium behavior in the supplier competition game). (a) The supplier competition game has at most one equilibrium in \(\text{int}(W)\).

(b) When \(w^o \notin W\) is an equilibrium of the supplier competition game, so is \(\Theta(w^o) \in W\). Moreover, \(w^o\) and \(\Theta(w^o)\) are equivalent in the sense of generating identical product assortments, sales volumes and profit values for the suppliers.

(c) If \(c \in \text{int}(C)\), \((w^*|c) = w^*(c) \in \text{int}(W)\) is the unique equilibrium of the supplier competition game, and the associated assortment \(N^+(w^*|c) = N\).

(d) If \(c \in \mathbb{R}_+^N \setminus \text{int}(C)\), all equilibria are outside \(\text{int}(W)\) and \((w^*|c) = w^*(\Gamma(c)) \in \partial W\) is one such equilibrium.

Similar to our observations on the second-stage retailer competition game, the equilibrium \((w^*|c)\) is either categorically unique, see part (c), or the equilibrium \((w^*|c)\) can be shown to stand out as the unique equilibrium that is *globally stable* under the suppliers’ robust best-response operator. This means that, irrespective of the suppliers’ initial price choices, the market converges to \((w^*|c)\) when suppliers iteratively adjust their prices as robust best-responses to the competitors’ prices. Therefore \((w^*|c)\) and the associated sales volumes are the natural predicted equilibrium outcome of the first-stage suppliers’ competition game.
We also obtain the following relationships between the effective price polyhedron \( P \), \( W \) and \( C \):

\[
P \subseteq W \subseteq C.
\]

Finally we return to the symmetry assumption (S). As mentioned, this assumption may be weakened considerably. For example, Theorem 5 in Federgruen and Hu (2016) together with Theorem 1 in Federgruen and Hu (2018) can be applied to show that Assumption (S) may be weakened to:

Assumption (S'): The matrices \( T^r(R) \) and \( T^s(S) \) are symmetric.

Symmetry of the matrix \( T^r(R) \), for example, means that only the price sensitivity coefficients of pairs of products sold by the same retailer need to be symmetric. See Federgruen and Hu (2016) for a discussion of even weaker partial symmetry conditions.

4. The price and variety implications of a vertical merger: All retail prices determined at the second stage

In this section, we explain the implications of a vertical merger between one of the suppliers and a group of retailers which this supplier sells to, possibly along with other suppliers. We assume that, after the merger, any outside supplier continues to offer the merged company all products that used to be offered to any one of the merging retailers. In this section, we assume that the retail prices of all products sold by the merged firm continue to be determined as part of the second stage retailer competition game, i.e., after all wholesale prices are selected and disclosed in the first stage supplier competition game. We believe this to be the main prevalent setting. However, in Section 6, we analyze mergers under the alternative assumption that the retail prices of the products, potentially sold by the merging supplier to her merging retailers, are determined as part of the first stage price competition game among the suppliers.

The implications in terms of equilibrium prices, sales volumes, product assortment and profit values, can easily be ascertained by solving the two-stage competition model for the following modified network, price sensitivity matrix \( \hat{R} \) and intercept vector \( \hat{a} \).

Without loss of generality, assume supplier \( j = J \) merges with the last \(|\mathcal{I}^o|\) retailers. We obtain a new network with the same supplier set \( \mathcal{J} \), but a new retailer set in which all of \((\mathcal{I} \setminus \mathcal{I}^o)\) is represented by a single retailer with index \( I - |\mathcal{I}^o| + 1 \). The product set \( \mathcal{N} \) remains unaltered except for the following re-indexing of the products:

\[
\mathcal{L}(i,j,k) = \begin{cases} (i,j,k) & \text{if } i \leq I - |\mathcal{I}^o|, \\ (I - |\mathcal{I}^o| + 1,j,k) & \text{if } i = I - |\mathcal{I}^o| + 1, \\ (I - |\mathcal{I}^o| + 1,j,|K(I - |\mathcal{I}^o| + 1,j)| + \cdots + |K(i - 1,j)| + k) & \text{if } i = I - |\mathcal{I}^o| + 2, \ldots, I. \end{cases}
\]
Note that the renumbering operator is a one-to-one mapping and let $L^{-1}(\cdot)$ denote the inverse mapping. Define a new price sensitivity matrix $\hat{R}$ and a new intercept vector $\hat{a}$ as follows: for all $l,l'$,

$$\hat{R}_{ll'} = R_{L^{-1}(l)L^{-1}(l')} \quad \text{and} \quad \hat{a}_l = a_{L^{-1}(l)}.$$ 

The post-merger retailer competition game, defined by the pair $(\hat{a}, \hat{R})$, satisfies the same fundamental properties as the pre-merger competition game. In particular, $\hat{a} \geq 0$ and $\hat{R}$ is a positive definite, symmetric, $Z$-matrix. (All these properties are invariant to any renumbering of the products.)

Let $D$ denote the set of products offered by supplier $J$ to the retailers $\{I - |I^o| + 1, \ldots, I\}$ and $\overline{D} = N \setminus D$ is the complementary set. Because of the merger, we have $w_D = c_D$. For any given wholesale price vector $w = (w_D, c_D)$, define:

$$\hat{p}^*(w) = [\hat{R} + \hat{T}^r(\hat{R})]^{-1}\hat{a} + [\hat{R} + \hat{T}^r(\hat{R})]^{-1}\hat{T}^r(\hat{R})w.$$  

(14)

Note that the mapping $\hat{T}^r(\cdot)$ itself differs from the mapping $T^r(\cdot)$, in general, since the merger generates a new partition of the products among the now $(I - |I^o| + 1)$ remaining retailers. Define, as in (1), (6)-(7),

$$\hat{q}(p) = \hat{a} - \hat{R}p, \quad \hat{P} = \{p \geq 0 \mid \hat{a} - \hat{R}p \geq 0\},$$

$$\hat{\Psi}(\hat{R}) = [\hat{R} + \hat{T}^r(\hat{R})]^{-1},$$

$$\hat{b} = \hat{\Psi}(\hat{R})\hat{a}, \quad \hat{S} = \hat{\Psi}(\hat{R})\hat{R}.$$ 

Since $\hat{R}$ is a symmetric, positive definite $Z$-matrix, the projection operator $\hat{\Omega}$ onto $\hat{p}$ is well defined, while $\hat{T}^r(\hat{R})$ is symmetric as well. It follows from Theorem 1(a) in Federgruen and Hu (2016) that the projection $\hat{\Theta}$ onto the polyhedron

$$\hat{W} = \{w \geq 0 \mid \hat{b} - \hat{S}w = \hat{\Psi}(\hat{R})\hat{q}(w) \geq 0\}$$

is well-defined in complete analogy to the projection operator $\Theta(\cdot)$ in the pre-merger market.

We are, thus, able to apply Theorem 1 to the post-merger retailer competition game specified by the pair $(\hat{a}, \hat{R})$ and product sets $\{\hat{N}(1), \ldots, \hat{N}(I - |I^o| + 1)\}$ under a general wholesale price vector $w = (w_D, c_D)$.

**Proposition 2.** Fix an arbitrary wholesale price vector $w = (w_D, c_D) \in \mathbb{R}_+^N$. Theorem 1 applies to the post-merger retailer game, with $P, W, (p^*|w), \Omega$ and $\Theta$ replaced by $\hat{P}$, $\hat{W}$, $(\hat{p}^*|w)$, $\hat{\Omega}$ and $\hat{\Theta}$. In particular, $(\hat{p}^*|w)$ is, in the post-merger game, the unique globally stable equilibrium (under the robust best-response operator).
The post-merger first stage supplier competition game has a somewhat different structure from its pre-merger counterpart. The reason is that in the post-merger game, the prices for the products in the set \( D \) are pre-specified and fixed as \( w_D = c_D \). This implies that the supplier competition game only involves the products in the set \( \bar{D} \). The raw demand functions are given by the (affine) functions:

\[
\hat{Q}_D(w_D, c_D) = (\hat{b}_D - \hat{S}_{DD}c_D) - \hat{S}_{DD}w_D,
\]

\[
\hat{Q}_\bar{D}(w_\bar{D}, c_D) = (\hat{b}_D - \hat{S}_{\bar{D}D}c_D) - \hat{S}_{\bar{D}D}w_\bar{D}.
\]

(The actual demand functions \( \hat{D}(\cdot) \) are the unique regular extension of \( \hat{Q}(\cdot) \), i.e., for an arbitrary vector \( w = (w_D, c_D) \), we have \( \hat{D}(w_D, c_D) = \hat{Q}(\hat{\Theta}(w_D, c_D)) \).) To characterize the equilibrium behavior in this game, we proceed as in the competition game without any price restrictions, see Section 3. Assuming the demand functions for the products in \( D \) are given by the raw demand functions \( \hat{Q}_D(\cdot) \), we specify the system of First Order Conditions (FOC) for a Nash equilibrium, which has the unique solution:

\[
\hat{w}_D^*(c) = c_D + [\hat{S}_{DD} + \hat{T}^*(\hat{S}_{DD})]^{-1}[(\hat{b}_D - \hat{S}_{DD}c_D) - \hat{S}_{DD}c_D].
\]  

(15)

Since \( \hat{w}_D = c_D \), the associated full price vector is:

\[
\hat{w}^*(c) = c + \left( \begin{array}{cc}
[\hat{S}_{DD} + \hat{T}^*(\hat{S}_{DD})]^{-1} & 0 \\
0 & 0
\end{array} \right) (\hat{b} - \hat{S}c).
\]  

(16)

Following the proof of Theorem 1 in Federgruen and Hu (2018), this wholesale price vector is indeed a Nash equilibrium, if and only if, \( \hat{w}^*(c) \geq 0 \) and \( \hat{Q}(\hat{w}^*(c)) = \hat{b} - \hat{S}\hat{w}^*(c) \geq 0 \). The former is verified, as in the proof of Theorem 1 ibid, since the matrix \( \hat{S} \) is a symmetric positive definite Z-matrix, while \( \hat{b} \geq 0 \) and \( c \geq 0 \). The latter system of inequalities can be reduced to

\[
\Xi^*(\hat{S})(\hat{b} - \hat{S}c) \geq 0,
\]

where

\[
\Xi^*(\hat{S}) = \begin{pmatrix}
I - \hat{S}_{DD}[\hat{S}_{DD} + \hat{T}^*(\hat{S}_{DD})]^{-1} & 0 \\
-\hat{S}_{DD}[\hat{S}_{DD} + \hat{T}^*(\hat{S}_{DD})]^{-1} & I
\end{pmatrix}.
\]

The inequalities for the products in \( \bar{D} \) can be verified, in complete analogy to the characterization of the pre-merger wholesale price polyhedron \( W \) in Theorem 1. For the products in \( D \),

\[
0 \leq \hat{Q}_D(\hat{w}^*(c)) = \hat{b}_D - \hat{S}_{DD}c_D - \hat{S}_{DD}\hat{w}_D^*(c)\]

\[
= \hat{b}_D - \hat{S}_{DD}c_D - \hat{S}_{DD}[c_D + [\hat{S}_{DD} + \hat{T}^*(\hat{S}_{DD})]^{-1}(\hat{b}_D - \hat{S}_{DD}c_D - \hat{S}_{DD}c_D)]
\]
\[\begin{align*}
&= (\hat{b}_D - \hat{S}_{DD}c_D - \hat{S}_{DD}c_D) - \hat{S}_{D\Gamma}[\hat{S}_{D\Gamma} + \hat{T}^*(\hat{S}_{D\Gamma})]^{-1}(\hat{b}_D - \hat{S}_{D\Gamma}c_D - \hat{S}_{D\Gamma}c_D) \\
&= \hat{T}^*(\hat{S}_{D\Gamma})[\hat{S}_{D\Gamma} + \hat{T}^*(\hat{S}_{D\Gamma})]^{-1}(\hat{b}_D - \hat{S}_{D\Gamma}c_D - \hat{S}_{D\Gamma}c_D) \\
&= (I - \hat{S}_{D\Gamma}[\hat{S}_{D\Gamma} + \hat{T}^*(\hat{S}_{D\Gamma})]^{-1})(\hat{b}_D - \hat{S}_{D\Gamma}c_D - \hat{S}_{D\Gamma}c_D),
\end{align*}\]

thus verifying the second part of the system of inequalities. Thus defining the post-merger effective cost rate polyhedron

\[\hat{C} \equiv \{c \geq 0 \mid \Xi^s(\hat{S})(\hat{b} - \hat{S}c) \geq 0\}, \quad (17)\]

we have shown that the unique solution \(\hat{w}^*_c(c)\) of the system of FOC, given by (15), is indeed a Nash equilibrium, as long as \(c \in \hat{C}\). As with the operator \(\Omega(\cdot)\), in view of Theorem 3.3.7 in Cottle et al. (1992), the following two properties suffice to verify that the associated projection operator \(\hat{\Gamma}(\cdot)\) is well defined, see (2).

