Some Large Sample Results for the Method of Regularized Estimators

PRELIMINARY DRAFT

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Abstract

We present a general framework for studying regularized estimators; i.e., estimation problems wherein “plug-in” type estimators are either ill-defined or ill-behaved. We derive primitive conditions that imply consistency and asymptotic linear representation for regularized estimators, allowing for slower than $\sqrt{n}$ estimators as well as infinite dimensional parameters. We also provide data-driven methods for choosing tuning parameters that, under some conditions, achieve the aforementioned results. We illustrate the scope of our approach by studying a wide range of applications, revisiting known results and deriving new ones.

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1 Introduction

Most of econometrics and statistics is concerned with estimation or constructing inferential procedures like tests or confidence regions for some parameter of interest, and in order to approximate the properties of these procedures we often rely on large sample theory.

In general, one can think of a parameter as a mapping, $\psi$, from the probability distribution generating the data, $P$, to some parameter space, which could be of finite or infinite dimension depending on the application.

The general large sample theory for estimators relies on local smoothness properties of this mapping. For instance, consistency of the “plug-in” estimator, which is given by $\psi(P_n)$ where $P_n$ is the empirical distribution,\(^1\) follows from continuity properties (under a suitable metric) of the mapping $\psi$ at the true probability distribution $P$; e.g., see Wolfowitz [1957], Donoho and Liu [1991]. Another example are asymptotic linear representations, where the estimator can be approximately expressed as the parameter plus a sample average of an unknown function, i.e.,

$$\hat{\psi}_n = \psi(P) + n^{-1} \sum_{i=1}^{n} \nu(Z_i) + \text{“small error”},$$

(1)

where $\nu$ is referred as the influence function and it often depends on the unknown $P$. Such representations are of interest because, among other things, are the building blocks for

\(^1\)Since $P_n$ is sufficient for the data $Z_1, ..., Z_n$, the only restriction here imposed by our definition of “plug-in” estimator is that it does not vary with $n$. 

constructing the large sample distribution of the estimator; also the shape of the influence function matters for robustness considerations (e.g. Hampel et al. [2011]) as well as efficiency considerations (e.g. Bickel et al. [1998]) of the estimator. This representation hinges, among other things, on local smoothness conditions of the parameter mapping around the true probability measure; e.g. see Bickel et al. [1998], Newey [1990] and reference therein.

These large sample results are general and powerful, but they rely on two additional assumptions — oftentimes only implicitly stated — that limit their scope. First, when using “plug-in” type estimators, it is assumed that the parameter mapping is well-defined when evaluated at the empirical distribution. Second, the aforementioned linear representation assumes that the parameter is root-n estimable — i.e., the influence function has finite variance.

The two aforementioned assumptions rule out many problems. For instance, it was noted as early as Stein [1956] that in many applications the parameter mapping is not well-defined when evaluated at the empirical distribution; i.e., the natural “plug-in” estimator is ill-defined. Also, the parameter of interest — even if it is a finite dimensional — is, either, known to be not root-n estimable because the model’s efficiency bound is infinite, or, it is simply hard to check a-priori whether the parameter is root-n estimable or not because of the complexity of the model. Hence, a linear representation of the form 1 either does not exist or does not provide a good description of the behavior of the estimator; see Bickel et al. [1998]. These problems are quite common in semi-/non-parametric models as well as in High-Dimensional models, which are prevalent in econometrics and statistics, especially since the advent of “big data”.

In situations where the “plug-in” estimator is ill-defined, the widespread solution is to replace the original estimation problem by a sequence of “nice” problems. This be viewed as replacing the original parameter mapping, \( \psi \), by a sequence of parameter mappings, \( (\psi_k)_{k=0}^{\infty} \), where each element is well-behaved and its limit coincides with the original mapping. The index of this sequence (denoted by \( k \)) represents what is often referred as the tuning (or regularization) parameter; e.g. it is the (inverse of the) bandwidth for kernels, the number of terms in a series expansion, or the (inverse of the) scale parameter in penalizations.

Even though this solution technique is ubiquitous, being used in a wide range of examples (see the Related Literature and Section 2 for references), the large sample properties for the associated regularized estimators have only been derived in a case-by-case basis; to our knowledge, there is no general large sample theory that would parallel the large sample theory — mainly developed for “plug-in” estimators — mentioned at the beginning. In particular, there is no general method for guiding the “standardization” of the estimator to
obtain the asymptotic distribution in slower than root-n cases. The goal of the paper is to fill this gap by providing the basis for a general large sample theory for regularized estimators.

While our proposed theory uses the specific structure of the regularized estimator — the fact that it is constructed using a sequence of pre-specified mappings —, our examples and results suggest that it is general enough to cover many applications of interest. Thus, we view our results as extending the scope of the general large sample theory mentioned at the beginning, and by doing so, creating an unifying set of results that can be used to address future and existing problems where either one of the two aforementioned implicit assumptions does not hold.

Our point of departure is the general conceptual framework put forward by Bickel and Li [2006], wherein the authors present an unifying framework that covers a wide range of problems. We complement their analysis, first, by providing additional examples, ranging from density estimation, bootstrapping parameters at the boundary and non-parametric IV regression. Second, and more importantly, we establish consistency and asymptotic (linear) representations for the regularized estimator. Our approach is akin to the one used in the standard large sample theory for “plug-in” estimators, in the sense that it relies on local properties of the mapping used for estimation. The key difference, however, is that in our case, the natural mapping is the (sequence of) regularized parameter mappings, \((\psi_k)_{k=1}^{\infty}\), as opposed to the “original” one, \(\psi\); this difference — in particular, the fact that the regularized mapping is actually a sequence of mappings — introduces nuances that are not present in the standard “plug-in” estimation case.

First, we establish consistency and convergence rates for regularized estimators. We show that the key component of the convergence rate is the modulus of continuity of the regularized mapping, which, in general will deteriorate as one moves further into the sequence of regularized mappings, thus yielding a generalized version of the well-known “variance-(squared) bias” trade-off. While this result, by itself, does not constitute a big leap from Bickel and Li’s framework, we use the underlying insights to propose a data-driven adaptive method to choose the tuning parameter that — under mild conditions — yields consistency and a convergence rate proportional to the choice that balances the “variance-(squared) bias” trade-off.

Second, we derive an asymptotic linear representation (ALR) for regularized estimators. A priori, how to formulate the notion of ALR for regularized estimators is not as intuitive as formulating the notion of consistency. Our approach for formulating and deriving ALR is analogous to the one behind expression 1 in the sense that it relies on a notion of (directional) differentiability, but contrary to the approach behind expression 1, it relies on differentiability
of each element in the sequence of regularized mappings, $\psi_k^\infty_{k=1}$, not on differentiability of the original mapping $\psi$. By doing so, we derive the concept of influence function of the regularized estimator based on the (sequence of) regularized mappings. We view this influence function as the natural departure from the traditional influence function as it is the sequence of regularized mappings, $\psi_k^\infty_{k=1}$, and not the original mapping, $\psi$, the one used for constructing the estimator.

By using the variation of the influence function of the regularized estimator, we provide a systematic way of standardizing the regularized estimator that admits parameters that are both root-n and not root-n estimable. This result follows because, while we require the derivative of each element of the sequence of regularized mappings to be well-behaved and have finite variation, we do not impose any uniform restrictions on its behavior as we move further into the sequence of regularized mappings (i.e., as $k$ diverges).

In addition to differentiability of the sequence of the regularized mappings, we also require that the reminder of the linear approximation is asymptotically negligible. In the standard case this reminder — which was denoted as “small error” in expression 1 — is simply required to be of smaller order than $n^{-1/2}$, but for regularized estimators the order of the reminder is more nuance, since often times the estimator converges at slower than root-n. It turns out that in order to obtain the ALR for regularized estimators, it suffices for the reminder to be of order smaller than the variation of the influence function.

We also present low-level sufficient conditions for asymptotic control of the reminder. We think these conditions could offer some guidance on the construction of regularized mappings, which, after all, are chosen by the researcher. Also, by studying particular examples as well as providing general results we first show that these conditions (and differentiability) are satisfied by many commonly used applications, and second provide insights on the type of properties that the sequence $\psi_k^\infty_{k=1}$ should have in order to satisfy these conditions (and differentiability).

The asymptotic linear representation is derived for both finite and infinite dimensional parameters. While the former case is the prevalent one in most semi-parametric models, we show that the latter case is still of interest since it can be used as the basis for the construction of confidence bands for unknown functions.

**Related Literature.** Examples of regularizations are so ubiquitous in econometrics and statistics that providing a thorough review is outside the scope of the paper. Other than Bickel and Li [2006], we refer the reader to Härdle and Linton [1994] and Chen [2007] which are excellent reviews for regularization methods such as kernel and series/sieves.

As mentioned above, in many semi-/non-parametric models the parameter of interest is
not root-n estimable so the estimator needs to be properly scaled — by something other than \( \sqrt{n} \). For density and regression estimation problems there is a large literature, especially for particular functionals like evaluation at a point; e.g. Eggermont and LaRiccia [2001] Vol I and II for references and results. In more general contexts such as M-estimation and GMM-based models, to our knowledge, the literature is much more sparse with only a few papers allowing for slower than root-n parameters in particular settings. Closest to ours are the papers by Chen and Liao [2014] in the context of M-estimation models with series/sieve-based estimators; Newey [1994] in a two-stage moment model using kernel-based estimators; and Chen and Pouzo [2015] in conditional moment models with sieve-based estimators.

The method for choosing tuning parameters proposed in Section 4 is based on the method presented in Pereverzev and Schock [2006] for ill-posed inverse problems. Similar versions has been used in particular applications by Pouzo [2015] for regularized M-estimators; by Chen and Christensen [2015] for non-parametric IV regressions; and by Gine and Nickl [2008] for estimation of the integrated square density.

**Notation.** Throughout, for I.I.D. data we use \( P^\infty \) to denote a probability measure over \((Z_1, Z_2, ...)\) when \( Z_i \sim P \); we will typically used \( P \) to denote probability statements of \( P^\infty \), i.e., \( o_P \) instead of \( o_{P^\infty} \) and so on. The term “wpa1-P” is short for with probability approaching 1 under \( P \), so for a generic sequence of I.I.D. random variables \((Z_n)_n\) with \( Z_n \sim P \), the phrase “\( Z_n \in A \) wpa1-P” formally means \( P(Z_n \notin A) = o(1) \). For a real-valued sequence \((x_n)_n\), \( x_n \uparrow a \in \mathbb{R} \cup \{ \infty \} \) means that the sequence is non-decreasing and its limit is \( a \); \( x_n \downarrow a \) is defined analogously. For any random variables \((X, Y)\) we use \( p_X \) and \( p_{XW} \) to denote the pdf (w.r.t. Lebesgue) corresponding to \( X \) and \( X, Y \) resp. For any linear normed spaces \( A \) and \( B \), let \( A^* \) be the dual of \( A \), and for any continuous, homogeneous of degree 1 function \( f : (A, ||.||_A) \mapsto (B, ||.||_B) \), \( ||f||_* = \sup_{|a| \neq 0} \frac{||f(a)||_B}{|a|_A} \). For an Euclidean set \( S \), we use \( L^p(S) \) to denotes the set of \( L^p \) functions with respect to Lebesgue. For any other measure \( \mu \), we use \( L^p(S, \mu) \) or \( L^p(\mu) \). The norm \( ||.|| \) denotes the Euclidean norm and when applied to matrices it corresponds to the operator norm. For any matrix \( A \), let \( e_{\min}(A) \) denote the minimal eigenvalue. The symbol \( \lesssim \) denotes less or equal up to universal constants; \( \gtrsim \) is defined analogously.

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Examples 2.1 and 2.2 also contain references for the density evaluation and the integrated square density problems.
2 Examples

In this section we present several canonical examples which will be studied throughout the paper. We start, however, by presenting a classical example — density estimation at a point — so as to motivate and illustrate our results. The examples illustrates two key points of the paper: The fact that in many non-parametric models the parameters may not be root-n estimable, but, at the same time, the parameter is not being directly used for estimation, rather a regularized version of the parameter is used instead.

Example 2.1 (Density Evaluation). The parameter of interest is the density function evaluated at a point, which can be formally viewed as a mapping from the space of probability distributions to \( \mathbb{R} \), given by \( P \mapsto \psi(P) = p(0) \), where \( p \) denotes the pdf of \( P \). This mapping is only defined over the class of probabilities that admit a continuous pdf; since the empirical probability distribution, \( P_n \), does not belong to this class one cannot implement the standard “plug-in” estimator \( \psi(P_n) \). To circumvent this shortcoming, the parameter mapping is replaced by the following sequence

\[
P \mapsto \psi_k(P) = (\kappa_k \ast P)(0) \equiv \int_{\mathbb{R}} h(k)^{-1} \kappa \left( \frac{z}{h(k)} \right) P(dz), \quad \forall k = 0, 1, 2, ...
\]

where \( \kappa_k(.) = h(k)^{-1} \kappa(. / h(k)) \) and \( \kappa \) is uniformly bounded, smooth, symmetric at 0 pdf. Also \( h(k) > 0 \) and \( h(k) = o(1) \). Intuitively, for any fixed \( k \), \( \psi_k \) evaluated at \( P_n \) is well-defined and well-behaved, given by \( n^{-1} \sum_{i=1}^{n} \kappa_k(Z_i) \); and as \( k \) diverges, \((\kappa_k \ast P)(0)\) will approximate \( p(0) \). That is, \((\psi_k)_{k \in \mathbb{N}_0}\) regularizes the parameter \( \psi \). In Section 3 we provide a general definition of regularization that encompasses this case and many others and provides a conceptual framework to derive general asymptotic results.

Consistency. One such asymptotic result is consistency of the estimator, for some diverging sequence \((k(n))_{n \in \mathbb{N}}\). In this case, it is easy to see that it follows from ensuring that both the “sampling error” \(|\psi_k(n)(P_n) - \psi_k(n)(P)|\) and the “approximation error” \(|\psi_k(n)(P) - \psi(P)|\) vanish. While the convergence of the latter term follows directly from the construction of the regularization, convergence of the sampling error is more delicate. The main challenge stems from the fact that even though \( P_n \) is expected to converge to \( P \) as \( n \) diverges, the modulus of continuity of \( \kappa_k \) deteriorates as \( k = k(n) \) diverges. The desired result follows by having a good estimate of the modulus of continuity of the mapping of \( \kappa_k - \psi_k \) in general — as a function of \( k \). In Section 4 we apply these ideas to postulate sufficient conditions for consistency and convergence rates for general regularizations.

For this example there are many ways of choosing the tuning parameter adaptively in a data-driven way (e.g. Härdle and Linton [1994] and reference therein). In order to obtain optimal (or close to optimal) rates of convergence, the choice has to balance the approxima-
tion and the sampling errors (or bounds for them); see Birge and Massart [1998]. In Section 4 we present a data-driven way for choosing the tuning parameter that achieves this balance and can be applied to a large class of regularizations.

**Asymptotic Linear Representation.** It is well-known that the efficiency bound for this problem is infinite (e.g. Bickel et al. [1998] p. 48). Even though this result is informative about the behavior of the parameter, it does not provide the whole description of the (asymptotic) behavior of the estimator. Because, to construct such bound one studies the local curvature of the parameter $\psi(.)$ at $P$, however, the (local) shape of $\psi$ does not provide extensive information about the asymptotic behavior of the estimator. After all, the mapping used for estimation is not $\psi$ but $\psi_k$ (for $k = k(n)$ diverging as $n$ diverges).

An alternative approach relies on the following linear representation

$$
\psi_k(P_n) - \psi(P) = n^{-1} \sum_{i=1}^{n} \{ \kappa_k(Z_i) - E_P[\kappa_k(Z)] \} + ((\kappa_k \ast P)(0) - p(0)),
$$

where, by drawing an analogy with the standard approach for root-n estimable parameters (see Hampel et al. [2011], Bickel et al. [1998], Newey [1990]), the term in the curly brackets can be thought as the influence function of the regularization at $k$. This term plays a crucial role on determining the asymptotic distribution of the estimator and on determining the proper way of standarizing it. For general regularizations, exact representations of this form are not always possible; however, in Section 5 we identify a class of regularizations — satisfying a certain differentiability notion (see Definition 5.2) — that admit, asymptotically, an analogous linear representation, with the influence function being a function of the derivative of the regularization.

It is well-known that, in this case, the scaling is given by $\sqrt{n}/h(k)$ which is slower (as $k$ diverges with $n$) than the “standard” $\sqrt{n}$. The $h(k)^{-1/2}$ correction arises because it is the correct order of the influence function, i.e., $\sqrt{Var_P \left( n^{-1/2} \sum_{i=1}^{n} \{ \kappa_k(Z_i) - E_P[\kappa_k(Z)] \} \right)} = \sqrt{Var_P(\kappa_k(Z))} \propto h(k)^{-1/2}$. Our results extend this simple observation to a large class of regularizations, thereby providing a canonical way for “standarizing” the estimator: By using $\sqrt{n}$ divided the standard deviation of the influence function.

It is worth to point out that the feature of slower than root-n convergence rate is pervasive and it is shared by many regularized parameters, especially in semi-/non-parametric models. Our method can be thus viewed as extending the standard approach for root-n estimable parameters to a larger class of problems.

The result in Section 5 focus on asymptotic linear representations for finite dimensional parameters; e.g. vector-valued functionals of the density. Even though this representation is enough to cover this example and many others, in other instances the parameter of interest
can be infinite dimensional. For instance, one could be interested on the asymptotic behavior of
\( \sup_z |(\kappa_k * P)(z) - p(z)| / \sigma_P(z) \) for some \( \sigma_P > 0 \) (e.g. Bickel and Rosenblatt [1973]). To
cover this case, the results in Section 5 must be extended since they rely on approximations
that are valid pointwise on \( z \), not uniformly. In Section 6 we extend our results to cover
such cases, and, by using insights from dual spaces, show how the results can be used as the
basis for constructing confidence bands for functions. △

We now provide additional canonical examples that will be developed throughout the
paper. These examples further illustrate the scope of our approach and additional nuances
of the type of problems we can study.

Example 2.2 (Integrated Square Density). Consider a similar setup as in example 2.1, but
now the parameter of interest in this case is given by
\[
P \mapsto \psi(P) = \int p(x)^2 dx.
\]
This mapping is well-defined over probabilities with density in \( L^2(\text{Leb}) \), but not when eval-
uated at the empirical probability distribution, \( P_n \), since \( P_n \) does not have a density in
\( L^2(\text{Leb}) \); it needs to be regularized. Bickel and Ritov [1990] showed that even though the
efficiency bound is finite, no estimator converges at root-n rate; thereby illustrating that in
some circumstances studying the local shape of \( \psi \) can be quite misleading. Our approach
does not suffer from this criticism since it directly captures the (local) behavior of the esti-
imator at hand. Our approach is also general enough to encompass many of the proposed
estimators in the literature, including “leave-one-out” types.

We think our method complements the literature (e.g. Bickel and Ritov [1990], Bickel and
Ritov [1988], Hall and Marron [1987] and Gine and Nickl [2008]) by providing an unifying
framework that, among other things, allow us to better understand how certain aspects of the
model/regularization affect the behavior of the estimator’s convergence rate. Also, from a
technical standpoint, this example illustrates how our method handles non-linear parameter
mappings. △

Example 2.3 (Non-Parametric IV Regression (NPIV)). Consider the Non-Parametric IV
Regression model characterized by
\[
E_{P_{Y|W,X}(\cdot|X)}[Y - h(W)] = 0,
\]
where \( h \) is some function in \( L^2(P) \), \( Y \) is the outcome variable, \( X \) is the endogenous regressor
and \( W \) is the IV.\(^3\) It is well-known that the problem needs to be regularized; see Darolles
\[^3\text{The notation } P_{Y|W,X} \text{ denotes the condition probability of } Y \text{ given } W, X.\]
et al. [2011], Hall and Horowitz [2005], Ai and Chen [2003], Newey and Powell [2003], Florens [2003] among others.

We focus on the case where the parameter of interest is a linear functional of $h$, which it is root-n estimable only under certain conditions (see Severini and Tripathi [2012]). We show how our method encompasses commonly used regularizations schemes. For each, we derive the influence function of the regularization and show how its standard deviation can be used to appropriately scaled the estimator to obtain asymptotic linear approximation regardless of whether the parameter is root-n or not. This last result, illustrates how our method can be used to generalize the approach proposed in Chen and Pouzo [2015] to general regularizations. Finally, in the spirit of Ackerberg et al. [2014], we link the influence function of the regularization to simpler, fully parametric, misspecified GMM models.

**Example 2.4** (Bootstrap when the parameter is on the boundary). Consider the case of bootstrapping the mean of a distribution when it is known to be non-negative as in Andrews [2000]. In this case, the parameter of interest is the Law of the estimator of the mean, 

$$
\psi_n(P)(A) \equiv P\{T_n(P) \in A\}, \quad \forall A \text{ Borel and } \forall n \in \mathbb{N}_0,
$$

where $T_n(P) = \sqrt{n}(\max\{n^{-1}\sum_{i=1}^{n} Z_i, 0\} - \max\{E_P[Z], 0\}) \in \mathbb{R}$ and $Z_i \sim P$.

The bootstrap estimator is simply given by $\psi_n(P_n)$, i.e., the plug-in type estimator of $\psi_n$. Andrews [2000] showed that this estimator is inconsistent and proposed several methods for correcting this. In this paper, we consider one — the m-out-of-n bootstrap — and show that it can be viewed as a regularization scheme.\footnote{Bickel and Li [2006] also proposed a similar exercise but for a different case: estimation of largest order statistic.} We apply our results to derive the convergence rate of the m-out-of-n estimator and propose a data-driven choice of tuning parameter that achieves the rate.

The next example is not really an example, it is rather a canonical estimation technique. It illustrates how our high level conditions translates to a particular estimation technique.

**Example 2.5** (Regularized M-Estimators). Given some model $\mathcal{M}$, the parameter mapping is defined as

$$
\psi(P) = \arg \min_{\theta \in \Theta} E_P[\phi(Z, \theta)], \quad \forall P \in \mathcal{M},
$$

where $\Theta$ and $\phi : \Omega \times \Theta \to \mathbb{R}_+$ are primitives of the problem and are such that the argmin is non-empty for any $P \in \mathcal{M}$. Many models of interest fit in this framework: High-dimensional linear and quantile regressions, non-parametric regression and likelihood-based models among others. In all of these cases, $\psi(P_n)$ is ill-defined or ill-behaved so it needs to be regularized.
We show how our results provide a general way of scaling the regularized estimator — even if the parameter is not root-n estimable —, and how they can be employed to get new limit theorems for confidence bands for general M-estimators, as well as a data-driven method to choose the tuning parameter.

3 Setup

Let $\Omega \subseteq \mathbb{R}^d$, and let $\omega \equiv (z_1, z_2, \ldots) \in \Omega^\infty$ denote a sequence of i.i.d. data drawn from some $P \in \mathcal{P}(\Omega) \subset \text{ca}(\Omega)$, where $\mathcal{P}(\Omega)$ is the set of Borel probability measures over $\Omega$ and $\text{ca}(\Omega)$ is the space of signed Borel measures of finite variation. For each $P \in \mathcal{P}(\Omega)$, let $P^\infty$ be the induced probability over $\Omega^\infty$. We define a model as a subset of $\mathcal{P}(\Omega)$, and it will typically be denoted as $\mathcal{M}$.

Remark 3.1. Since we only consider i.i.d. random variables, it is enough to define a model as a family of probabilities over marginal probabilities. For richer data structures, one would have to define the model as a family of probabilities over $(Z_1, Z_2, \ldots)$. See Appendix A for a discussion about how to extend our results to general stationary models.

Topology over $\text{ca}(\Omega)$. For the results in this paper, we need to endow $\text{ca}(\Omega)$ with some topology. For the results in Section 4 it suffices to work with a distance, $d$, under which the empirical distribution (defined below) converges to $P$.

For the results in Section 5 and beyond, however, it is convenient to have more structure on the distance function, and thus, we work with a distance of the form

$$\|P - Q\|_S \equiv \sup_{f \in S} \left| \int f(z) P(dz) - \int f(z) Q(dz) \right|$$

where $S$ is some class of Borel measurable and uniformly bounded functions (bounded by one). For instance, the total variation norm can be viewed as taking $S = \{1_A: A \text{ Borel}\}$ and its denoted directly as $\|\|_\text{TV}$; the weak topology over $\mathcal{P}(\Omega)$ is metrized by taking $S$ as the unit ball in the space of bounded Lipschitz functions, its norm is denoted directly as $\|\|_\text{Lib}$; see van der Vaart and Wellner [1996] for a more thorough discussion.