**Lemma 1.** (a) \(Y \equiv \Xi^s(\hat{S})\hat{S}\) is a symmetric positive definite \(Z\)-matrix. (b) \(\Xi^s(\hat{S})\hat{b} \geq 0\).

Thus, analogous to Proposition 1, we obtain:

**Proposition 3.** (Characterization of equilibria in the post-merger supplier competition game)

Fix an arbitrary cost rate vector \(c \in \mathbb{R}^N_{++}\).

(a) \(\hat{C} \neq \emptyset\), since \(0 \leq \hat{c}^e = (\Xi^s(\hat{S})\hat{S})^{-1}\Xi^s(\hat{S})\hat{b} = \hat{S}^{-1}\hat{b} \in \hat{C}\).

(b) Proposition 1 applies to the post-merger supplier competition game, with \(W, \ C, \ \Theta(\cdot), \ \Gamma(\cdot)\) and \((w^*|c)\) replaced by \(\hat{W}, \ \hat{C}, \ \hat{\Theta}(\cdot), \ \hat{\Gamma}(\cdot)\) and \((\hat{w}^*|c)\), respectively, see (16). In particular, \((\hat{w}^*|c)\) is the unique globally stable equilibrium (under the robustly best-response operator) in the post-merger supplier competition game.

In summary, we have shown how the post-merger sequential equilibrium in the first stage supplier competition game and the second stage retailer competition game can be characterized: in the pre-merger setting, for any given value of cost rates \(c\), there exists a unique globally stable wholesale price equilibrium, as well as a unique globally stable retail price equilibrium, which can be computed in closed form, with the help of a few matrix multiplications and inversions.

In general, it is impossible to predict whether a vertical merger will result in price increases or decreases, or what impact the merger has on the product variety offered in the market. The reason is that if a supplier merges with several retailers, the merger involves both a vertical and a horizontal component. The vertical aspect of the merger eliminates double marginalization and is therefore likely to reduce equilibrium prices. However, at the same time, the acquisition also merges multiple retailers who are competitors in the pre-merger world. It is well known that horizontal mergers tend to drive prices up, see, e.g., Deneckere and Davidson (1985) and Federgruen and
Pierson (2011). To isolate the impact of a vertical merger, we now focus on the special case where a supplier merges with only one retailer with whom she has an exclusive relationship, i.e., the supplier, without loss of generality (w.l.o.g.) supplier $J$, sells all of her products via this retailer, again w.l.o.g. retailer $I$, and supplier $J$ is the sole supplier of this retailer $I$.

4.1. Merger of a supplier with an exclusive retailer

In the special case where $|I_o| = 1$, the matrix $\hat{R} = R$ and the mapping $\hat{T}^r(\cdot) = T^r(\cdot)$. All quantities and mappings associated with the retailer competition game, therefore remain unaffected by the merger, eliminating the need to attach a $\hat{\cdot}$ sign for them. In contrast, the first stage supplier competition game is affected, as in the general case developed above. As one of our main results, we show that the vertical merger either leaves the equilibrium product assortment unchanged or it reduces the product variety offered in the market. We first show:

**Theorem 2.** (Merger of a supplier with an exclusive retailer: comparison of the effective cost rate polyhedron) $\hat{C} \subseteq C$.

We are now ready to prove one of our main results: the vertical merger (weakly) reduces the product variety in the market. Let $N^* (\hat{N}^*)$ denote the set of products sold in the pre-merger (post-merger) world. We show:

**Theorem 3.** (Merger of a supplier with an exclusive retailer: comparison of the equilibrium product assortment) (a) $\hat{N}^* \subseteq N^*$. (b) For any $c \in \mathbb{R}^N_+$, $\hat{\Gamma}(c) \leq \Gamma(c)$.

Any vertical merger thus reduces consumer welfare in the sense of (weakly) reducing product variety in the market. At the same time, we will show that the merger results in reduced wholesale and retail prices. We first need the following lemma.

**Lemma 2.** Fix $c \in \hat{C}$. $\hat{w}^*(c) \leq w^*(c)$.

For an arbitrary cost rate vector $c \in \mathbb{R}^N_{+}$, the pre-merger unique globally stable wholesale price equilibrium is given by $w^*(\Gamma(c))$ and the unique globally stable retail price equilibrium by $p^*(w^*(\Gamma(c)))$. The corresponding post-merger equilibria are $\hat{w}^*(\hat{\Gamma}(c))$ and $p^*(\hat{w}^*(\hat{\Gamma}(c)))$ respectively.

**Theorem 4.** (Merger of a supplier with an exclusive retailer: comparison of equilibrium retail and wholesale prices) Fix the cost rate vector $c > 0$.

\[ \hat{w}^*(\hat{\Gamma}(c)) \leq w^*(\Gamma(c)) \tag{18} \]
and
\[ p^*(\hat{w}^*(\hat{\Gamma}(c))) \leq p^*(w^*(\Gamma(c))), \tag{19} \]
i.e., the wholesale and retail price equilibria are (weakly) reduced due to the merger.
5. **Direct sales channel**

In this section, we return to the basic two-echelon model of Section 3, however, considering settings where some of the products are sold directly by some of the suppliers, as opposed to being distributed via the retailers. With a slight abuse of notation, in this section, let $\mathcal{D}$ denote the set of direct sales products in the consumer market. Products in the set $\overline{\mathcal{D}} \equiv \mathcal{N} \setminus \mathcal{D}$ are sold through retailers. As a concrete example, consider the network structure addressed in Hotkar and Gilbert (2018), see Figure 1.

![Network structure of Hotkar and Gilbert (2018) with supplier encroachment](image)

*Note.* $|\mathcal{I}| = 1, |\mathcal{J}| = 2, \mathcal{N} = \{(0, 1, 1), (1, 1, 1), (1, 2, 1)\}, \mathcal{D} = \{(0, 1, 1)\}$.

In this section, we characterize the equilibrium behavior in this generalized supply chain competition model, under the important assumption that the prices of the products in $\mathcal{D}$ are determined by the relevant suppliers as part of the first stage competition game. (For direct sales products, wholesale and retail prices coincide, i.e., $w_\mathcal{D} = p_\mathcal{D}$.) We assume that the retail prices of all remaining products ($\overline{\mathcal{D}}$) are determined by the retailers, independent from, but in response to the prices $w = (w_\mathcal{D}, w_{\overline{\mathcal{D}}})$. However, in some settings, the exact same product is sold via a supplier’s direct sales channel as well as one or more of the retailers and the retail prices are required to be the same. This setting can be handled by fixing the retail prices of these related products to the direct-sales channel prices in $w$, similar to the upfront restriction $w_\mathcal{D} = p_\mathcal{D}$. An example is the iPhone 8 sold at an identical price at the Apple Store (direct sales) and various independent retailers.

Note that the raw demand functions $q(\cdot)$, the effective retail price polyhedron $P$, the projection operator $\Omega(\cdot)$ and the full demand functions $d(\cdot)$ are all defined exactly as in the base model (where $\mathcal{D} = \emptyset$). However, since the retail prices of the products in $\mathcal{D}$ are determined in the first stage, the dynamics of the second stage retailer competition game differ from those in the basic, strictly hierarchical sequential oligopoly. In fact, they are analogous to those of the first-stage supplier competition game analyzed in Section 4, where $w_\mathcal{D}$, the wholesale prices of the products in $\mathcal{D}$ are predetermined and fixed ($w_\mathcal{D} = c_\mathcal{D}$).

In complete analogy to the analysis of the first stage competition model in Section 4, we therefore obtain that

$$p^\star(w) = w + \left( [R_{\mathcal{D}\mathcal{D}}^{-1} + T^r(R_{\mathcal{D}\mathcal{D}})]^{-1} 0 0 \right) (a - Rw)$$  \hspace{1cm} (20)
is the unique Nash equilibrium, if \( w \in \text{int}(\mathcal{W}) \), where

\[
\mathcal{W} = \{ w \geq 0 \mid \mathcal{Z}(R)(a - Rw) \geq 0 \}, \\
\mathcal{Z}(R) = \begin{pmatrix} 1 - R_{\mathcal{D}\mathcal{D}}[R_{\mathcal{D}\mathcal{D}} + T^r(R_{\mathcal{D}\mathcal{D}})]^{-1} & 0 \\ -R_{\mathcal{D}\mathcal{D}}[R_{\mathcal{D}\mathcal{D}} + T^r(R_{\mathcal{D}\mathcal{D}})]^{-1} & 1 \end{pmatrix}.
\]

To characterize the equilibrium behavior, when \( w \in \mathbb{R}^N_+ \setminus \mathcal{W} \), define the projection operator \( \check{\Theta}(\cdot) \) with respect to polyhedron \( \mathcal{W} \), analogous to (2). Invoking Theorem 3.3.7 in Cottle et al. (1992), once again, the projection operator is well defined, if the following two properties are satisfied. The proof of the following lemma is analogous to that of Lemma 1.

**Lemma 3.** (a) \( \check{S} \equiv \mathcal{Z}(R)R \) is a symmetric positive definite \( Z \)-matrix. (b) \( \check{b} \equiv \mathcal{Z}(R)a \geq 0 \).

Thus, following the proof of Theorem 1 in Federgruen and Hu (2018), we obtain:

**Proposition 4.** (Characterization of equilibrium behavior of the retailer competition game under direct sales).

(a) \( \mathcal{W} \neq \emptyset \), since \( 0 \leq w^* = (\mathcal{Z}(R)R)^{-1}\mathcal{Z}(R)a = R^{-1}a \in \mathcal{W} \).

(b) Theorem 1 applies with \( \mathcal{W}, \Theta(\cdot) \) and \( (p^*|w) \) replaced by \( \check{W}, \check{\Theta}(\cdot) \) and \( (\check{p}^*|w) \).

We now turn our attention to the first stage competition game among the suppliers. On the effective wholesale price polyhedron \( \check{W} \), the induced equilibrium demand functions are affine and given by \( \check{Q}(w) = \mathcal{Z}(R)(a - Rw) \). Outside of \( \check{W} \), we have, as in the base model of Section 3, that \( \check{D}(w) = \check{Q}(w') \) with \( w' = \check{\Theta}(w) \). In view of Lemma 3, the suppliers’ competition game has the same structural properties as that of the base model with \( \mathcal{D} = \emptyset \). Analogous to (11), define

\[
\check{S} \equiv \mathcal{Z}(R)R, \\
\check{P}^*(\check{S}) \equiv T^s(\check{S})[\check{S} + T^s(\check{S})]^{-1}, \\
\check{C} \equiv \{ c \geq 0 \mid 0 \leq \check{P}^*(\check{S})\check{Q}(c) = T^s(\check{S})[\check{S} + T^s(\check{S})]^{-1}\mathcal{Z}(R)(a - Rc) \}.
\]

Finally, let \( \check{\Gamma}(\cdot) \) denote the projection operator onto the polyhedron \( \check{C} \). In complete analogy to Proposition 1, we obtain:

**Proposition 5.** (Characterization of equilibrium behavior in the supplier competition game under direct sales). Fix an arbitrary cost rate vector \( c \in \mathbb{R}^N_+ \). Proposition 1 holds with \( \mathcal{W}, \Theta(\cdot) \) and \( \Gamma(\cdot) \) replaced by \( \check{W}, \check{\Theta}(\cdot), \check{\Gamma}(\cdot) \), respectively, and \( (w^*|c) \) by

\[
\check{w}^*(c) = [\check{S} + T^s(\check{S})]^{-1}\check{b} + [\check{S} + T^s(\check{S})]^{-1}T^s(\check{S})c = c + [\check{S} + T^s(\check{S})]^{-1}\check{Q}(c).
\]
6. **The price and variety implication of a vertical merger: first stage determination of retail prices of the products procured within the merged company**

In this section, we revisit the implications of a vertical merger between a supplier, w.l.o.g. supplier $J$, and a group of retailers to whom she sells possibly on a non-exclusive basis, i.e., along with one or more “outside” suppliers. Similarly, supplier $J$ may sell some of her products to retailers outside of the group with whom she merges. All products offered before the merger, continue to be offered thereafter.

Let $\mathcal{D}$ denote the set of products potentially sold by the merging supplier to her merging retailers; and $\overline{\mathcal{D}} = \mathcal{N} \setminus \mathcal{D}$ the complementary set of products, (potentially) offered in the market. Contrary to Section 4, we assume, here, that the retail/wholesale price decisions of the products in $\mathcal{D}$ are determined by the merging supplier, as part of the first stage competition game in which all of the suppliers’ product prices are selected.

Note that, under the above sequencing of price selections, the market dynamics in the supply chain are identical to those that arise when the products in the set $\mathcal{D}$ become “direct sales” products, and the retailer competition game becomes confined to the products in $\overline{\mathcal{D}}$, possibly involving fewer retailers, than in the pre-merger game. (As in Section 4, the complete group of merging retailers is replaced by a single retailer whose potential product set is the union of all product sets offered to the merging retailers, in the pre-merger world.) Section 5 characterizes in detail what the equilibrium behavior in the sequential oligopoly model looks like and how the equilibrium retail and wholesale prices, sales volumes and product assortment, can be computed efficiently, with the help of a few matrix additions, multiplications and inversions.

As explained in Section 4, when the merger involves multiple retailers, it is of a hybrid nature, both with a vertical and a horizontal component, since a group of retailers who are competing in the pre-merger model, now join forces. As explained, vertical mergers tend to reduce equilibrium prices, while horizontal mergers tend to have an opposite effect. To evaluate the impact of a vertical merger, we again consider a setting where supplier $J$ merges with a single retailer, w.l.o.g. retailer $I$, assuming there is an exclusive relationship between them: retailer $I$ receives all of his products from supplier $J$, and vice versa, supplier $J$ only sells through retailer $I$.

We start by comparing the retailers’ equilibrium behavior in the pre-merger and post-merger world, under an arbitrary given vector of wholesale prices $w \in \mathbb{R}^\mathcal{N}_+$. Note that in the post-merger world, $w_{\mathcal{D}}$, the wholesale prices for the products in $\mathcal{D}$, serve as the retail prices as well, as explained in the “direct sales” model of Section 5. We first show that the merger causes a (weak) shrinkage of
the effective wholesale price polyhedron along with a (weak) reduction of the equilibrium product assortment.

Let $\mathcal{N}^*$ and $\tilde{\mathcal{N}}^*$ denote the set of products sold in equilibrium, in the pre- and post-merger world, respectively. Recall from the results in Sections 3 and 5 that, while multiple price equilibria may exist, the globally stable equilibrium sales volumes and product assortment are uniquely specified.

**Theorem 5.** (Merger of a supplier with an exclusive retailer: comparison of retail price equilibrium under a given wholesale price vector $w$). Fix $w \in \mathbb{R}^N_{++}$. (a) $\tilde{W} \subseteq W$. (b) $\tilde{\mathcal{N}}^* \subseteq \mathcal{N}^*$. (c) $\tilde{\Theta}(w) \leq \Theta(w)$.

While, for a fixed vector of wholesale prices $w \in \mathbb{R}^N_{++}$, the (robustly and globally stable) equilibrium product variety is (weakly) reduced due to the merger, the (robustly and globally stable) retail price equilibrium is (weakly) reduced as well. Let $(p^*|w)$ and $(\tilde{p}^*|w)$ denote the (robustly and globally stable) retail price equilibrium under the wholesale price vector $w$, in the pre-merger and post-merger world, respectively.

**Theorem 6.** (Merger of a supplier with an exclusive retailer: comparison of retail price equilibria, under a given vector of wholesale prices $w$). Fix $w \in \mathbb{R}^N_{++}$.

$$(\tilde{p}^*|w) = \tilde{p}^*(\tilde{\Theta}(w)) \leq p^*(\Theta(w)) = (p^*|w).$$

Thus, assuming the suppliers’ wholesale price choices would not be affected by the merger, we have shown that the merger results in a (robustly and globally stable) retail price equilibrium which is lower than its pre-merger counterpart. At the same time, under a fixed vector of wholesale prices $w$, $\tilde{\mathcal{N}}^*$, the (robustly and globally stable) equilibrium product assortment after the merger, is more restricted than its pre-merger counterpart.

Of course, in the sequential two-echelon competition game, it is likely that the merger affects the suppliers’ wholesale price choices. In Section 4.1, we were able to show that $\hat{C} \subset C$ and in particular $\hat{C}^o \subset C^o$, see Theorem 2. By the definition of these sets, this implies that, if, in the post-merger world, the vector of cost rates $c$ results in all products being sold in equilibrium, the same is true in the pre-merger world. This allowed us to show that, for an arbitrary cost rate vector $c$, the (robustly and globally stable) price equilibrium is lower, after the merger, as compared to the pre-merger counterparts; similarly, the equilibrium product assortment after the merger is a subset of the products sold before the merger; see Theorems 3 and 4. Unfortunately, when the prices of the products in $\mathcal{D}$, i.e., the products sold by the merging firm, are selected as part of the
first stage competition game, the inclusion \( \hat{C} \subseteq C \), while in general still valid, can no longer be guaranteed, see Example 1 in Section 7.

The implications for the equilibrium product assortment are therefore dependent on the position of the cost rate vector \( c \) with respect to the polyhedra \( C \) and \( \hat{C} \):

(I) if \( c \in C \cap \hat{C} \): In this case, all products are sold before the merger, but some products are taken off the market, after the merger.

(II) if \( c \in C \cap \hat{C} \): In this case, the merger results in increased product variety; before the merger, some products are insufficiently competitive and fail to be sold, but, after the merger, all products are sold.

(III) if \( c \notin (\hat{C} \cup C) \): In this case, a restricted product assortment is sold both before and after the merger; the relationship between the product assortments is ambiguous.

We are, nevertheless, able to prove that both the wholesale and retail price equilibrium are reduced due to the merger, as long as \( c \in C \).

We first need the following lemma:

**Lemma 4.** (a) \( \hat{w}^*(c) \leq w^*(c) \), if \( c \in \hat{C} \). (b) \( \hat{w}^*(c) \leq w^*(c) \), if \( c \in C \).

Note that both \( w^*(\cdot) \) and \( \hat{w}^*(\cdot) \) represent affine mappings and do not, in general, coincide with \( (w^*|c) \) and \( (\hat{w}^*|c) \), respectively.

Recall in the pre-merger world,

\[
S = \Psi^*(R)R, \quad w^*(c) = c + [S + T^*(S)]^{-1}(\Psi^*(R)a - Sc) = c + [S + T^*(S)]^{-1}\Psi^*(R)(a - Rc),
\]

\[
C = \{c \geq 0 | T^*(S)[S + T^*(S)]^{-1}\Psi^*(R)(a - Rc) \geq 0\},
\]

and in the post-merger world,

\[
\hat{S} = \Xi^*(R)R = \hat{X}S, \quad \hat{X} = \Xi^*(R)[\Psi^*(R)]^{-1},
\]

\[
\hat{w}^*(c) = c + [\hat{S} + T^*(\hat{S})]^{-1}(\Xi^*(R)a - \hat{S}c) = c + [\hat{S} + T^*(\hat{S})]^{-1}\Xi^*(R)(a - Rc),
\]

\[
\hat{C} = \{c \geq 0 | T^*(\hat{S})[\hat{S} + T^*(\hat{S})]^{-1}\Xi^*(R)(a - Rc) \geq 0\}.
\]

**Theorem 7.** (Merger of a supplier with an exclusive retailer: comparison of equilibrium retail and wholesale prices). Fix \( c \in \mathbb{R}_+^N \cap C \).

\[
\hat{p}^*(\hat{w}^*|c) = \hat{p}^*(\hat{w}^*(\hat{\Gamma}(c))) \leq p^*(w^*|c) = p^*(w^*(\Gamma(c))) = p^*(w^*(c)).
\]
7. Numerical examples

In this section, we illustrate our results with the help of several examples:

Example 1. Consider a market with two suppliers and three products (i.e., $N = 3$). The raw demand functions $q(p) = a - Rp$ are specified as:

$$a = \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & -0.5 & -0.4 \\ -0.5 & 1 & -0.4 \\ -0.4 & -0.4 & 1 \end{pmatrix}.$$  

The $R$ matrix is easily verified to be a symmetric $Z$-matrix which is positive definite due to dominant diagonals, see Federgruen and Hu (2018). The network structures before and after vertical integration are displayed in Figure 2. This example describes a situation in which supplier 1 offers two products, product 1 and product 2 that are symmetrically differentiated: in the absence of price differentials, products 1 and 2 attract the same market size, presumably because of different product features. However the supply cost for product 1 is significantly lower than that of product 2. The competing supplier, supplier 2, distributes a third product, via a dedicated third retail organization. The product is the most expensive to manufacture, but is preferred by a large segment of the market, at least in the absence of price differentials.

![Figure 2](image)

We compare the equilibria in the pre-merger market with those in the post-merger market, when the retail price of product 3, the only product distributed by the merging firm, is determined in the second stage and first stage, respectively. Consistent with the notation employed throughout the paper, quantities associated with the former (latter) are differentiated with a $\hat{\cdot}$ (`.)

We first display the 8 vertices of the three effective cost rate polyhedra in Figure 3. Every vertex of $\hat{C}$ either coincides with one of $C$, or is pointwise smaller. Hence $\hat{C} \subset C$, confirming what has been proved in Theorem 2. In contrast, we observe that $\hat{C} \not\subset C$. The pre-merger effective cost rate polyhedron $C$ is given by

$$C = \left\{ c \geq 0 : \begin{cases} 1.7983 - 0.2064c_1 + 0.0936c_2 + 0.0414c_3 \geq 0 \\ 1.7983 + 0.0936c_1 - 0.2064c_2 + 0.0414c_3 \geq 0 \\ 4.0056 + 0.0414c_1 + 0.0414c_2 - 0.1958c_3 \geq 0 \end{cases} \right\},$$
while

\[ \tilde{C} = \begin{cases} 
  c \geq 0 & \left[ 1.4183 - 0.2203c_1 + 0.0797c_2 + 0.0771c_3 \geq 0 \\
  1.4183 + 0.0797c_1 - 0.2203c_2 + 0.0771c_3 \geq 0 \\
  6.4013 + 0.0771c_1 + 0.0771c_2 - 0.3316c_3 \geq 0 
\end{cases} \].