A parameter. A parameter on model $\mathcal{M}$ is a mapping $\psi : \mathcal{M} \to 2^{\Theta}$ with $(\Theta, \|\|_\Theta)$ being a normed space. By defining the parameter mapping as a set-valued function we allow for cases wherein a probability distribution does not uniquely pin down an element of the parameter set.
3.1 Regularization

Let $D \subseteq ca(\Omega)$ be the set of all discretely supported probability distributions. Let $P_n(\omega) \in D$ be the empirical distribution, where $P_n(\omega)(A) = n^{-1} \sum_{i=1}^{n} 1\{\omega: Z_i(\omega) \in A\}$ for any $A \subseteq \mathbb{R}^d$ Borel.\(^5\) As illustrated by our examples, in many situations — especially in non-/semi-parametric models — the parameter mapping might be either ill-defined (e.g., if $P_n \notin \mathcal{M}$) or ill-behaved when evaluated at the empirical distribution $P_n$, so it has to be regularized.

The following definition of regularization is based on the first part of the definition in Bickel and Li [2006] p. 7. In what follows, and with a slight abuse of notation, we define $d_\Theta(\cdot, \psi(P)) \equiv \inf_{\theta' \in \psi(P)} \|\cdot - \theta'\|_\Theta$.\(^6\)

**Definition 3.1.** Given a model $\mathcal{M}$, a regularization of the parameter mapping $\psi$ at $P \in \mathcal{M}$ is a sequence $\psi \equiv (\psi_k)_{k \in \mathbb{N}_0}$ such that

1. For any $k \in \mathbb{N}_0$, $\psi_k: \mathbb{D}_\psi \subseteq ca(\Omega) \to \Theta$ where $\mathbb{D}_\psi \supseteq \mathcal{M} \cup D$.

2. $\lim_{k \to \infty} d_\Theta(\psi_k(P), \psi(P)) = 0$.

Condition 1 ensures that $\psi_k(P_n)$ is well-defined and that it is a singleton for all $k \in \mathbb{N}_0$. Condition 2 ensures that, in the limit, the regularization approximates the original parameter mapping. While these conditions are general enough to encompass a wide array of commonly used methods, these conditions are not enough to obtain “nice” asymptotic properties of the regularized estimator such as consistency and asymptotic normality. In analogy to the standard asymptotic theory for $\psi$-based estimators, these properties will be obtained by essentially imposing different degrees of smoothness on the regularization.

3.2 Examples (cont.)

The following examples illustrate that the Definition 3.1 encompasses a wide array of commonly used methods.

**Example 3.1** (Integrated Square Density (cont.)). In this case $\Theta = \mathbb{R}$ and the model, $\mathcal{M}$, is given by the class of all probabilities $P$ over $\mathbb{R}$ that have a continuous density (with respect to Lesbegue) $p$ such that $p \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and

$$|p(x + t) - p(x) - p'(x)t| \leq C(x)t^\theta, \forall t, x \in \mathbb{R},$$  \(3\)

\(^5\)For each $i = 1, 2, ...$, $Z_i(\omega)$ is understood as the $i$–th component of $\omega$; we will typically omit $\omega$ from the notation.

\(^6\)The abuse of notation arises because $d_\Theta(\cdot, \psi(P))$ doesn’t satisfy the properties of a distance. However, it does satisfy triangle inequality, i.e., $d_\Theta(\theta, \psi(P)) \leq d_\Theta(\theta, \tau) + d_\Theta(\tau, \psi(P))$ for any $\tau, \theta \in \Theta$ which is the property we use throughout the paper.
with $C \in L^2(\mathbb{R})$ and $\varrho \geq 0$; this last restriction is commonly used in the literature, e.g. Bickel and Ritov [1988], Hall and Marron [1987] and Powell et al. [1989].

We consider a class of regularizations given by

$$P \mapsto \psi_k(P) = \int (\kappa h(k) * P)(x) P(dx), \forall k \in \mathbb{N}_0,$$

where $k \mapsto h(k) \in \mathbb{R}_{++}$ is such that $\lim_{k \to \infty} h(k) = 0$ and for all $h > 0$, $t \mapsto \kappa h(t) \equiv (1/h)\kappa(t/h)$ where $\kappa \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ is symmetric at 0, continuously differentiable over $\mathbb{R} \setminus \{0\}$ with bounded derivatives, and satisfies $\int \kappa(u) du = 1$. Depending on the form of $\kappa$, this regularization encompasses many estimators proposed in the literature. For instance, for $\kappa = \rho + \lambda(\rho - \rho \ast \rho)$ with some $\lambda \in \mathbb{R}$ and some pdf $\rho$, symmetric at 0 and smooth, it follows that:

1. For $\lambda = 0$, the implied estimator is $n^{-1} \sum_{i=1}^n \hat{p}_h(Z_i)$, with $z \mapsto \hat{p}_h(z) = \frac{1}{nh} \sum_{i=1}^n \rho((Z_i - z)/h)$.

2. For $\lambda = -1$, the implied estimator is $\int (\hat{p}_h(z))^2 dz$.

3. For $\lambda = 1$, the implied estimator is $\int (\hat{p}_h(z))^2 dz + 2(n^{-1} \sum_{i=1}^n \hat{p}_h(Z_i) - \int (\hat{p}_h(z))^2 dz)$.

The first two estimators are standard; the third estimator is inspired by the one considered in Newey et al. [2004]. The main difference between 1-2 and 3 is that in the latter the kernel $\kappa$ is a twicing kernel; i.e., $\kappa$ is of the form $\kappa = -\rho \ast \rho + 2\rho$ (cf. Newey et al. [2004]).

Condition 1 in Definition 3.1 holds since we can set $D_{\psi} = ca(\mathbb{R})$; Condition 2 follows from the next proposition, which establishes a bound for the approximation error.

**Proposition 3.1.** For any $k \in \mathbb{N}$ and any $P \in \mathcal{M}$,

$$|\psi_k(P) - \psi(P)| \leq h(k)^{\rho} E_{|\kappa|}[|U|^\rho] \times ||C||_{L^1}.$$

In the particular case that $\kappa$ is a twicing kernel, it follows

$$|\psi_k(P) - \psi(P)| \leq h(k)^{2\rho} (E_{|\kappa|}[|U|^\rho])^2 ||C||_{L^2}^2.$$

**Proof.** See Appendix B. □

This proposition reflects the fact that twicing kernel yields better bounds for the approximation error (cf. Newey et al. [2004]). △

**Example 3.2** (NPIV (cont.)). The model $\mathcal{M}$ is defined as the class of probabilities over $Z = (Y, W, X) \in \mathbb{R} \times [0, 1]^2$ with pdf with respect to Lebesgue, such that: $p_X = p_W = U(0, 1)$,

\footnote{Details of the claims 1-3 are shown in the Appendix B.}

\footnote{This restriction is mild and can be changed to accommodate discrete variables simply by requiring pdf’s with respect to the counting measure.}
second stage" regularization is summarized by an operator \( R_k \) such that, for any \( T_r \) regularized; e.g. when \( T_r \) regularization of \( T \) to regularize the "first stage parameters" — the operator \( T_r \) commonly used regularizations such as Tikhonov-/Penalization-based regularization (e.g., Newey and Powell [2003]) regularizations, and the second stage regularization encompasses \( \) and \( \) and the Series-Based (e.g., Ai and Chen [2003] and Hall and Horowitz [2005]) and the estimation problem needs to be regularized. First, we need to regularize the “first stage parameters” — the operator \( T_r \) and \( r_P \); second, given the regularization of \( T_r \) and \( r_P \), the inverse problem for finding \( \psi(P) \) typically needs to be regularized; e.g. when \( T_r \) is compact or when \( \psi(P) \) is not a singleton.

The regularization of the “first stage” is given by a sequence of mappings \( (T_{k,P},r_{k,P})_{k\in\mathbb{N}_0} \) such that, for any \( k \in \mathbb{N}_0 \), \( T_{k,P} : \Theta \subseteq L^2([0,1]) \rightarrow L^2([0,1]) \) and \( r_{k,P} \in L^2([0,1]) \). The “second stage” regularization is summarized by an operator \( R_{k,P} : L^2([0,1]) \rightarrow L^2([0,1]) \) for which

\[
\psi(P) = \{h \in L^2([0,1]) : r_P = T_P[h]\}.
\]

To illustrate our method, we consider the estimation of a linear functional of \( \psi(P) \) of the form \( \gamma(P) \equiv \int \pi(w)\psi(P)(w)dw \) for some \( \pi \in L^2([0,1]) \), which by the Riesz representation theorem covers any linear bounded functional on \( L^2([0,1]) \).

It is well-known that the estimation problem needs to be regularized. First, we need to focus on the objects of interest that are \( h \) and \( P \); it implies that \( L^2([0,1],p_X) = L^2([0,1],p_W) = L^2([0,1]) \) which simplifies the derivations.9\footnote{To restrict the support to [0, 1] is common in the literature (e.g. Hall and Horowitz [2005]). At this level of generality, one can always re-define \( h \) as \( h \circ F_W^{-1} \) so that \( p_W = U(0,1) \); of course this will affect the smoothness properties of \( h \). The restriction \( p_X = U(0,1) \) is really about \( p_X \) being known, since in that case, one can always take \( F_X(X) \) as the instrument.}

The restriction (2) is what defines an IV non-parametric model. It implies that for any \( P \in \mathcal{M} \), \( r_P(\cdot) \equiv \int yP_{Y|X}(dy,\cdot) \) is well-defined and belongs to the range of the operator \( T_P : \Theta \subseteq L^2([0,1]) \rightarrow L^2([0,1]) \) given by \( T_P[h](\cdot) = \int h(w)p_{W|X}(w,\cdot)dw \) for any \( h \in L^2([0,1]) \).\footnote{Alternatively, we can define \( T_P[h](X) = \int h(w)p(w|X)dw \) and \( r_P(X) = \int yp(y|X)dy \). Depending on the type of the regularization one has at hand, it is more convenient to use one or the other.} From these conditions, it can be shown that for any \( P \in \mathcal{M} \),

\[
\psi(P) = \{h \in L^2([0,1]) : r_P = T_P[h]\}.
\]

We assume that the regularization structure \( (T_{k,P},r_{k,P},R_{k,P})_{k\in\mathbb{N}_0} \) is such that: (1) \( \lim_{k \to \infty} ||R_{k,P}[T_{k,P}[g]]-T_kT_P[g]||_{L^2([0,1])} = 0 \) pointwise over \( g \in L^2([0,1]) \); (2) \( \lim_{k \to \infty} ||R_{k,P}[T_{k,P}[r_{k,P}-r_P]]||_{L^2([0,1])} = 0 \). We relegate a more thorough discussion and particular examples of the regularization to Appendix B.1. For now, it suffices to note that the first stage regularization encompasses commonly used regularizations such as the Kernel-based (e.g., Darolles et al. [2011], Hall and Horowitz [2005]) and the Series-Based (e.g., Ai and Chen [2003] and Newey and Powell [2003]) regularizations, and the second stage regularization encompasses commonly used regularizations such as Tikhonov-/Penalization-based regularization (e.g.,

\[
E_{P_n}|Y|^2 < \infty \text{ and } ||p_{X|W}||_{L^\infty} < \infty; \text{ and (2) there exists } h \in \Theta \text{ where } \Theta \text{ is a subspace of } L^2([0,1],p_W) \text{ that satisfies 2.}
\]
Darolles et al. [2011] and Hall and Horowitz [2005]) and Series-based regularization (e.g., Ai and Chen [2003] and Newey and Powell [2003]). For these combinations, conditions (1)-(2) have been verified, under primitive conditions, in the literature; e.g. see Engl et al. [1996] Ch. 3-4.

It is easy to see that under conditions (1)-(2), the expression in 5 is in fact a regularization for $\psi(P)$ with $D_\psi \supseteq M \cup D$ being a linear subspace specified in expression 17 in Appendix B.1. From this result, it also follows that

$$\{\gamma_k(P) \equiv \int \pi(w)\psi_k(P)(w)dw\}_{k \in \mathbb{N}_0}$$

is a regularization for $\gamma(P)$ (in this case, $\Theta = \mathbb{R}$).