It is easy to verify that \( c = (1, 14, 22) \in \tilde{C}^o \), but fails to satisfy the second constraint in the set of constraints in \( C \), i.e., \( c \in \tilde{C}^o \setminus C \). This means that under this cost rate vector \( c = (1, 14, 22) \), if the merging firm’s price is selected in the \textit{first} stage, the merger \textit{expands} the product variety from the product set \( \{1, 3\} \) to all three products. On the other hand, if \( c \in C^o \setminus \hat{C} \) and the merging firm’s price is selected in the \textit{second} stage, the merger results in a \textit{reduction} of the product variety.

Now consider \( c = (1, 14, 22) \); see Table 1. In equilibrium, before vertical integration, product 1 is in the market because of its cost advantage and product 3 is in the market because its features are preferred by a large segment of the market. Product 2 does not survive against the simultaneous competition of products 1 and 3. Given its $22 cost rate and double marginalization, product 3 is in the pre-merger market, offered at the retail price of $22.99, with supplier 2 capturing a profit margin of 69 cents and retailer 3 one of 30 cents. However, supplier 2 and retailer 3’s very modest margins notwithstanding, their price points allow product 1’s retail price to be set at $15.36 while still capturing the vast majority of the market. While supplier 1 has the potential of creating a market share for product 2, by offering retailer 2 a price modestly above its cost rate of $14, the supplier maximizes her aggregate profit by foreclosing retailer 2, offering her product at the wholesale price of $17.88 instead.

Consider now the post-merger world, with the retail price of product 3 continuing to be determined in stage two, along with the other retail prices. The new equilibrium maintains the same
Table 1  Comparison of pre- and post-merger in (robustly and globally stable) equilibrium at $c = (1, 14, 22)$

<table>
<thead>
<tr>
<th></th>
<th>pre-merger</th>
<th>post-merger</th>
<th>post-merger</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$p_3$ set in 2nd stage</td>
<td>$p_1$ set in 1st stage</td>
</tr>
<tr>
<td>assortment</td>
<td>${1, 3}$</td>
<td>${1, 3}$</td>
<td>${1, 2, 3}$</td>
</tr>
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<td>$p_1$</td>
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<td>15.09</td>
<td>14.44</td>
</tr>
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<td>17.55*</td>
<td>17.04</td>
</tr>
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<td>22.53</td>
<td>22.33</td>
</tr>
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<td>10.43</td>
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</tr>
<tr>
<td>$d_1$</td>
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<td>3.7028</td>
<td>4.0111</td>
</tr>
<tr>
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<td>0</td>
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<tr>
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<td>0.5278</td>
<td>0.2622</td>
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<td>NA</td>
</tr>
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<td>$\pi_{S2+R3}$</td>
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<td>0.2785</td>
<td>0.0874</td>
</tr>
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<td>total firm profits</td>
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<td>54.3528</td>
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<td>consumer surplus</td>
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<tr>
<td>social welfare</td>
<td>79.3466</td>
<td>78.4903</td>
<td>82.9864</td>
</tr>
</tbody>
</table>

*: demand-choke price.

product assortment, i.e., only products 1 and 3 are sold in the market. As proven by Theorem 4, all equilibrium wholesale and retail prices are reduced. Indeed, the vertical merger forces the merging firm to offer product 3 at the reduced price of $22.53$, in response to wholesale price reductions by supplier 1. While the merger expands its sales volume by more than 75%, the net result is a reduction of the aggregate profits of supplier 2 and her retailer 3.

When the merging firm is required to select and disclose the retail price of its product, upfront, i.e., at the first stage, this induces a different dynamic: all retail and wholesale prices are reduced beyond the reduction in the previous post-merger scenario. In this setting, all three products are sold in equilibrium, as it is now advantageous for supplier 1 to offer product 2 to retailer 2 at the reduced wholesale price of $16.93$, opening up a small market niche for this product. In this scenario, the vertical merger results in a decline of both the sales volume and aggregate profits of the merging parties, along with the aggregate profit of the competing (and non-merging) supplier.
In this example, the consumers benefit unequivocally from the vertical merger, regardless of whether prices, product variety or consumer surplus are used as the yard stick. The benefit is greatest if the merging firm is required to select and disclose the price of its product at the first stage of the competition game. The vertical merger results in a decline of total firm profits in the industry, but more so when the merging firm continues to select its price, during the second stage. In the latter scenario (only), social welfare = consumer welfare + total firm profits, actually takes a hit. The reduction of total firm profits, due to the merger, is consistent with a finding in McGuire and Staelin (1983) that the existence of intermediaries may help fend off competition.

Example 2 (2 products). Consider a market with $N = 2$ products, with raw demand functions $q(p) = a - Rp$, where

$$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}.$$ 

We consider three network structures.

Network structure 1: Before vertical integration, a single supplier sells through two competing retailers (see Figure 4(a)). The supplier merges with retailer 2 (see Figure 4(b)).

![Network structure 1: 1 supplier and 2 retailers](image)

![Effective cost rate polyhedra for network structure 1](image)
Under network structure 1, we have $\hat{C}_1 \subset C_1 \subset \hat{C}_1$, see Figure 5. $\hat{C}_1$ is the quadrangle (II); $C_1$ is the union of (II) and the triangle (III) while $\hat{C}_1$ is the union of (II) and both triangles (I) and (III). Recall from Theorem 2 that if the merging parties had an exclusive relationship with each other, the opposite relationship $\hat{C}_1 \subseteq C_1$ would apply. In view of $C_1 \subset \hat{C}_1$, this network structure provides another example where vertical integration may increase product variety. More specifically, fix a vector $c \in \hat{C}_1 \setminus C_1$. Before the merger, product 2 with a relatively high marginal cost is priced out of the market, but after the merger, product 2 becomes available. In contrast, if, after the merger, the retail price of product 2 is determined during the first stage supplier game, the merger may have the opposite effect of decreasing product variety. This occurs when the cost rate vector $c$ is in triangle (III), i.e., $c \in C_1 \setminus \hat{C}_1$.

Since we consider a symmetric $R$ matrix, the demand function may be derived from a representative consumer maximizing the quadratic utility function, i.e., $\max_{d \geq 0} U(d; p) = (R^{-1} a - p)^\top d - \frac{1}{2} d^\top R^{-1} d$. We can compute the consumer surplus as the maximized consumer utility under the equilibrium retail prices. We can also compute the total society-wide welfare which, as mentioned, is the sum of consumer surplus and total firm profits.

**Figure 6** Welfare impact under network structure 1

![Graphs showing welfare impact](image)

*Note.* Fix $c_1 = 1.5$.

From Figure 6, we see that vertical integration leads to higher consumer surplus and social welfare, regardless of whether the integrated firm sets its retail price in the first or second stage. Comparing the two post-merger scenarios, the consumer surplus and social welfare are lower if the integrated firm sets the retail price in the first stage, as opposed to the second stage. However, the opposite relative ranking applies under network structure 2, see Figure 9 below.
Network structure 2: Before vertical integration, two competing suppliers sell through one common retailer (see Figure 7(a)). Supplier 2 merges with the retailer (see Figure 7(b)).

Under network structure 2, we have $\hat{C}_2 \subset C_2 \subset \check{C}_2$. In view of $C_2 \subset \check{C}_2$, this network structure provides another example where vertical integration may increase product variety. More specifically, consider a cost rate vector $c \in \check{C}_2 \setminus C_2$. Before the merger, product 2 with a relatively high marginal cost is priced out of the market. After the merger, product 2 becomes available, if the retail price of the merging firm’s product is announced as part of the first stage competition game. In contrast, if it is selected in the second stage and $c \in C_2 \setminus \hat{C}_2$, product 1 is sold before the merger but the merger succeeds to elbow product 1 out of the market, creating a monopoly for the merging firm.

From Figure 9, we see that vertical integration, again, leads to higher consumer surplus and social welfare, regardless of whether the integrated firm sets the retail price in the first or second stage. Comparing the two post-merger scenarios, the consumer surplus and social welfare are higher if the integrated firm sets the retail price in the first stage, as opposed to the second stage, in contrast to what we saw under network structure 1.

Network structure 3: Before vertical integration, there are two parallel competing supply chains each with one supplier selling exclusively to a retailer, and after vertical integration, one supply chain becomes an integrated one (see Figure 10). This network structure has been studied
Figure 9  Welfare impact under network structure 2

Note. Fix $c_1 = 1.5$.

in McGuire and Staelin (1983), assuming $c = 0$. With the tools developed in this paper, we are able to study the impacts of vertical integration on price and variety.

Figure 10  Network structure 3: 2 parallel supply chains (see McGuire and Staelin 1983)

Figure 11  Effective cost rate polyhedra for network structure 3

Under network structure 3, we have $\hat{C}_3 = \hat{C}_3 \subseteq C_3$, see Figure 11. That is, the post-merger
effective cost rate polyhedron is the same regardless of whether the retail price of the integrated firm is determined in the first or second stage, and it is contained in its pre-merger counterpart.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{welfare_impact.png}
\caption{Welfare impact under network structure 3}
\end{figure}

Note. Fix $c_1 = 1.5$.

From Figure 12, we see that vertical integration leads to higher consumer surplus and social welfare, regardless of whether the integrated firm sets the retail price in the first or second stage. Comparing the two post-merger scenarios, the consumer surplus is lower if the integrated firm sets the retail price in the first stage as opposed to in the second stage. However, the social welfare of the two scenarios is exactly the same.

8. Conclusion

In this paper, we have characterized the impact a vertical merger has on various equilibrium performance measures, including the equilibrium prices, sales volumes, product assortment, firm profit levels as well as a consumer and a social welfare measure. In doing so, we have responded to the growing discussion among antitrust policy makers, economists and business executives alike, to understand the implications of mergers, both in terms of its price and its non-price effects.

To this end, we have analyzed a very general two-echelon model with an arbitrary number of suppliers and retailers at the upper and lower echelon, respectively. The upstream suppliers offer one or multiple products to some or all of the retailers or directly to the end consumer. We have stipulated a general, albeit parsimonious consumer demand model under which the product assortment sold in the market depends on the equilibrium wholesale and retail prices; this is in contrast to almost all traditional demand models under which all products attain some market share, under any such price combination.
Among our main results, we have identified general conditions under which a (strictly) vertical merger results in a *decline* of product variety, along with reductions of equilibrium prices. One important additional observation is that the relative timing of various price decisions and disclosures has a significant impact on the merger effects. This shows that various merger effects reported in “Nash-in-Nash” models may be colored by the intrinsic assumption that all price decisions are made simultaneously.

Several generalizations of our model should be pursued. First, we have assumed that subsequent to a vertical merger, firms do not deliberately withdraw products from the market as a foreclosure measure; instead, any such product withdrawals are the result of their being “priced out” of the market in the new post-merger equilibrium. It will be interesting to incorporate such targeted foreclosure measures into our model.

Second, we have ignored the fact that vertical mergers may result in cost savings due to cost synergies or economies of scale. One may conjecture that our proven implications for price and product variety reductions continue to apply and become even more pronounced, when the integrated firm benefits from cost savings, because it will be able to set even lower prices and elbow out even more non-competitive products from the market. Future work should confirm (or qualify) this conjecture with a formal analysis.

Finally, we have focused on two-echelon markets. However, our results can easily be extended to mergers within markets with more than two echelons, following the results in Section 4 of Federgruen and Hu (2016).

**References**


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Online Appendix to
“The Price and Variety Effects of Vertical Mergers”

A. Preliminary Linear Algebra Results

In this Appendix, we present some properties of square matrices of special structure that are needed in the analysis of the paper. A matrix is a $P$-matrix, if all of its principal minors are positive. It is well known that a positive definite matrix is a $P$-matrix. A matrix which is both a $Z$-matrix and a $P$-matrix is referred to as a $ZP$-matrix.

**Lemma A.1** (Properties of $ZP$-matrices). Let $M$ be a $ZP$-matrix and $N$ be a $Z$-matrix such that $M \leq N$, i.e., $N - M \geq 0$. Then

(a) $M^{-1}$ exists and $M^{-1} \geq 0$.
(b) $N$ is a $ZP$-matrix and $N^{-1} \leq M^{-1}$.
(c) $MN^{-1}$ and $N^{-1}M$ are $ZP$-matrices.
(d) If $D$ is a positive diagonal matrix, then $DM$, $MD$ and $M + D$ are $ZP$-matrices.

**Proof of Lemma A.1.** (a)-(d). By Horn and Johnson (1991, Theorem 2.5.3), a $ZP$-matrix is a nonsingular, so-called, $M$-matrix. Properties (a)-(d) of $ZP$-matrices can be found in Horn and Johnson (1991, Section 2.5) as properties of $M$-matrices. □

**Lemma A.2** (Harville 1997, Theorems 14.8.4 and 14.8.5). Suppose $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$. Suppose $M$ is positive definite, then the Schur complement $M_{22} - M_{21}M_{11}^{-1}M_{12}$ of the block $M_{11}$ in $M$ is positive definite;

(b) Suppose $M$ is symmetric except for the block $M_{22}$. If $M_{11}$ and its Schur complement $M_{22} - M_{21}M_{11}^{-1}M_{12}$ in $M$ are positive definite, then $M$ is positive definite.

**Lemma A.3.** If

$$M = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix},$$

where $M_{11}$ and $M_{22}$ are $ZP$-matrices, and $M_{12} \leq 0$, $M$ is a $ZP$-matrix.

**Proof of Lemma A.3.** First, since $M_{11}$ and $M_{22}$ are $Z$-matrices and $M_{12} \leq 0$, $M$ is a $Z$-matrix. Second, we show that $M$ is a $P$-matrix, i.e., all of its principal minors are positive. For any principal minor, corresponding with an index set $\mathcal{I}$, let $\mathcal{I}_1$ ($\mathcal{I}_2$) denote the intersection of the set $\mathcal{I}$ and the index set corresponding with the first (second) set of columns in the partitioned matrix $M$. Then

$$M_{\mathcal{I}, \mathcal{I}} = \begin{pmatrix} M_{\mathcal{I}_1, \mathcal{I}_1} & M_{\mathcal{I}_1, \mathcal{I}_2} \\ 0 & M_{\mathcal{I}_2, \mathcal{I}_2} \end{pmatrix};$$

this implies that $\det(M_{\mathcal{I}, \mathcal{I}}) = \det(M_{\mathcal{I}_1, \mathcal{I}_1})\det(M_{\mathcal{I}_2, \mathcal{I}_2}) > 0$. The equality can, for example, be found in Problem 23 of Chapter 2.6 in Shores (2007). The inequality follows from $\det(M_{\mathcal{I}_1, \mathcal{I}_1}) > 0$ and $\det(M_{\mathcal{I}_2, \mathcal{I}_2}) > 0$, since $M_{11}$ and $M_{22}$, by assumption, are $P$-matrices. □
## B. Proofs

**Proof of Lemma 1.** For notational simplicity, we denote index set $\overline{D}$ by 1 and index set $D$ by 2.

(a) We will show that $Y$ is (a1) symmetric, (a2) positive definite and (a3) a $Z$-matrix. Note that

$$
\begin{align*}
\hat{S}_{12} - \hat{S}_{11}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{11} &= T^*(\hat{S}_{11})[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{11}, \\
\hat{S}_{21} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{11} &= 0.
\end{align*}
$$

Similarly, we can write

$$
\hat{S}_{21} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{11} = \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} T^*(\hat{S}_{11}).
$$

Hence by decomposing the matrix $\hat{S}$ into blocks and performing matrix multiplications, we write

$$
Y \equiv \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} I - \hat{S}_{11}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} & 0 \\ -\hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} & I \end{pmatrix} \begin{pmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{pmatrix}
$$

$$
= \begin{pmatrix} \hat{S}_{11} - \hat{S}_{11}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{11} & \hat{S}_{12} - \hat{S}_{11}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{12} \\ \hat{S}_{21} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{11} & \hat{S}_{22} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{12} \end{pmatrix}
$$

$$
= \begin{pmatrix} T^*(\hat{S}_{11})[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} T^*(\hat{S}_{11}) & T^*(\hat{S}_{11})[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{12} \\ \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} T^*(\hat{S}_{11}) & \hat{S}_{22} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{12} \end{pmatrix}.
$$

(B.1)

$\hat{S}_{11}$ is positive definite and a $Z$-matrix. It follows that $T^*(\hat{S}_{11})$ is positive definite and a $Z$-matrix, as well, see the proof of Theorem 1 in Federgruen and Hu (2016). It then follows from Lemma A.1(a) that both $\hat{S}_{11}$ and $T^*(\hat{S}_{11})$ are invertible. The Schur complement of the block $Y_{11}$ in $Y$ is therefore given by:

$$
Y_{22} - Y_{21}Y_{11}^{-1}Y_{12}
$$

$$
= \hat{S}_{22} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{12} - \{\hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} T^*(\hat{S}_{11})\}
$$

$$
-\{\hat{S}_{11}^{-1}[\hat{S}_{11} + T^*(\hat{S}_{11})]T^*(\hat{S}_{11})^{-1}\} \{T^*(\hat{S}_{11})[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{12}\}
$$

$$
= \hat{S}_{22} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \hat{S}_{12} - \{\hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} T^*(\hat{S}_{11})\}(\hat{S}_{11}^{-1} \hat{S}_{12})
$$

$$
= \hat{S}_{22} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} \{I + T^*(\hat{S}_{11})\} \hat{S}_{12}
$$

$$
= \hat{S}_{22} - \hat{S}_{21} \hat{S}_{12}^{-1} \hat{S}_{12}.
$$

(B.2)

(a1) Since $\hat{S}$ is symmetric, by the first representation in (B.1), $Y$ is symmetric because: Since $\hat{S}$ is symmetric, $T^*(\hat{S})$, $\hat{S}_{11}$ and $T^*(\hat{S}_{11})$ are symmetric as well. Symmetry of the block $Y_{11} =$
$T^*(\hat{S}_{11})[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}\hat{S}_{11} = [[\hat{S}_{11}]^{-1} + [T^*(\hat{S}_{11})]^{-1}]^{-1}$ follows from the fact that the transpose of the inverse equals the inverse of its transpose and the transpose of the sum of two matrices equals the sum of the transposes. Moreover, 

$$[\hat{S}_{21} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}\hat{S}_{21} = \hat{S}_{12} - \hat{S}_{11}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}\hat{S}_{12}, \text{ i.e., } Y_{21} = Y_{12}. \text{ Finally, } Y_{22} = [\hat{S}_{22} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}\hat{S}_{12} = Y_{22}.$$

(a2) Given (a1) is proven, to show that $Y$ is positive definite, by Lemma A.2(b) in Appendix A, it is sufficient to show that $Y_{11}$ and its Schur complement $Y_{22} - Y_{12}Y_{11}^{-1}Y_{12}$ are positive definite. First, $Y_{11} = T^*(\hat{S}_{11})[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}\hat{S}_{11} = \Psi^*(\hat{S}_{11})\hat{S}_{11}$ is positive definite by Federgruen and Hu (2015, Proposition 2(d)), since $\hat{S}$ and, hence $\hat{S}_{11}$, are positive-definite. Second, since $\hat{S}$ is positive definite, then the Schur complement of the block $\hat{S}_{11}$ in $\hat{S}$, $\hat{S}_{22} - \hat{S}_{21}\hat{S}_{11}^{-1}\hat{S}_{12}$, is positive definite by Lemma A.2(a). Hence, by (B.2), the Schur complement of $Y_{11}$, $Y_{22} - Y_{12}Y_{11}^{-1}Y_{12} = \hat{S}_{22} - \hat{S}_{21}\hat{S}_{11}^{-1}\hat{S}_{12}$ is positive definite.

(a3) We examine $Y$’s four block sub-matrices, one by one.

First, note that $Y_{11} = T^*(\hat{S}_{11})[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}\hat{S}_{11} = \Psi^*(\hat{S}_{11})\hat{S}_{11}$. By the proof of Federgruen and Hu (2016, Theorem 1(a)), $Y_{11}$ is a $Z$-matrix, since $\hat{S}_{11}$ is a symmetric positive-definite $Z$-matrix.

Second, since $\hat{S}_{11}$ is a $ZP$-matrix, $T^*(\hat{S}_{11})$ is a $Z$-matrix and $\hat{S}_{11} \leq T^*(\hat{S}_{11})$. Thus, $T^*(\hat{S}_{11})$ is also a $ZP$-matrix by Lemma A.1(b). Then by Lemmas A.1(c) and A.1(d), $\hat{S}_{11}[T^*(\hat{S}_{11})]^{-1} + I$ is also a $ZP$-matrix. Hence, by Lemma A.1(a),

$$T^*(\hat{S}_{11})[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} = [\hat{S}_{11}[T^*(\hat{S}_{11})]^{-1} + I]^{-1} \geq 0. \quad \text{(B.3)}$$

Since $\hat{S}_{22} \leq 0$, $Y_{12} = T^*(\hat{S}_{11})[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}\hat{S}_{12} \leq 0$.

Similarly, we can show that $Y_{21} = \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}T^*(\hat{S}_{11}) \leq 0$.

Lastly, note that

$$[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} = T^*(\hat{S}_{11})^{-1} \cdot T^*(\hat{S}_{11})[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} - T^*(\hat{S}_{11})^{-1}\Psi^*(\hat{S}_{11}) \geq 0,$$

since both $T^*(\hat{S}_{11})^{-1} \geq 0$ and $\Psi^*(\hat{S}_{11}) \geq 0$. The former follows from the positive definiteness of $T^*(\hat{S}_{11})$ as a $Z$-matrix, see Lemma A.1(a), while the latter follows from Federgruen and Hu (2015, Proposition 2(e)). Moreover, since $\hat{S}_{21} \leq 0$ and $\hat{S}_{12} \leq 0$, then $\hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}\hat{S}_{12} \geq 0$. Hence, $Y_{22} = \hat{S}_{22} - \hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1}\hat{S}_{12} \leq \hat{S}_{22}$. Since $\hat{S}_{22}$ is a $Z$-matrix, then $Y_{22}$ is also a $Z$-matrix.

We have shown that $Y_{11}$ and $Y_{22}$ are $Z$-matrices, and $Y_{12}$ and $Y_{21}$ are non-positive matrices. Hence, $Y$ is a $Z$-matrix.