Example 3.3 (Bootstrap Estimation (cont.)). Let $\mathcal{M}$ be the class of Borel probability measures over $\mathbb{R}$ with non-negative mean, unitary second moment and finite third moments. Let $\Theta = \mathcal{P}(\mathbb{R})$ — the class of finite Borel probability measures over $\mathbb{R}$ — and let $||\theta - \theta'||_\Theta \equiv ||\theta - \theta'||_{L,ib}$ for any $\theta, \theta'$ in $\Theta$, where $Lip$ is the class of real-valued functions such that $|f(x) - f(x')| \leq ||x - x'||$. This norm is one of the notions of distance typically used to establish validity of the Bootstrap.

For any $n \in \mathbb{N}_0$, $\omega \in \mathbb{R}^\infty$ and $P \in \mathcal{M}$, let $T_n(\omega, P) = \sqrt{n}(\max\{n^{-1}\sum_{i=1}^n Z_i(\omega), 0\} - \max\{E_P[Z], 0\}) \in \mathbb{R}$. This is a regularization for $T_n(P)$, $\forall A$ Borel. The bootstrap approach can be interpreted as a regularization scheme in the following sense: For any $k \leq n$ and any $P \in \mathcal{M}$, let

$$\psi_k(P)(A) = P^\infty \left( \sqrt{k} \left( \max\{k^{-1}\sum_{i=1}^k Z_i, 0\} - \max\{E_P[Z], 0\} \right) \in A \right), \forall A \text{ Borel}.$$ 

In particular

$$\psi_k(P_n)(A) = P_n^\infty \left( \sqrt{k} \left( \max\{k^{-1}\sum_{i=1}^k Z_i^*, 0\} - \max\{n^{-1}\sum_{i=1}^n Z_i, 0\} \right) \in A \right), \forall A \text{ Borel}$$

where $(Z_i^*)_{i=1}^n$ is an i.i.d. sample drawn from $P_n$. It is easy to see that $(\psi_k(P))_k$ is indeed a regularization.

Example 3.4 (Regularized M-Estimators (cont.)). We impose the following assumptions over $(\mathcal{M}, \Theta, \phi)$: $\Theta$ is a subspace of $L^q$ where $L^q \equiv L^q(\mathbb{Z}, \mu)$ for any $q \in [1, \infty)$ and some

\[\text{Our definition of parameter mapping does not depend on } n \text{ explicitly; but it can easily be extended to be applied for each } n.\]

\[\text{Again, formally condition 2 in the Definition 3.1 does not depend } n, \text{ but it should be understood that it applies for each } n.\]
finite measure \( \mu \), and for \( q = \infty \), \( L^\infty = C(\mathbb{Z}, \mathbb{R}) \), and \( \theta \mapsto E_P[|\phi(Z, \theta)|] \) bounded and continuous, for all \( P \in \mathcal{M} \).

The regularization is lifted from Pouzo [2015] and is defined by: a sequence of linear subspaces of \( L^q \), \((\Theta_k)_k\), such that \( \text{dim}(\Theta_k) = k \) and the union is dense in \( \Theta \); a vanishing real-valued sequence \((\lambda_k)_k\) with \( \lambda_k \in (0, 1] \) and a lower-semi compact function \( \text{Pen} : L^q \to \mathbb{R}_+ \) such that, for each \( k \in \mathbb{N}_0 \)

\[
\psi_k(P) \equiv \arg \min_{\theta \in \Theta_k} E_P[|\phi(Z, \theta)|] + \lambda_k \text{Pen}(\theta)
\]
is a singleton for any \( P \in \mathcal{M} \cup \mathcal{D} \).

It is clear that condition 1 in Definition 3.1 holds; we now show by contradiction that Condition 2 also holds. Suppose that there exists a \( \epsilon > 0 \) such that \( d_\Theta(\psi_k(P'), \psi_k(P)) \geq \epsilon \) for all \( k \) large. Let \( \Pi_k \psi_k(P) \) be the projection of some element of \( \psi_k(P) \) onto \( \Theta_k \); eventually, \( d_\Theta(\Pi_k \psi_k(P), \psi_k(P)) \leq \epsilon \). Then, by optimality of \( \psi_k(P) \) and some algebra, for large \( k \), \( \inf_{\theta \in \Theta : d_\Theta(\theta, \psi_k(P)) \leq \epsilon} E_P[|\phi(Z, \theta)|] \leq E_P[|\phi(Z, \psi_k(P))|] + \{ E_P[|\phi(Z, \psi_k(P)) - \phi(Z, \Pi_k \psi_k(P))|] + \lambda_k \text{Pen}(\Pi_k \psi_k(P)) \} \). By continuity of \( E_P[|\phi(Z, \cdot)|] \), \( \lambda_k \downarrow 0 \) and convergence of \( \Pi_k \psi_k(P) \) to \( \psi_k(P) \) the term in the curly bracket vanishes as \( k \) diverges, leading to the contradiction \( \inf_{\theta \in \Theta : d_\Theta(\theta, \psi_k(P)) \leq \epsilon} E_P[|\phi(Z, \theta)|] \leq E_P[|\phi(Z, \psi_k(P))|] \). △

4 Consistency and convergence rates for Regularized Estimators

The results in this section extend the program started by Wolfowitz [1957] to regularized estimators. It also presents a data driven method for choosing tuning parameters that yields consistent estimators as well as providing an explicit rate of convergence.

We say a function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is a modulus of continuity if \( f \) is continuous, non-decreasing and such that \( f(t) = 0 \) iff \( t = 0 \).

**Definition 4.1 (Continuous Regularization).** A regularization \( \psi \) of \( \psi \) is continuous at \( P \in \mathbb{D}_\psi \) with respect to \( d \), if there exists a family of modulus of continuity \((\delta_k)_{k \geq 0}\) such that for any \( k \in \mathbb{N}_0 \)

\[
||\psi_k(P') - \psi_k(P)||_\Theta \leq \delta_k(d(P', P)) \tag{6}
\]
for any \( P' \in \mathbb{D}_\psi \).

The definition is equivalent to the standard “\( \delta/\epsilon \)”-definition of continuity because the modulus of continuity of \( \psi_k \), \( \delta_k \), can converge to 0 arbitrarily slowly. Moreover, the definition

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14The class \( C(\mathbb{Z}, \mathbb{R}) \) is the class of continuous and uniformly bounded real-valued functions on \( \mathbb{Z} \).

15A lower-semi compact function is one with lower contour sets compact.
does not impose any uniform bounds on $\delta_k$ across different $k \in \mathbb{N}_0$. While such restriction would simplify the proofs considerably, it is too strong for many applications. Recall that the regularization is introduced precisely due to the poor behavior of $\psi$ at $P$.

By imposing continuity with respect to a metric that ensures convergence of the empirical distribution $P_n$ to $P$, the definition readily implies consistency of $\psi_k(P_n)$ to $\psi_k(P)$ for any fixed $k \in \mathbb{N}_0$. However, in view of Condition (2), unless there exists a $k \in \mathbb{N}_0$ such that $\psi_k(P) = \psi(P)$ this result is of limited interest. In order to guarantee consistency to $\psi(P)$ in general cases, $k$ must be allowed to depend on $n$. That is, we need to extend convergence results that hold pointwise in $k$ to ones that hold over (diverging) sequences $(k(n))_n$; by means of a diagonalization argument the next theorem shows that such sequences exist.

**Theorem 4.1 (Consistency of Regularized Estimators).** Suppose a regularization (at $P$), $\psi$, is continuous with respect to a distance $d$ such that $d(P_n, P) = o_P(1)$. Then there exists a $(k(n))_{n \in \mathbb{N}}$ such that

$$d_\Theta(\psi_{k(n)}(P_n), \psi(P)) = o(1) \text{ and } \delta_{k(n)}(d(P_n, P)) = o_P(1),$$

and

$$d_\Theta(\psi_{k(n)}(P_n), \psi(P)) = o_P(1).$$

**Proof.** See Appendix C. \qed

By the triangle inequality it is easy to see that the distance between the regularized estimator, $\psi_{k(n)}(P_n)$, and the true parameter is bounded by the sum of two terms: the “sampling error”, $\delta_{k(n)}(d(P_n, P))$ and the “approximation error”, $d_\Theta(\psi_{k(n)}(P), \psi(P))$, which generalizes the well-known “variance-bias” trade-off present in many applications. This observation suggests a criterion to construct tuning sequence $(k(n))_n$ that yields a consistent estimators: For a given rate of convergence of $P_n$ to $P$ under $d$, i.e., a $(r_n)_n$ such that $d(P_n, P) = o_P(r_n)$, choose a diverging sequence $(k(n))_n$ such that $\delta_{k(n)}(r_n^{-1}) = o(1)$. An example of a choice that satisfies this criterion is what we call the oracle choice,

$$n \mapsto k^*(n) = \arg\min_{k \in \mathbb{N}_0} \{\delta_k(r_n^{-1}) + d_\Theta(\psi_k(P), \psi(P))\},$$

which minimizes the trade-off between the approximation and the sampling errors. This choice represents commonly used heuristics and it is a good prescription to obtain approximately optimal rate of convergences (Birge and Massart [1998]). However, often times it is unfeasible, since it relies on knowledge of the approximation error, which is typically unknown because it depends on features of the unknown $P$.

It is thus desirable to construct a choice of tuning parameter that sidestep this issue while still providing approximately the same rates of convergence. We now show that an adaptation
of the Lepski method (e.g. Pereverzev and Schock [2006]) provides a data-driven choice that satisfies these properties under some conditions. This method only uses knowledge of the sampling error, \( k \mapsto \delta_k(r^{-1}) \) — or a bound of it — which is typically known given the sample. The cost to pay for this feature is that we need to impose additional — but, in our view, mild — conditions over \( k \mapsto \delta_k(t) \).

To construct this choice of tuning parameter we first define a grid \( G_n \subseteq \mathbb{R}_+ \) for each \( n \). Also, we introduce a monotone envelope of the approximation error, given by a \( k \mapsto \bar{B}_k(P) \) continuous non-increasing function from \( \mathbb{R}_+ \) to itself such that \( \bar{B}_k \geq d_\Theta(\psi_k(P), \psi(P)) \) for all \( k \geq 0 \) and \( \lim_{k \to \infty} \bar{B}_k(P) = 0 \). We impose the following restrictions on the grid and \( k \mapsto \delta_k(t) \).

**Assumption 4.1.** There exists a \( C \in [1, \infty) \) and a \( N \) such that for all \( n \geq N \): (i) for any \( t \in \mathbb{R}_+ \) and two consecutive elements in \( G_n \), \( k' < k \),

\[
\delta_k(t) \leq C \delta_{k'}(t); 
\]

(ii) There exists a \( k \) in \( G_n \) such that \( \delta_k(r^{-1}) \geq \bar{B}_k(P) \).

**Assumption 4.2.** \( k \mapsto \delta_k(t) \) is non-decreasing and continuous for each \( t \geq 0 \).

The data-driven choice of tuning parameter is constructed as follows. For any \( r > 0 \) let \( \tilde{k}_n(r) = \min\{k: k \in F_n(r) \} \) a.s.-\( \mathbb{P}^\infty \), where

\[
r \mapsto F_n(r) \equiv \{k \in G_n: ||\psi_k(P_n) - \psi_{k'}(P_n)||_\Theta \leq 4\delta_{k'}(r^{-1}), \forall k' \geq k \ in \ G_n \}.
\]

**Theorem 4.2.** Suppose a regularization (at \( P), \psi, is continuous with respect to a distance \( d \), and that there exists a \( (r_n)_n \) such that \( d(P_n, P) = o_P(r_n) \). Suppose further that Assumption 4.2 holds and that \( (G_n)_n \) satisfies Assumption 4.1. Then

\[
d_\Theta(\psi_{\tilde{k}_n(r_n)}(P_n), \psi(P)) = O_P \left( \inf_{k \in G_n} \{\delta_k(r^{-1}) + \bar{B}_k(P)\} \right).
\]

**Proof.** See Appendix C.1. \( \square \)

The following corollary is a direct consequence and its proof is omitted.

**Corollary 4.1.** Suppose \( k \mapsto d_\Theta(\psi_k(P), \psi(P)) \) is continuous and non-increasing. Then under the conditions of Theorem 4.2, it follows

\[
d_\Theta(\psi_{\tilde{k}_n(r_n)}(P_n), \psi(P)) = O_P \left( \inf_{k \in G_n} \{\delta_k(r^{-1}) + d_\Theta(\psi_k(P), \psi(P))\} \right).
\]

\(^{16}\)The inequality only needs to hold over all \( t \) in some compact interval containing 0.

\(^{17}\)If \( F_n(r) \) is empty, define \( k_n(r) \) as \( \infty \).
Theorem 4.2 and its corollary show that under some conditions on the grid and the approximation error, our data-driven choice of tuning parameter achieves the same rate as the one corresponding to the “oracle” choice, provided the approximation error decreases monotonically.\(^{18}\) If the approximation is not monotonically decreasing, our data-driven choice can “under-smooth”, i.e., the sampling error can dominate the approximation error, even asymptotically. Whether this is the case, it depends on the tail behavior of the approximation error vis-a-vis \(k \mapsto \bar{B}_k(P)\).

The following corollary extends the “oracle” result in Theorem 4.2 to an un-restricted “oracle” result — one where the infimum is not restricted to the grid \(G_n\). Unsurprisingly, in order to achieve this result we need to ensure that the grid is not far from the choice of tuning parameter that balances the sampling error and the monotone envelope of the approximation error. A sufficient condition for this is to strengthen Assumption 4.1(ii) to ensure that the grid “surrounds this choice.

**Corollary 4.2.** Suppose the conditions of Theorem 4.2 hold and there exists a \(N\) such that for all \(n \geq N\), there exists a \(k \in G_n\) such that \(\delta_k(r_n^{-1}) \leq \bar{B}_k(P)\). Then

\[
d_{\Theta} \left( \psi_{k_n(P)}(P_n), \psi(P) \right) = O_P \left( \inf_{k \in \mathbb{R}_+} \{ \delta_k(r_n^{-1}) + \bar{B}_k(P) \} \right).
\]

**Proof.** See Appendix C.1.

### 4.1 Examples (cont.)

Example 4.1 illustrates how our results can be applied to derive the rate of convergence of the “k-out-of-n” bootstrap procedure, and example 4.2 provides primitive conditions for establishing continuity in M-estimation problems.

**Example 4.1** (Bootstrap Estimation (cont.).) Let \(\mathcal{W}(\cdot, \cdot)\) denote the Wasserstein distance over \(\mathcal{P}(\Omega)\), that is \(\mathcal{W}(P, Q) \equiv \inf_{\zeta \in H(P, Q)} \int |z - z'| \zeta(dz, dz')\), where \(H(P, Q)\) is the set of Borel probabilities over \(\Omega^2\) with marginals \(P\) and \(Q\). The following proposition suggests the form of \(\delta_k\) under which the regularization is continuous under \(\mathcal{W}\).

**Proposition 4.1.** For any \(k \in \mathbb{N}\), \(\|\psi_k(P) - \psi_k(Q)\|_{\Theta} \leq 2\sqrt{k}\mathcal{W}(P, Q)\) for any \(P\) and \(Q\) in \(\mathcal{M} \cup \mathcal{D}\).\(^{19}\)

**Proof.** See Appendix C.2

\(^{18}\)The continuity restriction is technical to ensure that certain quantities are well-defined.

\(^{19}\)For the display, \(Z \sim P\) and \(Z' \sim Q\).
The previous results suggest \( t \mapsto 2\sqrt{k}t \) as a choice of \( \delta_k \) in Definition 4.1, and also suggests a natural distance over \( \mathcal{P}(\Omega) \), the Wasserstein distance. Theorem 1 in Fournier and Guillin [2015] (their results are applied with \( d = 1, p = 1 \) and \( q = 2 \)) shows that \( \mathcal{W}(P_n, P) = O_P(n^{-1/2}) \), i.e., \( r_n = l_n n^{-1/2} \) where \( (l_n)_n \) diverges arbitrary slowly. This result and our previous proposition illustrates that \( k(n) = n \) will not yield a good control for the estimation error, \( k(n) = o(n) \) is needed.

The other key component for our consistency result is the approximation error, \( ||\psi_{k(n)}(P) - \psi_n(P)||_\Theta \); the next proposition provides an estimate for it.

**Proposition 4.2.** For any \( \eta \in \mathbb{N} \) and \( k(n) \leq n \) and \( P \in \mathcal{M} \),
\[
||\psi_{k(n)}(P) - \psi_n(P)||_\Theta \leq 14k(n)^{-1/2}E_P[|Z|^3]
\]

**Proof.** See Appendix C.2 \( \square \)

In view of our results in Section 4, this proposition and proposition 4.1 imply that the regularized bootstrap procedure is consistent, i.e., \( ||\psi_{k(n)}(P_n) - \psi_n(P)||_\Theta = o_P(1) \) for any divergent sequence \( (k(n))_n \) such that \( k(n)/n = o(1) \). Since the approximation error term is decreasing in \( k \), it follows that our choice of tuning parameter achieves a rate of convergence given by \( \min_k \left\{ \sqrt{l_n k/n} + k^{-1/2}E_P[|Z|^3] \right\} = l_n^{1/4}n^{-1/4}E_P[|Z|^3] \).

We conclude by noting that Propositions 4.1 and 4.2 jointly with the bounds in Fournier and Guillin [2015] are non-asymptotic results, thereby providing a thorough description of the behavior of the k-out-of-n bootstrap. \( \triangle \)

**Example 4.2** (Regularized M-Estimators (cont.)). The following proposition shows that the regularization is continuous and more importantly it provides a “natural” choice of distance and illustrates the role of the regularization structure \( ((\lambda_k, \Theta_k)_k, Pen) \) and primitives \((\Theta, \phi)\) for determining the rate of convergence of the regularized estimator. Henceforth, let \((\theta, P, k) \mapsto Q_k(P, \theta) \equiv E_P[\phi(Z, \theta)] + \lambda_k Pen(\theta)\).

**Proposition 4.3.** For each \( \eta \in \mathbb{N}_0 \) and \( P \in \mathcal{M} \cup \mathcal{D} \),
\[
||\psi_k(P) - \psi_k(P')||_{L^\eta} \leq \Gamma_k^{-1}(d(P, P')), \quad \forall P' \in \mathcal{M} \cup \mathcal{D},
\]

where for all \( t > 0 \)
\[
\Gamma_k(t) = \inf_{s \geq t} \left\{ \min_{\theta \in \Theta_k} \frac{Q_k(P, \theta) - Q_k(P, \psi_k(P))}{s} \right\}
\]

and \( d(P, P') \equiv \max_{k \in \mathbb{N}} ||P - P'||_{S_k} \), where \( S_k \equiv \left\{ \frac{\phi(\cdot, \theta) - \phi(\cdot, \psi_k(P))}{||\theta - \psi_k(P)||_G} : \theta \in \Theta_k \right\} \).

**Proof.** See Appendix C.2 \( \square \)

\(^{20}\)At \( t = 0 \), we define \( \Gamma_k(0) = 0. \)
The proof applies the standard arguments due to Wald — for establishing consistency of estimators — to “strips” of the sieve set $\Theta_k$; by doing so, one improves the rates obtained from the standard Wald approach.\footnote{The idea behind splitting the sieve set into strips comes from Shen and Wong [1994].}

The proposition suggests the natural notion of distance over the space of probabilities, that is defined by the class of “test functions” given by $\phi(z, \theta)$. By imposing additional conditions on $\phi$ and $\Theta_k$ one can embed the class $S_k$ into well-known classes of functions for which one has a bound for the supremum of the empirical process $f \mapsto n^{-1} \sum_{i=1}^n f(Z_i) - EP[f(Z)]$, and thus bounds for $d(P_n, P)$. For instance, if $\theta \mapsto \frac{d\phi(z, \theta)}{dz}$ is Lipschitz uniformly in $z$, then by using the mean value theorem and some algebra it follows that $S_k \subseteq Lip$ for every $k$, and thus $d(P_n, P) = O_P(n^{-1/2})$ (see van der Vaart and Wellner [1996]).

The modulus of continuity, $\Gamma_k^{-1}$, is non-decreasing and is continuous over $t > 0$ (see the proof), and by definition $\Gamma_k(0) = 0$.\footnote{The “$max_{k\in\mathbb{N}}$” comes from the fact that $d$ cannot depend on $k$ in the definition of continuity.} Its behavior is determined by how well the criterion separates points in $\Theta_k$ relative to the norm $||.||_{L^q}$; the flatter $Q_k(P, \cdot)$ is around its minimizer, the larger $\Gamma_k^{-1}$. Importantly, even though $\Gamma_k(t) > 0$ for each $k$ (recall that $\psi_k(P)$ is assumed to be unique), as $k$ diverges, $\Gamma_k(t)$ may approach zero. This phenomena relates to the potential ill-posedness of the original problem, and will affect the rate of convergence of the estimator.

To shed some more light on the behavior of $\Gamma_k$ and on the potential ill-posedness, consider the case where, $q = 2$, $Q(P, \cdot)$ is strictly concave and smooth, and $Pen(.) = ||.||_{L^2}^2$. Since $\psi_k(P)$ is a minimizer, $Q_k(P, \cdot)$ behaves locally as a quadratic function, in particular $\Gamma_k(t) \geq 0.5(C_k + \lambda_k)t$ for some non-negative constant $C_k$ related to the Hessian of $Q(P, \cdot)$, and thus $\Gamma_k^{-1}(t) \leq (C_k + \lambda_k)^{-1}t$. If $C_k \geq c > 0$ then $\Gamma_k^{-1}(t) \leq t$; we deem this case to be well-posed as $||\psi_k(P') - \psi_k(P)||_{L^q} \leq d(P', P)$.\footnote{The “inf_{k\geq t}” in the definition of $\Gamma_k$ ensures that it is non-decreasing; it can be omitted if such property is not needed.} On the other hand, if $\lim \inf_{k\to\infty} C_k = 0$ then, while the previous bound for the modulus of continuity is not possible, the following bound $\Gamma_k^{-1}(t) \leq \lambda_k^{-1}t$ is. This case is deemed to be ill-posed and $||\psi_k(P') - \psi_k(P)||_{L^q} \leq \lambda_k^{-1}d(P', P)$.

Finally, under the conditions discussed in the previous paragraph, in the ill-posed case, Assumption 4.1 holds for a grid chosen as $\{1, \ldots, j(n)\}$ provided $\lim_{k\to\infty} \lambda_k/\lambda_{k+1} < \infty$ and that $j(n)$ is such that $\lambda_{j(n)} \asymp r_n^{-1}$. Assumption 4.2 holds if $k \mapsto \lambda_k$ is chosen to be decreasing and lower semi-continuous.\footnote{This case relates to the so-called identifiable uniqueness condition (see White and Wooldridge [1991]).} Under these conditions Theorem 4.2 delivers a choice of tuning parameter that achieves consistency and a rate of $\text{min}_{k} \{\lambda_k^{-1} \times r_n + \inf_{l \geq k} ||\psi_l(P) - \psi(P)||_{L^q}\}$.

\footnote{For the well-posed case these assumptions hold trivially.}
5 Asymptotic Linear Representations for Regularized Estimators

We derive asymptotic linear representations (ALR) for the regularized estimator. In the standard large sample theory for “plug-in estimators” ALR is commonly used as the basis for large sample distribution theory of the estimator and robustness and efficiency considerations. In this section, we pursue an analogous approach but for regularized estimators. In what follows, let $L_0^2(P) = \{ f \in L^2(P) : E_P[f] = 0 \}$ and let $\nu \equiv (\nu_k)_{k \in \mathbb{N}_0}$ where, for all $k \in \mathbb{N}_0$, $\nu_k : \mathbb{Z} \to \Theta$.