(b) Since

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \equiv \Xi^*(\hat{S})\hat{b} = \begin{bmatrix} I - \hat{S}_{11}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} & 0 \\ -\hat{S}_{21}[\hat{S}_{11} + T^*(\hat{S}_{11})]^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix},$$
\[ z_1 = [I - S_{11} [S_{11} + T^*(S_{11})]^{-1}] b_1 = T^*(S_{11})[S_{11} + T^*(S_{11})]^{-1} b_1 \geq 0 \] by (B.3) and \( b_1 \geq 0 \), and \( z_2 = -S_{21} [S_{11} + T^*(S_{11})]^{-1} b_1 + b_2 \geq 0 \) because \( S_{21} \leq 0, b_1, b_2 \geq 0 \) and \( S_{11} + T^*(S_{11}) \) is a ZP-matrix and hence \( [S_{11} + T^*(S_{11})]^{-1} \geq 0 \) by Lemma A.1(a). \( \square \)

**Proof of Theorem 2.** Recall from (17) that \( \bar{C} \equiv \{ c \geq 0 \mid \Xi^*(S) (b - Sc) \geq 0 \} \) while \( C = \{ c \geq 0 \mid \Psi^*(S) (b - Sc) \geq 0 \} \). We will show that \( \Xi^*(S) \) is invertible and that

\[ \Psi^*(S) [\Xi^*(S)]^{-1} \geq 0. \] (B.4)

(B.4) proves the theorem: if \( c \in \bar{C} \), \( \Xi^*(S) (b - Sc) \geq 0 \). Under (B.4), we can pre-multiply both sides of this inequality with \( \Psi^*(S) [\Xi^*(S)]^{-1} (\geq 0) \) to conclude that \( \Psi^*(S) (b - Sc) \geq 0 \), i.e., \( c \in C \). To show (B.4), it suffices, by Lemma A.1(a), to show that the inverse matrix \( X \equiv \Xi^*(S) [\Psi^*(S)]^{-1} \) is a ZP-matrix. \( \Psi^*(S) = T^*(S) [S + T^*(S)]^{-1} \); \( \det(\Psi^*(S)) = \det(T^*(S)) \det([S + T^*(S)]^{-1}) > 0 \), since both \( T^*(S) \) and \( [S + T^*(S)]^{-1} \) are invertible; thus, \( \Psi^*(S) \) is invertible as well. This implies in particular that

\[ \Xi^*(S) \) is invertible, \] (B.5)

since otherwise \( \det(X) = \det(\Xi^*(S)) \det([\Psi^*(S)]^{-1}) = 0 \), contradicting the invertibility of \( X \).

Since supplier \( J \) sells all of her products exclusively to retailer \( I \), the sets \( \bar{D} \) and \( D \) do not share a supplier. For notational simplicity, we denote the index set \( \bar{D} \) by 1 and the index set \( D \) by 2. Thus, \( T^*(S) = \begin{pmatrix} T^*(S_{11}) & 0 \\ 0 & T^*(S_{22}) \end{pmatrix} \). Recall that \( S \) is a positive definite Z-matrix, \( S_{11} \) and \( S_{22} \) therefore have the same properties. Hence, all of the matrices \( T^*(S), T^*(S_{11}) \) and \( T^*(S_{22}) \) are positive definite Z-matrices, and are hence invertible by Lemma A.1(a). Thus, 

\[ [T^*(S)]^{-1} = \begin{pmatrix} [T^*(S_{11})]^{-1} & 0 \\ 0 & [T^*(S_{22})]^{-1} \end{pmatrix}. \] Then,

\[
X = \Xi^*(S) [\Psi^*(S)]^{-1} = \Xi^*(S) [S + T^*(S)] [T^*(S)]^{-1} = \Xi^*(S) [S [T^*(S)]^{-1} + I] \\
= \begin{pmatrix} I - S_{11} [S_{11} + T^*(S_{11})]^{-1} & 0 \\ -S_{21} [S_{11} + T^*(S_{11})]^{-1} & I \end{pmatrix} \begin{pmatrix} S_{11} S_{12} \\ S_{21} S_{22} \end{pmatrix} \begin{pmatrix} [T^*(S_{11})]^{-1} & 0 \\ 0 & [T^*(S_{22})]^{-1} \end{pmatrix} + I \\
= \begin{pmatrix} S_{11} [T^*(S_{11})]^{-1} + I \end{pmatrix}^{-1} \begin{pmatrix} S_{11} [T^*(S_{11})]^{-1} + I & S_{12} [T^*(S_{22})]^{-1} \\ S_{21} [T^*(S_{11})]^{-1} & S_{22} [T^*(S_{22})]^{-1} + I \end{pmatrix} \begin{pmatrix} I & X_{12} \\ 0 & X_{22} \end{pmatrix}. \] (B.6)

To show that \( X \) is a ZP-matrix, it is by Lemma A.3 sufficient to show that \( X_{12} \leq 0 \) and \( X_{22} \) is a ZP-matrix.

First, we show that \( X_{12} \leq 0 \). Recall, again, \( S \) is a positive definite Z-matrix; \( S_{11} \) therefore has the same properties. Since \( T^*(S_{11}) \) is symmetric, \( S_{11} \leq T^*(S_{11}) \). Since \( S_{11} \) is a ZP-matrix, \( T^*(S_{11}) \) is a Z-matrix. Thus by Lemma A.1(b), \( T^*(S_{11}) \) is a ZP-matrix and by Lemma A.1(c), \( S_{11} [T^*(S_{11})]^{-1} \)
is a ZP-matrix. By Lemma A.1(d), \([T^*(S_{11})]^{-1} + I\) is a ZP-matrix, as well. By Lemma A.1(a), 
\([S_1[T^*(S_{11})]^{-1} + I]^{-1} \geq 0\) while \([T^*(S_{22})]^{-1} \geq 0\), since \(T^*(S_{22})\) is a ZP-matrix. Since \(S_{12} \leq 0\), \(X_{12} = [S_1[T^*(S_{11})]^{-1} + I]^{-1} S_{12}[T^*(S_{22})]^{-1} \leq 0\).

Next, we show that \(X_{22}\) is a ZP-matrix. Write

\[
X_{22} = \left[ S_{22} + T^*(S_{22}) - S_{21}[S_{11} + T^*(S_{11})]^{-1} S_{12} \right][T^*(S_{22})]^{-1}.
\]

Since both \(S\) and \(T^*(S)\) are positive definite \(Z\)-matrices, \(S + T^*(S)\) is also a positive definite \(Z\)-matrix, and a fortiori, a ZP-matrix. By Lemma A.2(a), the Schur complement \(S_{22} + T^*(S_{22}) - S_{21}[S_{11} + T^*(S_{11})]^{-1} S_{12}\) of \(S_{11} + T^*(S_{11})\) in the positive definite matrix \(S + T^*(S)\) is also positive definite. (Note, since \(\mathcal{D}\) and \(\mathcal{D}\) have no common supplier, \(T^*(S_{12}) = T^*(S_{21}) = 0\).) Since \(S_{22} + T^*(S_{22})\) is a \(Z\)-matrix, \(S_{21}, S_{12} \leq 0\) and \([S_{11} + T^*(S_{11})]^{-1} \geq 0\), it follows that \(S_{22} + T^*(S_{22}) - S_{21}[S_{11} + T^*(S_{11})]^{-1} S_{12}\) is also a \(Z\)-matrix, hence a ZP-matrix. Moreover, since \(S_{22} \leq T^*(S_{22})\) due to the symmetry of \(T^*(S_{22})\), we have \(S_{22} + T^*(S_{22}) - S_{21}[S_{11} + T^*(S_{11})]^{-1} S_{12} \leq 2T^*(S_{22})\). Since \(2T^*(S_{22})\) is a \(Z\)-matrix, by Lemma A.1(c), \(X_{22} = \left[ S_{22} + T^*(S_{22}) - S_{21}[S_{11} + T^*(S_{11})]^{-1} S_{12} \right][T^*(S_{22})]^{-1}\) is a ZP-matrix. \(\square\)

\textbf{Proof of Theorem 3.} (a) In view of Theorem 2, we distinguish between 3 cases:

1. \(c \in \hat{C}^o\): Since \(\hat{C} \subseteq C\) by Theorem 2, we have \(\hat{N}^* = N^* = N\), i.e., all products are sold in both the pre-merger and post-merger equilibrium.

2. \(c \in C^o \setminus \hat{C}^o\): Since \(c \in C^o\), \(N^* = N\); since \(c \notin \hat{C}^o\), \(\hat{N}^* \subset N\); in this case, all products are sold in the pre-merger setting, but some are withdrawn from the market, after the merger.

3. \(c \notin C^o\): In this case, only a subset of the full product assortment \(N\) is sold before the merger.

We now show that an even (weakly) smaller subset is sold in the post-merger equilibrium. Note that in the \textit{pre-merger} setting, the (suppliers’) equilibrium sales volumes are given by the (unique) regular extension of the (raw) affine functions:

\[
\Psi^*(S)(b - Sc).
\]

Similarly, in the \textit{post-merger} world, the (suppliers’) equilibrium sales volumes are given by the (unique) regular extension of the (raw) affine functions:

\[
\Xi^*(S)(b - Sc).
\]

This means that the pre-merger equilibrium sales volumes are given by: \(\Psi^*(S)(b - S(c - t^*))\), and the post-merger equilibrium volumes by: \(\Xi^*(S)(b - S(c - \hat{t}^*))\), where \(t^*\) is the unique solution to LCP: \(\Psi^*(S)(b - S(c - t)) \geq 0\), \(t^*[\Psi^*(S)(b - S(c - t))] = 0\) and \(t \geq 0\), and \(\hat{t}^*\) is the
unique solution to LCP: \( \Xi^*(S)(b - S(c - \hat{\ell})) \geq 0, \ hat{\ell}^T[\Xi^*(S)(b - S(c - \hat{\ell}))] = 0 \) and \( \hat{\ell} \geq 0 \). Note from the conditions of the LCP that \( \mathcal{N}^* = \{ l \in \mathcal{N} | t^*_l = 0 \} \) and \( \tilde{\mathcal{N}}^* = \{ l \in \mathcal{N} | \hat{\ell}^*_l = 0 \} \). To prove \( \tilde{\mathcal{N}}^* \subseteq \mathcal{N}^* \), it therefore suffices to show that

\[
\hat{\ell}^* \geq t^*. \tag{B.7}
\]

Fix \( l \in \mathcal{N} \). By Mangasarian (1976, Theorem 3), the (unique) solution to the LCP: \( \Psi^*(S)(b - S(c - t)) \geq 0, \ t^T[\Psi^*(S)(b - S(c - t))] = 0 \) and \( t \geq 0 \) is the (unique) solution of any LP of the form:

\[
\min \pi^T t \quad \text{s.t.} \quad \Psi^*(S)(b - S(c - t)) \geq 0, \quad t \geq 0, \tag{B.8}
\]

where \( \pi \) is any vector of positive coefficients. Similarly, the (unique) solution to the LCP: \( \Xi^*(S)(b - S(c - \hat{\ell})) \geq 0, \ \hat{\ell}^T[\Xi^*(S)(b - S(c - \hat{\ell}))] = 0 \) and \( \hat{\ell} \geq 0 \) is the (unique) solution of any LP of the form:

\[
\min \pi^T \hat{\ell} \quad \text{s.t.} \quad \Xi^*(S)(b - S(c - \hat{\ell})) \geq 0, \quad \hat{\ell} \geq 0, \tag{B.9}
\]

where \( \pi \) is any vector of positive coefficients. Select the vector \( \pi \) such that \( \pi_l = 1 \) and \( \pi_{l'} = \epsilon \) for all \( l' \neq l \), for some \( \epsilon > 0 \). In the proof of Theorem 2, we showed that \( \Psi^*(S)[\Xi^*(S)]^{-1} \geq 0 \), see (B.4). Then any feasible solution to LP (B.9) is a feasible solution to LP (B.8), so that

\[
t^* + \epsilon \sum_{l' \neq l} t^*_{l'} = \pi^T t^* \leq \pi^T \hat{\ell}^* = t^*_l + \epsilon \sum_{l' \neq l} \hat{\ell}^*_l. \]

Thus \( t^*_l \leq \hat{\ell}^*_l \) follows by taking the limit for \( \epsilon \downarrow 0 \).