**Definition 5.1 (Asymptotic Linear Representation: ALR($\Xi, k$)).** We say that a regularization $\psi$ admits a (weak) asymptotic linear representation for $k \equiv (k(n))_{n \in \mathbb{N}}$ under $\Xi \subseteq \Theta^*$ at $P \in \mathbb{D}_\psi$ with influence $\nu$, if for all $(n, \ell) \in \mathbb{N} \times \Xi$, $\ell[\nu_{k(n)}] \in L_0^2(P)$ and

$$|\ell[\psi_{k(n)}(P_n) - \psi_{k(n)}(P)] - n^{-1} \sum_{i=1}^n \ell[\nu_{k(n)}](Z_i)| = o_P(n^{-1/2}\|\ell[\nu_{k(n)}]\|_{L^2(P)}). \quad (8)$$

If a regularization admits a ALR($\Xi, k$) then, in order to study its asymptotic behavior, it suffices to study the behavior of the empirical process $n^{-1/2} \sum_{i=1}^n \ell[\nu_{k(n)}](Z_i)$ for each $\ell \in \Theta^*$. Also, the reminder term is smaller than $n^{-1/2}\|\ell[\nu_{k(n)}]\|_{L^2(P)}$ as opposed to, say, $n^{-1/2}$, because the former is the proper order of the leading term, $n^{-1} \sum_{i=1}^n \ell[\nu_{k(n)}](Z_i)$. That is, the natural scaling in the ALR is given by $\sqrt{n}/\|\ell[\nu_{k(n)}]\|_{L^2(P)}$ as opposed to just $\sqrt{n}$; as the examples in Section 5.3 shows, in many situations $\lim_{k \to \infty} \|\ell[\nu_k]\|_{L^2(P)} = \infty$.

This definition captures the natural asymptotic representation needed when the parameter of interest is a vector-valued functional of $\psi(P)$; that is, when $\Xi$ is “small”, like the evaluation of functions at a point and integrated square density among others. In other cases, however, where the parameter of interest is infinite-dimensional this representation is too weak and a stronger notion is needed, which we develop in Section 6.

Finally, in situations where the parameter of interest is a non-linear functional, the asymptotic linear representation can be applied to the linear approximation of it and additional results will be needed to control the approximation error.

The differentiability of the regularization is the key property that allow us to derive ALR.
5.1 Differentiable Regularizations

For any set \( P \subseteq ca(\Omega) \) and any \( P \in ca(\Omega) \) a **curve at \( P \) in \( P \) is an isomorphism \( t \mapsto P \in \mathcal{P} \) such that \( P(0) = P \). Given any subset \( \mathcal{T} \subseteq (ca(\Omega), ||\cdot||_S) \), a **curve, \( P(\cdot) \), in \( P \) is tangent to \( \mathcal{T} \) at \( P \) if there exists a \( \dot{P}(0) \in \mathcal{T} \) such that

\[
\lim_{t \to 0} \frac{\|P(t) - P - \dot{P}(0)\|_S}{t} = 0.
\]

We typically denote a curve at \( P \) by \( t \mapsto P(t) \) and its derivative by \( \dot{P}(0) \). In what follows we focus our attention on curves in \( \mathcal{D}\psi \), since we need to ensure that \( \psi_k \) evaluated at a curve to be well-defined. We use \( C(P, \mathcal{T}) \) to denote the class of curves in \( \mathcal{D}\psi \) that are tangent to \( \mathcal{T} \) at \( P \).

**Definition 5.2** (Differentiable Regularization: \( \text{DIFF}(P(\cdot), \mathcal{T}) \)). *We say a regularization \( \psi \) is directionally differentiable at \( P \in \mathcal{D}\psi \) tangentially to \( \mathcal{T} \) under a curve \( P(\cdot) \in C(P, \mathcal{T}) \), if for any \( k \in \mathbb{N}_0 \), there exists a \( D\psi_k(P) : (\mathcal{T}, ||\cdot||_S) \to \Theta \) continuous and homogeneous of degree 1 such that

\[
\lim_{t \downarrow 0} \frac{\psi_k(P(t)) - \psi_k(P)}{t} = D\psi_k(P)[\dot{P}(0)].
\]

**Remark 5.1.** The definition of derivative is akin to the concept of Lie Derivative (see Lang [1996]), but it does not impose any linear structure on \( \mathcal{T} \) nor in \( D\psi_k(P) \), and \( t \) is restricted to be non-negative. This feature of the definition is analogous to the idea of directional derivative in Shapiro [1990]. It has been shown to be sufficient for showing the validity of the Delta Method (see Shapiro [1990]), and turns out to be enough to also carry out our analysis.\(^{26}\) Of course, the definition boils down to the “standard” notion of differentiability if \( \mathcal{T} \) and \( D\psi_k(P) \) are linear. △

5.2 Main Result

In order to obtain an asymptotic linear representation for the regularization, it is enough for the regularization to be differentiable under curves of the form \( t \mapsto P + t\sqrt{n}(P_n - P) \). In view of this fact, a reasonable tangent set to consider is \( \mathcal{T}_0(P) \equiv \text{Cone}(\mathcal{D} - \{P\}) \).\(^{27}\) Henceforth, we use \( \mathcal{L}(P, \mathcal{T}_0(P)) \) to denote the class of all curves of the form \( t \mapsto P + tQ \) for \( Q \in \mathcal{T}_0(P) \).

If the regularization is in fact differentiable under all linear curves, then the mapping \( z \mapsto \varphi_k(P)(z) \equiv D\psi_k(P)[\delta_z - P] \) suggests itself as a candidate for the influence function in the ALR. The next lemma shows that this candidate satisfies the necessary condition.

\(^{26}\)See also Fang and Santos [2014] and Cho and White [2017] for further references, examples and discussion.

\(^{27}\)For any set \( S \) of a vector space \( X \), \( \text{Cone}(S) \equiv \{a x : x \in S \text{ and } a \geq 0\} \).
Lemma 5.1. For each $k \in \mathbb{N}_0$ and each $\ell \in \Theta^*$, $\ell[\varphi_k(P)] \in L^2(P)$.

Proof. See Appendix D.

Non surprisingly, to obtain the ALR property in Definition 5.1 one needs a better control on the remainder of the linear approximation 9, because, while expression 9 implies that
\[
\sqrt{n}(\psi_k(P + n^{-1/2}Q) - \psi_k(P)) - D\psi_k(P)[Q] = o(1) \text{ pointwise on } Q \in \mathcal{T}_0(P),
\]
one needs this result to hold for a sequence, $(\sqrt{n}(P_n - P))_{n \in \mathbb{N}}$, in $\mathcal{T}_0(P)$. The next theorem formalizes this discussion.

Theorem 5.1. Suppose a regularization, $\psi$, is $DIFF(P(\cdot), \mathcal{T}_0(P))$ under all $P(\cdot) \in \mathcal{L}(P, \mathcal{T}_0(P))$ and suppose that, for each $k \in \mathbb{N}_0$ and $\ell \in \Xi$, there exists a $\eta_{k,\ell} : \mathbb{D}_\psi \times \mathbb{D}_\psi \to \mathbb{R}_+$ such that
\[
|\ell[\psi_k(P + tQ) - \psi_k(P) - tD\psi_k(P)[Q]]| \leq \eta_{k,\ell}(P + tQ, P), \forall t \geq 0 \text{ and } Q \in \mathcal{T}_0,
\]
and
\[
\sqrt{n}\eta_{k,\ell}(P_n, P) = o_P (||\ell[\varphi_k(P)]||_{L^2(P)}) \label{eq:10}
\]
Then there exists a diverging sequence $(k(n))_n$ for which $\psi$ admits a ALR($\Xi, k$).

Proof. See Appendix D.

This theorem shows existence of a sequence of tuning parameters for which ALR holds; below, in Section 5.4.1, we expand on this result by proposing a way for constructing such sequence. The proof of the Theorem is quite straightforward. First, it is shown that $\psi$ admits a ALR($\Xi, k$) for every constant sequence $(k(n))_n$; this claim follows from condition 10 and definition 5.2. The result then follows from a diagonalization argument.

Existence of $\eta_{k,\ell}$ is ensured by Definition 5.2, the non-trivial restriction it is imposed in condition 10; in Section 5.4.2 we provide sufficient conditions for it.

5.3 Examples (cont.)

Example 5.1 (Integrated Square Density (cont.)). The next proposition shows that Definition 5.2 is satisfied by our class of regularizations and it also establishes a bound for the remainder term which is used to verify condition 10.

Proposition 5.1. For any $P \in \mathcal{M}$, the regularization defined in 4 is $DIFF(P(\cdot), ca(\mathbb{R}))$ under any curve $P(\cdot) \in \mathcal{L}(P, ca(\mathbb{R}))$ with
\[
Q \mapsto D\psi_k(P)[Q] = 2 \int (\kappa_{h(k)} \ast P)(z) Q(dz)
\]
linear and continuous with respect to \( \| \cdot \|_{L^2} \). Also, for any \( Q \in ca(\mathbb{R}) \) and \( t \geq 0 \), \( \eta_k(P + tQ, P) = t^2 \int (\kappa_{h(k)} \ast Q)(z)Q(dz) \), and
\[
\eta_k(P_n, P) = O_P \left( \frac{\kappa_{h(k)}(0)}{n} + 2\| \kappa \|_{L^2} \sqrt{\| p \|_{L^\infty}} \right. \left. + \frac{\| p \|_{L^\infty}}{n\sqrt{h(k)}} \right) + \frac{2\| \kappa \|_{L^2} \sqrt{\| p \|_{L^\infty}}}{n^2\sqrt{h(k)}}.
\]

**Proof.** See Appendix D.2.

From the proposition it follows that the influence function is given by \( z \mapsto \varphi_k(P)(z) \equiv 2\{ (\kappa_{h(k)} \ast P)(z) - E_P[(\kappa_{h(k)} \ast P)(Z)] \} \), and satisfies \( \sup_k \| \varphi_k(P) \|_{L^2(P)} \leq 2\| p \|_{L^\infty(\mathbb{R})} \| \kappa \|_{L^1(\mathbb{R})} \) (see Lemma D.2 in Appendix D.2). Hence, the natural scaling in the ALR — if it holds — is \( \sqrt{n} \).

Proposition 5.1 also illustrates the role of \( \kappa_{h(k)}(0) \) in the convergence rates for this class of regularizations. By Proposition 5.7(1) (for which all conditions hold) and Theorem 5.1, regularizations with \( \kappa_{h(k)}(0) = O(1/h(k)) \) (e.g. cases 1-3) admit an ALR provided that \( h(k)^{-1} = o(\sqrt{n}) \), whereas regularizations with \( \kappa_{h(k)}(0) = 0 \) admit an ALR provided that \( h(k)^{-1} = o(n) \). For the latter case, the estimator is of the “leave-one-out”-type, i.e.,
\[
\psi_k(P_n) = n^{-2} \sum_{i \neq j} \kappa_{h(k)}(Z_i - Z_j).
\]

For instance, the “leave-one-out” version of the estimators discussed in cases 1-3 above, one has \( t \mapsto \kappa(t) = \rho(t) + \lambda(\rho(t) - \rho \ast \rho(t))1\{t \neq 0\} \) and thus\(^{28}\)

1’. For \( \lambda = 0 \), the implied estimator is \( n^{-2} \sum_{i \neq j} h^{-1}\rho((Z_i - Z_j)/h) \).

2’. For \( \lambda = -1 \), the implied estimator is \( n^{-2} \sum_{i \neq j} h^{-1}\rho((Z_i - Z_j)/h) \).

3’. For \( \lambda = 1 \), the implied estimator is \( \int (\hat{\rho}_h(z))^2dz + 2(n^{-1} \sum_{i=1}^{n} \hat{\rho}_h(Z_i) - \int (\hat{\rho}_h(z))^2dz) + \frac{\int (h^{-1}\rho(z/h))^2dz - 2h^{-1}\rho(0/h)}{n} \).

The estimator 1’ is essentially the one considered by Gine and Nickl [2008] (see also Powell and Stoker [1996]);\(^{29}\) the estimators in 2’-3’ are, to the best of our knowledge, new, although the estimator in 3’ is closely related to the one considered in Bickel and Ritov [1988].

Finally, Propositions 3.1 and 5.1 imply that, for cases 1-2, if \( q \leq 1 \), then there does not exist a \( (k(n))_{n \in \mathbb{N}} \) for which \( \psi_k(n)(P_n) - \psi(P) \) is root-n asymptotically normal centered at zero. However, for cases 1’-2’ — the “leave-one-out” type estimators — the situation is better and the statement is true only for \( q \leq 1/2 \). For cases 3 and 3’ the situation is different due to the better bound in the approximation error, and the cutoffs are \( q \leq 1/2 \)

\(^{28}\)Details of the claims 1’-3’ are shown in the Appendix B.

\(^{29}\)The only discrepancy is the scaling: Their estimator is scaled by \( n(n - 1) \) as opposed to \( n^{-2} \).
Example 5.2 (Regularized M-Estimators (cont.)). Recall that \( Q_k(\theta, P) = E_P[\phi(Z, \theta)] + \lambda_k \text{Pen}(\theta) \). We now present sufficient conditions over \((\mathcal{M}, \Theta, \phi)\) and the regularization structure that guarantees differentiability of the regularization. We restrict our attention to linear sieves, i.e., for each \( k \in \mathbb{N}_0 \), \( \Theta_k \) is the linear span of some basis functions \( \kappa^k \equiv (\kappa_j)_j=1 \). In view of the results in Example 4.2, we assume that there exists a positive vanishing sequence \((\delta_{k,n})_n\) for which, wpal-1-P, \( \psi_k(P_n) \in \Theta_k(\delta_{k,n}) \equiv \Theta_k \cap \Theta(\delta_{k,n}) \) where \( \Theta(\delta_{k,n}) \equiv \{ \theta \in L^q : \| \theta - \psi_k(P) \|_{L^q} \leq \delta_{k,n} \} \); henceforth, we can thus take \( \Theta_k(\delta_{k,n}) \) as the “relevant sieve” space. The next two assumptions impose smoothness restrictions on \((\phi, \text{Pen})\).

**Assumption 5.1.** (i) \( \text{Pen} \) is strictly convex and twice continuously differentiable; (ii) there exists a \( C_0 < \infty \) and a \( \varrho > 0 \) such that for any \( k \in \mathbb{N}_0 \),

\[
\sup_{h \in \Theta_k(\delta_{k,n})} \sup_{(v_1, v_2) \in \Theta_k^2} \frac{d^2 \text{Pen}(h)}{d\theta^2}[v_1, v_2] - \frac{d^2 \text{Pen}(\psi_k(P))}{d\theta^2}[v_1, v_2] \leq C_0 \| h - \psi_k(P) \|_{L^q}^\varrho.
\]

For the following assumption, let \( S_{1,k}(\delta_{k,n}) = \left\{ \frac{d\phi(z, \theta)}{d\theta}[v/\|v\|_{L^q}] : (v, \theta) \in \Theta_k \times \Theta_k(\delta_{k,n}) \right\} \) and let \( S_{2,k}(\delta_{k,n}) \) be defined analogously but using the second derivative; finally let \( \Phi_{r,k,n}(\cdot) = \sup_{f \in S_{r,k}(\delta_{k,n})} |f(\cdot)| \).

**Assumption 5.2.** (i) \( \theta \mapsto \phi(z, \theta) \) is convex and twice continuously differentiable; (ii) there exists a \( \mathcal{S} \subseteq L^\infty(\mathbb{Z}) \), such that for \( r \in \{1, 2\} \) and all \( (k, n) \in \mathbb{N}_0 \times \mathbb{N}_0 \), \( S_{r,k}(\delta_{k,n}) \subseteq \mathcal{S} \) and \( \Phi_{2,k,n} \in \mathcal{S} \); (iii) for any \( k \in \mathbb{N}_0 \) and any \( z \in \mathbb{Z} \),

\[
\sup_{h \in \Theta_k(\delta_{k,n})} \sup_{(v_1, v_2) \in \Theta_k^2} \frac{d^2 \phi(z, h)}{d\theta^2}[v_1, v_2] - \frac{d^2 \phi(z, \psi_k(P))}{d\theta^2}[v_1, v_2] \leq C_0 \| h - \psi_k(P) \|_{L^q}^\varrho.
\]

The class \( \mathcal{S} \) imposes restrictions on the first and second derivative of \( \phi \) as a function of \( z \), at least it requires boundedness of these functions and it is the relevant class for constructing the norm over \( ca(\mathbb{Z}) \).

Let \( \mathcal{T}_0 \equiv \cup_{k,n \in \mathbb{N}_0} \left\{ \zeta \in ca(\mathbb{Z}) : E_\zeta \left[ \left| \frac{d\phi(z, \theta)}{d\theta}[v/\|v\|_{L^q}] \right| \right] < \infty , \forall (v, \theta) \in \Theta_k \times \Theta_k(\delta_{k,n}) \right\} \), which is a linear subspace in \( ca(\mathbb{Z}) \). Assumption 5.2(ii) — \( \mathcal{S}_{1,k}(\delta_{k,n}) \subseteq \mathcal{S} \) for all \( k, n \) — ensures that elements of the form \( a(P_n - P) \) for \( a \geq 0 \), belong to \( \mathcal{T}_0 \).

**Proposition 5.2.** Suppose Assumptions 5.1 and 5.2 hold. Then, for each \( k \in \mathbb{N}_0 \), \( \psi_k \) is Frechet Differentiable with derivative given by,

\[
D \psi_k(P)[Q] = (E_Q[\nabla_k(P)(Z)])^T \Delta_k(P)^{-1} k^k, \quad \forall Q \in (\mathcal{T}_0, \| \cdot \|_S),
\]

and \( \varrho \leq 1/4 \), resp. These results are consistent with those in Bickel and Ritov [1988] and Hall and Marron [1987] among others. \( \triangle \)
where \( \Delta_k(P) \equiv E_P \left[ \frac{d^2 \phi(Z, \psi_k(P))}{d\theta^2} [k^k, k^k] \right] + \lambda_k \frac{d^2 P \psi_k(P)}{d\theta^2} [k^k, k^k] \in \mathbb{R}^{k \times k} \) and \( z \mapsto \nabla_k(P)(z) \equiv \frac{d \phi(z, \psi_k(P))}{d\theta} [k^k] \in \mathbb{R}^k \).

**Proof.** See Appendix D.3.

This result implies that the regularization is \( DIFF(P(.), T_0) \) for any linear curve, i.e., \( P(.) \in \mathcal{L}(P, T_0) \), and that the influence function in the ALR is given by \( z \mapsto \varphi_k(P)(z) \equiv (\nabla_k(P)(z) - E_P [\nabla_k(P)(Z)]) \Delta_k(P)^{-1} k^k \in \Theta_k \). For any, \( \ell \) a linear functional over \( L^q \),

\[
||\ell[\varphi_k(P)]||_{L^2(P)}^2 = (\ell[k^k])^T \Delta_k(P)^{-1} \Sigma_k(P) \Delta_k(P)^{-1} (\ell[k^k])
\]

where \( \Sigma_k(P) \in \mathbb{R}^{k \times k} \) is the covariance matrix of \( \nabla_k(P)(Z) \).\(^{30}\) Under our assumptions, for each \( k \), \( \Delta_k(P) \) is non-singular, so \( ||\ell[\varphi_k(P)]||_{L^2(P)}^2 < \infty \). But, whether this holds uniformly over \( k \) depends on the limit behavior of the eigenvalues \( \Delta_k(P)^{-1} \) and \( \Sigma_k(P) \) which may diverge as \( k \) diverges.\(^{31}\) By using \( ||\ell[\varphi_k(P)]||_{L^2(P)} \) as the scaling factor our approach adapts to either case, thus allowing the researcher to sidestep this discussion altogether.

Frechet differentiability of \( \psi_k \) already implies a bound for \( Q \mapsto \eta_{k,\ell}(P + tQ, P) \) that is uniform over \( ||.||_S \)-bounded sets. Unfortunately, this general result is silent about how this bound depends on \( (k, \ell) \). By using the Mean Value Theorem on the first derivative, the following proposition presents an explicit bound for \( \eta_{k,\ell} \).

**Proposition 5.3.** For any \( P \in \mathcal{M} \), any \( B_0 < \infty \) and any \( k \in \mathbb{N}_0 \):

\[
\eta_{k,\ell}(P + tQ, P) = t||Q||_S \sup_{t \in [0, 1]} ||\ell[D\psi_k(P + tQ) - D\psi_k(P)]||_*,
\]

for any \( \ell \in \Theta^* \), any \( Q \in \mathcal{Q} \equiv \{ \zeta \in ca(Z) : ||\zeta||_S \leq B_0 \} \) and any \( t \geq 0 \). Moreover, there exists a \( T \in (0, 1] \) such that

\[
||\ell[D\psi_k(P + tQ) - D\psi_k(P)]||_* \leq t \max\{ ||Q||_{S^2}, ||Q||_{S^2} ||\ell[k^k]|| \} \times \mathcal{C}(B_0, P, k)
\]

for any \( \ell \in \Theta^* \), any \( Q \in \mathcal{Q} \) and any \( t \leq T \), where \( \mathcal{C}(B_0, P, k) \) is given in expression 20 in Appendix D.3.\(^{33}\)

**Proof.** See Appendix D.3.

Admittedly the bound presented here might be loose, but it does hint at how \( k, \ell, t \) and \( ||Q||_S \) affect the reminder. For instance, if the class \( \mathcal{S} \) is P-Donsker, then Proposition 5.3, with \( Q = \sqrt{n}(P_n - P) \) — which is a.s.-\( P \) a bounded sequence (see Lemma D.1) — and \( t = n^{-1/2} \), implies Proposition 5.7(1); this in turn implies expression 10. \( \triangle \)

\(^{30}\)The expression \( \ell[k^k] \) should be understood as \( \ell \) applied to component-by-component to \( k^k \).

\(^{31}\)For instance, if \( E_P \left[ \frac{d^2 \phi(Z, \psi_k(P))}{d\theta^2} [k^k, k^k] \right] \) becomes singular or large \( k \), the maximal eigenvalue of \( \Delta_k(P)^{-1} \) diverges at rate \( \lambda_k^{-1} \).

\(^{32}\)Recall that the norm \( ||.||_* \) is the operator norm associated, in this case, to \( ||.||_{L^q} \).

\(^{33}\)\( T \) may depend on \( B_0, P \) and \( k \), but it does not depend on \( \ell \) or \( Q \).
In the NPIV example, it is not hard to see that the influence function of $\gamma$ will be given by $z \mapsto \int D\psi_k(P)[\pi](z) - E_P[D\psi_k(P)[\pi](Z)]$ provided $D\psi_k(P) : \mathcal{T}_0(P) \to L^2([0, 1])$ exists.\footnote{\cite{HallInoue2003}} In the next two example we characterize $D\psi_k(P)^*$ for two widely used regularizations: The Sieve-based and Penalization-based.

**Example 5.3** (NPIV (cont.): The sieve-based Case). We study the sieve-based regularization approach, which is constructed using two basis for $L^2([0, 1])$, $(u_k, v_k)_{k \in \mathbb{N}}$, and two indices $k \mapsto (J(k), L(k))$ such that\footnote{$u^k(x) \equiv (u_1(x), \ldots, u_k(x)), \quad v^k(w) \equiv (v_1(w), \ldots, v_k(w))$ and $Q_{uu} \equiv E_{Leb}[u^k(X)(u^k(X))^T], \quad Q_{uv} \equiv E_{Leb}[u^k(X)(v^k(W))^T]$ and $Q_{vv} \equiv E_{Leb}[v^k(W)(v^k(W))^T]$.}

\[
\begin{align*}
(g, x) & \mapsto T_{k,P}[g](x) = (u^{J(k)}(x))^T Q_u^{-1} E_P[u^{J(k)}(X)g(W)] \quad , \\
x & \mapsto r_{k,P}(x) = (u^{J(k)}(x))^T Q_u^{-1} E_P[u^{J(k)}(X)Y], \\
\mathcal{R}_{k,P} & = (\Pi_k^T T_{k,P} \Pi_k)^{-1}
\end{align*}
\]

where $\Pi_k : L^2([0, 1]) \to \text{lin}\{v^L\} \subseteq L^2([0, 1])$ is the projection operator, $g \mapsto \Pi_k[g] = (v^L)^T Q_u^{-1} \int v^L(w)g(w)dw$.

The next proposition proves differentiable of the regularization $\gamma$ and provides the expression for the derivative.