(b) Since \( \hat{\mathcal{C}} \subseteq C \), if \( c \in C \), then \( \hat{\Gamma}(c) \leq c = \Gamma(c) \). If \( c \notin C \), \( \hat{\Gamma}(c) = c - t^* \leq c - t^* = \Gamma(c) \), where the inequality follows from (B.7). \( \Box \)

Proof of Lemma 2. We showed in the proof of Theorem 2 that \( \Xi^*(S) \) is invertible, see (B.5). First, we want to show that

\[
\begin{pmatrix}
[S_{11} + T^*(S_{11})]^{-1} & 0 \\
0 & 0
\end{pmatrix}
[\Xi^*(S)]^{-1} \leq [S + T^*(S)]^{-1}[\Xi^*(S)]^{-1} = [\Xi^*(S)(S + T^*(S))]^{-1}. \tag{B.10}
\]

From the proof of Theorem 2, recall that \( X \equiv \Xi^*(S)[\Psi^*(S)]^{-1} = \Xi^*(S)[S + T^*(S)][T^*(S)]^{-1} \). Thus, employing the last representation of \( X \) in (B.6):

\[
\Xi^*(S)[S + T^*(S)] = XT^*(S)
\]

\[
= \begin{pmatrix}
I & 0 \\
0 & -S_{21}[S_{11} + T^*(S_{11})]^{-1}[S_{12}[T^*(S_{22})]^{-1} - S_{12} + T^*(S_{22})[T^*(S_{22})]^{-1} + 1]
\end{pmatrix}
\begin{pmatrix}
T^*(S_{11}) & 0 \\
0 & T^*(S_{22})
\end{pmatrix}
\]

\[
= \begin{pmatrix}
T^*(S_{11}) & 0 \\
0 & S_{22} + T^*(S_{22}) - S_{21} + T^*(S_{11})[S_{12}]^{-1}
\end{pmatrix}.
\]
Since $S_{22} + T^*(S_{22}) - S_{21}[S_{11} + T^*(S_{11})]^{-1}S_{12}$ is the Schur complement of $S_{11} + T^*(S_{11})$ in the positive definite matrix $S + T^*(S)$, it follows from Lemma A.2(a) that it is positive definite; hence invertible. Then the right hand side of (B.10) can be written as

$$
[S + T^*(S)]^{-1} = \begin{pmatrix} [S_{11} + T^*(S_{11})]^{-1} - S_{21}[S_{22} + T^*(S_{22}) - S_{21}[S_{11} + T^*(S_{11})]^{-1}S_{12}]^{-1} \\ 0 \\ 0 \end{pmatrix}
$$

(B.11)

Since

$$
\Xi^*(S) = \begin{pmatrix} T^*(S_{11})[S_{11} + T^*(S_{11})]^{-1} & 0 \\ -S_{21}[S_{11} + T^*(S_{11})]^{-1} & I \end{pmatrix},
$$

we can write the left hand side of inequality (B.10) as

$$
\begin{pmatrix} [S_{11} + T^*(S_{11})]^{-1} & 0 \\ 0 & 0 \end{pmatrix} [\Xi^*(S)]^{-1}
\begin{pmatrix} [S_{11} + T^*(S_{11})]^{-1} & 0 \\ 0 & 0 \end{pmatrix}^{-1}
\begin{pmatrix} T^*(S_{11}) & 0 \\ -S_{21}[S_{11} + T^*(S_{11})]^{-1} & I \end{pmatrix}
$$

(B.12)

Now let us verify block by block that $\Xi^*(S)[S + T^*(S)]$ is a $Z$-matrix. Consider the $(1,1)$-block: Since $T^*(S)$ is a $ZP$-matrix, $T^*(S_{11})$ is a $Z$-matrix. Consider the $(1,2)$-block: In the proof of Theorem 2, when we show $X_{12} \leq 0$, we have shown that $[S_{11}[T^*(S_{11})]^{-1} + \mathbb{I}]^{-1} \geq 0$; moreover, since $S_{12} \leq 0$, then the $(1,2)$-block of $\Xi^*(S)[S + T^*(S)]$ is non-positive. Consider the $(2,2)$-block: In the proof of Theorem 2, when we show $X_{22}$ is a $ZP$-matrix, we have shown that $S_{22} + T^*(S_{22}) - S_{21}[S_{11} + T^*(S_{11})]^{-1}S_{12}$ is a $Z$-matrix. Now we have verified that $\Xi^*(S)[S + T^*(S)]$ is a $Z$-matrix. Moreover, note that $2\Xi^*(S)S \leq \Xi^*(S)[S + T^*(S)]$, then by Lemma A.1(b), $\Xi^*(S)[S + T^*(S)]$ is a $ZP$-matrix since $\Xi^*(S)S$ is a $ZP$-matrix (see Lemma 1(a)). Hence, by Lemma A.1(a), $[\Xi^*(S)[S + T^*(S)]]^{-1} \geq 0$. Then by comparing (B.11) and (B.12), it is easy to see that inequality (B.10) holds.

For all $c \in \mathcal{C} = \{c \geq 0 \mid \Xi^*(S)(b - Sc) \geq 0\}$,

$$
\hat{w}^*(c) = c + \begin{pmatrix} [S_{11} + T^*(S_{11})]^{-1} & 0 \\ 0 & 0 \end{pmatrix} [\Xi^*(S)]^{-1} \cdot \Xi^*(S)(b - Sc)
\leq c + [S + T^*(S)]^{-1}[\Xi^*(S)]^{-1} \cdot \Xi^*(S)(b - Sc)
= c + [S + T^*(S)]^{-1}(b - Sc) = w^*(c),
$$

where the inequality is due to (B.10). □
Proof of Theorem 4. Note that \( \hat{\Gamma}(c) \in \hat{C} \). Then, by Lemma 2, \( \hat{w}^*(\hat{\Gamma}(c)) \leq w^*(\hat{\Gamma}(c)) \leq w^*(\Gamma(c)) \), where the second inequality from \( \hat{\Gamma}(c) \leq \Gamma(c) \), see Theorem 3(b), and the monotonicity of the mapping \( w^*(\cdot) \), see Theorem 3(c) in Federgruen and Hu (2016). This proves (18). Recall that for any given wholesale price vector, the pre- and post-merger retailer competition games are identical. Theorem 3(c) in Federgruen and Hu (2016) also establishes that the mapping \( p^*(\cdot) \) is monotone, thus verifying (19). \( \square \)

Proof of Theorem 5. (a) Recall from Theorem 1 and Section 5 that

\[
W = \{ w \geq 0 \mid \Psi^r(R)(a - Rw) \geq 0 \},
\]

\[
\tilde{W} = \{ w \geq 0 \mid \Xi^r(R)(a - Rw) \geq 0 \}.
\]

Thus, to show that \( \tilde{W} \subseteq W \), it suffices that \( \Psi^r(R)[\Xi^r(R)]^{-1} \geq 0 \), see the proof of Theorem 2. As shown there, the latter is guaranteed by showing that the inverse of this matrix, i.e., \( \Xi^r(R)[\Psi^r(R)]^{-1} \), is a \( ZP \)-matrix. The proof of the latter is analogous to the proof that the matrix \( X = \Xi^s(S)[\Psi^s(S)]^{-1} \) is a \( ZP \)-matrix, in the proof of Theorem 2. The proof of the latter uses the fact that the sets \( D \) and \( \overline{D} \) do not share a retailer, since the set \( \overline{D} \) consists of all products sold by retailers \( \{1, \ldots, I - 1\} \), and \( D \) of all products sold by retailer \( I \). Furthermore, since the merger involves the single retailer \( I \), we have that \( \hat{T}^r(R_{DD}) = T^r(R_{DD}) \) and \( \tilde{T}^r(R_{\overline{D}\overline{D}}) = T^r(R_{\overline{D}\overline{D}}) \).

(b) and (c): The proof of these two parts is analogous to the proof of Theorem 3. \( \square \)

Proof of Theorem 6. The proof is analogous to that of Theorem 4, invoking the analogue of Lemma 2. \( \square \)

Proof of Lemma 4. For notational simplicity, we denote index set \( \overline{D} \) by 1 and index set \( D \) by 2.

(a) Note that

\[
\tilde{X} \equiv \Xi^r(R)[\Psi^r(R)]^{-1} = \Xi^r(R)[R + T^r(R)][T^r(R)]^{-1} = \Xi^r(R)[R[T^r(R)]^{-1} + I]
\]

\[
= \begin{pmatrix}
1 - R_{11} [R_{11} + T^r(R_{11})]^{-1} & 0 \\
-R_{21} [R_{21} + T^r(R_{21})]^{-1} & 1
\end{pmatrix}
\begin{pmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{pmatrix}
\begin{pmatrix}
[T^r(R_{11})]^{-1} & 0 \\
0 & [T^r(R_{22})]^{-1}
\end{pmatrix}
+ I
\]

\[
= \begin{pmatrix}
|R_{11} [T^r(R_{11})]^{-1} + I|^{-1} & 0 \\
-R_{21} [R_{21} + T^r(R_{21})]^{-1} & 1
\end{pmatrix}
\begin{pmatrix}
R_{11} [T^r(R_{11})]^{-1} + I & R_{12} [T^r(R_{22})]^{-1} \\
R_{21} [T^r(R_{11})]^{-1} & R_{22} [T^r(R_{22})]^{-1}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
I & [R_{11} [T^r(R_{11})]^{-1} + I]^{-1} R_{12} [T^r(R_{22})]^{-1} \\
0 & -R_{21} [R_{21} + T^r(R_{21})]^{-1} R_{12} [T^r(R_{22})]^{-1} + R_{22} [T^r(R_{22})]^{-1} + I
\end{pmatrix}
\equiv \begin{pmatrix}
1 \hat{X}_{12} \\
0 \hat{X}_{22}
\end{pmatrix}.
\]

In analogy to the proof that the matrix \( X \) in the proof of Theorem 2 is a \( ZP \)-matrix, we have:

\( \tilde{X} \) is a \( ZP \)-matrix. (B.13)
First we show that

\[ \tilde{X}T^*(S)[T^*(\tilde{S})]^{-1} \leq I. \]  \hspace{1cm} (B.14)

Write

\[
\tilde{X}T^*(S)[T^*(\tilde{S})]^{-1} = \begin{pmatrix} I & \tilde{X}_{12} \\ 0 & \tilde{X}_{22} \end{pmatrix} \begin{pmatrix} T^*(S_{11}) & 0 \\ 0 & S_{22} \end{pmatrix} \left[ T^*(\tilde{S}_{11})^{-1} 0 \\ 0 \tilde{S}_{22}^{-1} \right] = \begin{pmatrix} T^*(S_{11})[T^*(\tilde{S}_{11})]^{-1} & \tilde{X}_{12}S_{22}\tilde{S}_{22}^{-1} \\ 0 & \tilde{X}_{22}\tilde{S}_{22}^{-1} \end{pmatrix},
\]

where the first equality uses the fact that \( D \) consists of the products sold by supplier \( J \) as the only supplier involved in the merger, and \( \overline{D} \) those of all other suppliers.

By comparing block by block, we verify that inequality (B.14) holds as follows.

The (1,1) block of \( \tilde{X}T^*(S)[T^*(\tilde{S})]^{-1} \): Since

\[ \tilde{S} = XS = \begin{pmatrix} I & \tilde{X}_{12} \\ 0 & \tilde{X}_{22} \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} S_{11} + \tilde{X}_{12}S_{21} & S_{12} + \tilde{X}_{12}S_{22} \\ \tilde{X}_{21}S_{21} & \tilde{X}_{22}S_{22} \end{pmatrix}, \]  \hspace{1cm} (B.15)

\( S_{11} \leq S_{11} + \tilde{X}_{12}S_{21} = \tilde{S}_{11} \), where the inequality is due to the fact that both \( \tilde{X} \) and \( S \) are \( Z \)-matrices, so that \( \tilde{X}_{12}, S_{21} \leq 0 \) (\( \tilde{X} \) is a \( Z \)-matrix by (B.13) and \( S \) is a \( Z \)-matrix). Hence

\[ T^*(S_{11}) \leq T^*(\tilde{S}_{11}). \]  \hspace{1cm} (B.16)

Moreover, since \( \tilde{S} \) is a symmetric positive definite \( Z \)-matrix, by Lemma 3(a), \( \tilde{S}_{11} \) has the same properties, so that \( T^*(\tilde{S}_{11}) \) is a \( ZP \)-matrix, and then \( [T^*(\tilde{S}_{11})]^{-1} \geq 0 \), by Lemma A.1(a). Therefore, by (B.16), \( [\tilde{X}T^*(S)[T^*(\tilde{S})]^{-1}]_{11} = T^*(S_{11})[T^*(\tilde{S}_{11})]^{-1} \leq T^*(\tilde{S}_{11})[T^*(\tilde{S}_{11})]^{-1} \leq I. \)

The (1,2) block of \( \tilde{X}T^*(S)[T^*(\tilde{S})]^{-1} \):

\[ [\tilde{X}T^*(S)[T^*(\tilde{S})]^{-1}]_{12} = \tilde{X}_{12}S_{22}\tilde{S}_{22}^{-1} = \tilde{X}_{12}S_{22}(\tilde{X}_{22}\tilde{S}_{22})^{-1} = \tilde{X}_{12}\tilde{X}_{22}^{-1} \leq 0, \]

where the second equality follows from the structure of the matrix \( \tilde{S} \), see (B.15), and the last equality is by (B.13) \( \tilde{X} \) is a \( ZP \)-matrix, and hence \( \tilde{X}_{12} \leq 0 \) and \( \tilde{X}_{22}^{-1} \geq 0 \), by Lemma A.1(a).

The (2,2) block of \( \tilde{X}T^*(S)[T^*(\tilde{S})]^{-1} \):

\[ [\tilde{X}T^*(S)[T^*(\tilde{S})]^{-1}]_{22} = \tilde{X}_{22}S_{22}\tilde{S}_{22}^{-1} = \tilde{X}_{22}S_{22}(\tilde{X}_{22}\tilde{S}_{22})^{-1} = I. \]

Hence,

\[ \tilde{X}[S + T^*(S)][T^*(\tilde{S})]^{-1} = \tilde{X}S[T^*(\tilde{S})]^{-1} + \tilde{X}T^*(S)[T^*(\tilde{S})]^{-1} \leq \tilde{S}[T^*(\tilde{S})]^{-1} + I = [\tilde{S} + T^*(\tilde{S})][T^*(\tilde{S})]^{-1}, \]  \hspace{1cm} (B.17)

where the inequality is due to \( \tilde{S} = \tilde{X}S \) and (B.14).
Next, we show that \([S + T^*(S)]^{-1}\bar{X}^{-1} \geq 0\). This is because \(S + T^*(S)\) is a ZP-matrix, and \(\bar{X}\) is a ZP-matrix by (B.13), hence \([S + T^*(S)]^{-1}\bar{X}^{-1} \geq 0\), by Lemma A.1(a).

Pre-multiplying both sides in inequality (B.17) by \([S + T^*(S)]^{-1}\bar{X}^{-1}(a - Rc) \geq 0\), we have

\[
[T^*(\tilde{S})]^{-1} \leq [S + T^*(S)]^{-1}\Psi'(R)[\Xi'(R)]^{-1}[\bar{S} + T^*(\tilde{S})][T^*(\tilde{S})]^{-1}.
\]  

(B.18)

For all \(c \in \mathcal{C} = \{c \geq 0 \mid T^*(\tilde{S})[\bar{S} + T^*(\tilde{S})]^{-1}\Xi'(R)(a - Rc) \geq 0\},

\[
\bar{w}^*(c) = c + [\bar{S} + T^*(\tilde{S})]^{-1}\Xi'(R)(a - Rc)
= c + [T^*(\tilde{S})]^{-1} \cdot \{T^*(\tilde{S})[\bar{S} + T^*(\tilde{S})]^{-1}\Xi'(R)(a - Rc)\}
\leq c + \{[S + T^*(S)]^{-1}\Psi'(R)[\Xi'(R)]^{-1}[\bar{S} + T^*(\tilde{S})][T^*(\tilde{S})]^{-1}\}
\cdot \{T^*(\tilde{S})[\bar{S} + T^*(\tilde{S})]^{-1}\Xi'(R)(a - Rc)\}
= c + [S + T^*(S)]^{-1}\Psi'(R)(a - Rc) = w^*(c),
\]

where the inequality is due to (B.18).

(b) First, we show that

\[
\bar{X} \leq T^*(\tilde{S})[T^*(S)]^{-1}.
\]  

(B.19)

Note that

\[
\bar{X}T^*(S) = \begin{pmatrix} I & \bar{X}_{12} \\ 0 & \bar{X}_{22} \end{pmatrix} \begin{pmatrix} T^*(S_{11}) & 0 \\ 0 & S_{22} \end{pmatrix} = \begin{pmatrix} T^*(S_{11}) \bar{X}_{12}S_{22} \\ 0 & \bar{X}_{22}S_{22} \end{pmatrix}
\]

and recall that

\[
\bar{S} = \begin{pmatrix} S_{11} + \bar{X}_{12}S_{21} & S_{12} + \bar{X}_{12}S_{22} \\ \bar{X}_{22}S_{21} & \bar{X}_{22}S_{22} \end{pmatrix}.
\]

By Lemma 3(a), both \(S\) and \(\bar{S}\) are symmetric and invertible matrices. Since \(\mathcal{D}\) consists of the products sold by supplier \(J\) and \(\mathcal{D}\) those of all other suppliers, we have

\[
T^*(S) = \begin{pmatrix} T^*(S_{11}) & 0 \\ 0 & S_{22} \end{pmatrix}
\]

and

\[
T^*(\tilde{S}) = \begin{pmatrix} T^*(\tilde{S}_{11}) & 0 \\ 0 & \tilde{S}_{22} \end{pmatrix}.
\]

Thus,

\[
\bar{X} - T^*(\tilde{S})[T^*(S)]^{-1}
= [\bar{X}T^*(S) - T^*(\tilde{S})][T^*(S)]^{-1}
= \left[ \begin{pmatrix} T^*(S_{11}) \bar{X}_{12}S_{22} \\ 0 \bar{X}_{22}S_{22} \end{pmatrix} - \begin{pmatrix} T^*(\tilde{S}_{11}) & 0 \\ 0 & \tilde{X}_{22}S_{22} \end{pmatrix} \right] \begin{pmatrix} T^*(S_{11})^{-1} & 0 \\ 0 & S_{22}^{-1} \end{pmatrix}
= \begin{pmatrix} T^*(S_{11}) - T^*(\tilde{S}_{11}) & \bar{X}_{12}S_{22} \\ 0 & \tilde{X}_{22}S_{22} \end{pmatrix} \begin{pmatrix} T^*(S_{11})^{-1} & 0 \\ 0 & S_{22}^{-1} \end{pmatrix}
= \begin{pmatrix} 1 - T^*(\tilde{S}_{11})[T^*(S_{11})]^{-1} & \bar{X}_{12} \\ 0 & 0 \end{pmatrix}
\leq 0.
\]
To verify the inequality, note that $\bar{X}_{12} \leq 0$ since $\bar{X}$ is a Z-matrix: $T^*(\bar{S}_{11})[T^*(S_{11})]^{-1} \geq I$ is verified by post-multiplying both sides of (B.16) with $[T^*(S_{11})]^{-1}$, a non-negative matrix since $T^*(S)$ is a ZP-matrix, see Lemma A.1(a). We have thus verified inequality (B.19).

Hence,

$$[\bar{S} + T^*(\bar{S})]^{-1}[\bar{S} + \bar{X}T^*(S)]T^*(S)^{-1}$$

$$= [\bar{S} + T^*(\bar{S})]^{-1}[[\bar{S} + T^*(\bar{S})] + [\bar{X}T^*(S) - T^*(\bar{S})]]T^*(S)^{-1}$$

$$= [T^*(S)]^{-1} + [\bar{S} + T^*(\bar{S})]^{-1}[\bar{X} - T^*(\bar{S})]T^*(S)]^{-1}$$

$$\leq [T^*(S)]^{-1}, \quad (B.20)$$

where the inequality is due to (B.19) and $[\bar{S} + T^*(\bar{S})]^{-1} \geq 0$, as the inverse of the ZP-matrix $\bar{S} + T^*(\bar{S})$, see Lemma A.1(a). (Since $2\bar{S} \leq \bar{S} + T^*(\bar{S})$ and $\bar{S} + T^*(\bar{S})$ is a Z-matrix, by Lemma A.1(b), $\bar{S} + T^*(\bar{S})$ is a ZP-matrix.)

For all $c \in C = \{c \geq 0 | T^*(S)[S + T^*(S)]^{-1}\Psi^r(R)(a - Rc) \geq 0\}$, it follows from (22) and (21) that

$$\bar{w}^*(c) = c + [\bar{S} + T^*(\bar{S})]^{-1}\Xi^r(R)(a - Rc)$$

$$= c + [\bar{S} + T^*(\bar{S})]^{-1}\Xi^r(R)[\Psi^r(R)]^{-1}[S + T^*(S)][T^*(S)]^{-1}\{T^*(S)[S + T^*(S)]^{-1}\Psi^r(R)(a - Rc)\}$$

$$= c + [\bar{S} + T^*(\bar{S})]^{-1}[\bar{S} + \bar{X}T^*(S)][T^*(S)]^{-1}\{T^*(S)[S + T^*(S)]^{-1}\Psi^r(R)(a - Rc)\}$$

$$\leq c + [S + T^*(S)]^{-1}\Psi^r(R)(a - Rc) = w^*(c),$$

where the inequality is due to (B.20). □

**Proof of Theorem 7.** Note that

$$\hat{p}^*(\mathbf{w}^*(\hat{\Gamma}(c))) \leq \bar{p}^*(\mathbf{w}^*(c)) \leq p^*(\Theta(\mathbf{w}^*(c))) \leq p^*(\Theta(w^*(c))) \leq p^*(w^*(c)) = p^*(w^*(\hat{\Gamma}(c))).$$

The first inequality is due to $\hat{\Gamma}(c) \leq c$ and the monotonicity of the mappings $\bar{w}^*(\cdot)$ and $\hat{p}^*(\cdot)$, following from Theorem 3(c) in Federgruen and Hu (2016), when applied to the post-merger two-stage competition model. The second inequality follows from Theorem 6, since $\bar{w}^*(c) \in \hat{\mathbf{W}}$ so that $\hat{\Theta}(\bar{w}^*(c)) = \bar{w}^*(c)$. The third inequality follows from the fact that $\bar{w}^*(c) \leq w^*(c)$, by Lemma 4(b), in combination with the monotonicity of the operators $\Theta(\cdot)$ and $p^*(\cdot)$; that of the projection operator $\Theta(\cdot)$ was shown in Lemma A.2 of Federgruen and Hu (2016) and the monotonicity of $p^*(\cdot)$ follows, again, from Theorem 3(c) ibid. The last inequality follows from $\Theta(w^*(c)) \leq w^*(c)$ and the monotonicity of $p^*(\cdot)$. Finally, the equality follows from the fact that $c \in C$, hence $\Gamma(c) = c$. □
References