**Proposition 5.4.** For any $P \in \mathcal{M}$, the sieve-based regularization $\gamma$ is $\text{DIFF}(P(.), \mathcal{D}_\psi)$ under all $P(.) \in \mathcal{L}(P, \mathcal{D}_\psi)$ and with $D\gamma_k(P)[\zeta] = \int D\psi_k(P)[\pi](z)\zeta(dz)$, where

\[
D\psi_k(P)^*[\pi](y, w, x) = (y - \psi_k(P)(w))(u^{J(k)}(x))^T Q_u^{-1} Q_w(Q_u^{-1} Q_w)^{-1} E_{Leb}[v^L(W)\pi(W)]
\]

\[+ \{E_P[(\psi_id(P)(W) - \psi_k(P)(W))(u^{J(k)}(X))^T]Q_u^{-1} u^{J(k)}(x)
\]

\[\times (v^L(w))^T (Q_u^{-1} Q_w)^{-1} E_{Leb}[v^L(W)\pi(W)]\}.
\]

\[
(11)
\]

**Proof.** See Appendix D.4. \hfill \square

Even though expression for $D\psi_k(P)^*[\pi]$ may look cumbersome, it has an intuitive interpretation: It is identical to the influence function of the parameter $\int \theta^T v^L(w)\pi(w)dw$ where $\theta$ is the estimand of a **misspecified linear GMM model** where the “endogenous variables” are $v^L(W)$ and the “instrumental variables” are $u^{J(k)}(X)$; cf. Hall and Inoue [2003]. The first term in the RHS of expression 11 also has an intuitive interpretation: It is the influence function of the parameter $\int \theta^T v^L(W)\pi(w)dw$ but in **well-specified linear GMM model**.

The proposition implies that for the “fix-$k$” case, expression 11 is the proper influence function to be considered. However, one can ask whether as $k$ diverges, the second term (the one in curly brackets) in RHS of expression 11 can be ignored. To shed light on this matter,
it is convenient to use operator notation for expression 11:
\[
D\psi^\ast_k(P)[\pi](y, w, x) = T_{k, P}\mathcal{R}_{k, P}\Pi_k[\pi](x) \times (y - \psi_k(P)(w))
\]
\[
+ \mathcal{R}_{k, P}\Pi_k[\pi](w) \times T_{k, P}[\psi_{id}(P) - \psi_k(P)](x)
\]
(we derive this equality in expression 21 in Appendix D.4). The term \(T_{k, P}[\psi_{id}(P) - \psi_k(P)]\) is multiplied by \(\mathcal{R}_{k, P}\Pi_k[\pi]\), which is different to \(T_{k, P}\mathcal{R}_{k, P}\Pi_k[\pi]\) — the factor multiplying \((y - \psi_k(P)(w))\). If \(\pi \in \text{Range}(T_P)\) both multiplying factors converge to bounded quantities as \(k\) diverges. Thus, since \(T_{k, P}[\psi_{id}(P) - \psi_k(P)]\) vanishes, the first summand in the RHS of expression 11 “asymptotically dominates” the second one. This is framework considered in Ackerberg et al. [2014]. However, if \(\pi \notin \text{Range}(T_P)\) — and thus \(\gamma(P)\) is not root-estimable (see Severini and Tripathi [2012]) — the situation is more subtle and without additional assumptions it is not clear which term in expression 11 dominates. The reason is that the aforementioned multiplying factors will no longer converge to a bounded quantity, and moreover, the rate of growth of \(T_{k, P}\mathcal{R}_{k, P}\Pi_k[\pi]\) can can be dominated by the rate of \(\mathcal{R}_{k, P}\Pi_k[\pi]\).

For this last case of \(\pi \notin \text{Range}(T_P)\), the results closest to ours are those in Chen and Pouzo [2015] wherein the influence function for slower than root-n sieve estimators is derived. Their expression for the influence function is simpler than ours, but this arises from a different set of assumptions and, more importantly, a different approach that directly focus on expressions for “diverging \(k\)”. \(\triangle\)

**Example 5.4** (NPIV (cont.): The Penalization-based Case). We study the penalization-based regularization case given by
\[
(x, g) \mapsto T_{k, P}[g](x) \equiv \int \kappa_k(x' - x) \int g(w)P(dw, dx')
\]
\[
x \mapsto r_{k, P}(x) \equiv \int \kappa_k(x' - x) \int yP(dy, dx')
\]
\[
\mathcal{R}_{k, P} = (T^\ast_{k, P}T_{k, P} + \lambda_k I)^{-1}
\]
where \(\kappa_h(.) = h^{-1}\kappa(. / h)\) and \(\kappa\) is a smooth, symmetric around 0 pdf.

As opposed to the previous case, there is no obvious link to a “simpler” problem like GMM and thus it is not obvious a-priori what the influence function would be and what the proper scaling should be when \(\gamma(P)\) is not root-n estimable. Theorem 5.1 suggests \(D\psi^\ast_k(P)[\pi]\) and \(\sqrt{n/\text{Var}_P(D\psi^\ast_k(P)[\pi])}\) as the influence function and scaling factor resp.; the next proposition characterizes it.

**Proposition 5.5.** For any \(P \in \mathcal{M}\), the Penalization-based regularization \(\gamma\) is \(DIFF(P(.), D_\psi)\).
under all \( P(.) \in \mathcal{L}(P, \mathbb{D}_\psi) \) with \( D\gamma_k(P)[\cdot] = \int D\psi_k(P)^* [\pi](z) \zeta(dz) \), where
\[
D\psi_k^*(P)[\pi](y, w, x) = \mathcal{K}_k^2 T_P T_k^2 T_P + \lambda_k I)^{-1}[\pi](x) \times (y - \psi_k(P)(w))
\]
\[+ \lambda_k (T_P T_k^2 T_P + \lambda_k I)^{-1}[\pi](w) \times \mathcal{K}_k^2 T_P T_k^2 T_P + \lambda_k I)^{-1}[\psi_id(P)](x). \tag{13}
\]
where \( \mathcal{K}_k \) is the convolution operator \( g \mapsto \mathcal{K}[g] = \kappa_k \ast g \).

Proof. See Appendix D.4. \( \square \)

If \( \pi \in \text{Range}(T_P) \), then the variance term converges to \( ||T_P[v^*](X)(Y - \psi(P)(W))||^2_{L^2(P)} = E_P[(T_P(T_P^* T_P)^{-1}[\pi](X))^2 E_P[(Y - \psi(P)(W))^2|X]] \) as \( k \) diverges, where \( v^* \equiv (T_P^* T_P)^{-1}[\pi]. \) The function \( (y, w, x) \mapsto T_P[v^*](x)(y - \psi(P)(w)) \) is the influence function one would obtained by employing the methods in Ai and Chen [2007] (with identity weighting) and \( v^* \) is the Riesz representer of the functional \( w \mapsto \int \pi(w)g(w)dw \) using their weak norm \( ||T_P[\cdot]||_{L^2(P)}. \)

If \( \pi \notin \text{Range}(T_P) \), the variance diverges, and, as in the sieve case, without additional assumptions it is not clear which term dominates the variance term \( \text{Var}_P(D\psi_k^*(P)[\pi]) \), as \( k \) diverges. This case illustrates how our results can be used to extend the results in Chen and Pouzo [2015] for irregular sieve-based estimators to more general regularization schemes. \( \triangle \)

### 5.4 Discussion

We discuss under which conditions the approximation error \( \sqrt{n}[\psi_{k(n)}(P) - \psi(P)] \) is asymptotically ignorable as well as sufficient conditions for expression 10 in Theorem 5.1.

#### 5.4.1 Undersmoothing

Theorem 5.1 implies that\(^{36} \)
\[
\frac{\sqrt{n}[\ell[\psi_{k(n)}(P_n)] - \ell[\psi(P)]]}{||\ell[\varphi_{k(n)}(P)]||_{L^2(P)}} = n^{-1/2} \sum_{i=1}^n \frac{\ell[\varphi_{k(n)}(P)](Z_i)}{||\ell[\varphi_{k(n)}(P)]||_{L^2(P)}}
\]
\[+ \frac{\sqrt{n}[\psi_{k(n)}(P) - \psi(P)]}{||\ell[\varphi_{k(n)}(P)]||_{L^2(P)}} + o_P(1) \tag{14}
\]
where \( (k(n))_n \) is either a constant sequence or the sequence in Theorem 5.1.

Expression 14 shows that the asymptotic behavior of the regularized estimator — once scaled and centered — is characterized by a term due to the approximation error and a stochastic term. Ideally, one would like to consider sequences \( (k(n))_n \) in the set in Theorem 5.1 for which the approximation term in expression 14 vanishes, but, unfortunately this result does not hold in general; e.g. see Example 5.1. This fact notwithstanding, the following result shows that if there exists a sequence of tuning parameter for which the ALR and the

\(^{36}\)Provided that \( ||\ell[\varphi_{k(n)}(P)]||_{L^2(P)} > 0. \)
asymptotic negligibility of the approximation error both hold, then the data-driven way of choosing tuning parameters described in Section 4 will also satisfy this property. To show this result we need to following assumption.

**Assumption 5.3.** For each \( \ell \in \Xi \), there exists a diverging real-valued sequence \((l_n)_{n \in \mathbb{N}}\) and a mapping \( k \mapsto C_{k,\ell} \in \mathbb{R}_+ \) such that, uniformly over all diverging sequences \((k(n))_n\):

1. Uniformly over all \((k'(n))_n\) such that \(k'(n) \leq k(n)\),
   \[
   \sqrt{n} \left| \ell \left[ D\psi_{k'(n)}(P_n) - D\psi_{k(n)}(P_n) \right] \right/ C_{k(n),\ell} = o_P(l_n^{-1}).
   \]
2. \( \sqrt{n} \left( \frac{\kappa_{k(n),\ell}(P_n, P) - D\psi_{k(n)}(P_n) - D\psi_{\tilde{k}(n)}(P_n)}{C_{k(n),\ell}} \right) = o_P(l_n^{-1}) \) and \( \frac{\|D\phi_{k(n)}(P)\|_{L^2(P)}}{C_{k(n),\ell}} \leq 1 \).
3. \( k \mapsto C_{k,\ell} \) is non-decreasing.
4. There exists a \( C \in \mathbb{R}_+ \) such that for any \( k \in \mathbb{N}_0 \), \( C_{k+1,\ell} \leq C_{k,\ell}(1 + C) \).

For all \( n \in \mathbb{N} \) and \( \ell \in \Xi \), let \( \tilde{k}(n) = \min \{ k : k \in F_{n,\ell} \} \), where

\[
F_{n,\ell} = \left\{ k \in \mathbb{N}_0 : \left| \ell \left[ \psi_k(P_n) - \psi_{k'}(P_n) \right] \right| \leq 5C_{k',\ell}/(l_n \sqrt{n}) \right\}, \quad \forall k' \geq k.
\]

Also, let \( k(n) = \min \{ k : \frac{C_{k,\ell}}{l_n} \geq B_{k,\ell}(P) \} \) where \( B_{k,\ell} = \sup_{l \geq k} |\ell[\psi_l(P) - \psi(P)]| \).

**Proposition 5.6.** Suppose all the conditions in Theorem 5.1 and Assumption 5.3 hold. Then, for any \( \ell \in \Xi \):

1. \( \sqrt{n} \left| \ell[\psi_{\tilde{k}(n)}(P_n) - \psi(P)] - n^{-1/2} \sum_{i=1}^{n} \ell[\varphi_{\tilde{k}(n)}(P)](Z_i) \right| \leq 8 \frac{C_{k(n),\ell}}{l_n}, \text{ wpa}1 - P. \)
2. If there exists a sequence \((j(n))_{n \in \mathbb{N}}\) such that \( \frac{C_{j(n),\ell}}{l_n} + \sqrt{n} |\ell[\psi_{j(n)}(P) - \psi(P)]| = o(1) \),
   then \((k(n))_{n \in \mathbb{N}}\) satisfies \( \frac{C_{k(n),\ell}}{l_n} = o(1) \).\(^{37}\)

**Proof.** See Appendix D.1.

The first statement provides bounds the error from a linear approximation under the data-driven choice of tuning parameter, \( \tilde{k}(n) \), in terms of the “infeasible” choice, \( k(n) \). The second statement shows that if there exists a sequence for which both the reminder of the ALR and the approximation error are “small”, then the same holds under \( k(n) \). Therefore, in this case

\[
\sqrt{n} \frac{\ell[\psi_{\tilde{k}(n)}(P_n) - \psi(P)]}{\|D\phi_{\tilde{k}(n)}(P)\|_{L^2(P)}} = n^{-1/2} \sum_{i=1}^{n} \ell[\varphi_{\tilde{k}(n)}(P)](Z_i) + o_P(1),
\]

\(^{37}\)By construction of \( k(n) \), if \( \frac{C_{k(n),\ell}}{l_n} = o(1) \), it automatically follows that \( \sqrt{n}|\ell[\psi_{k(n)}(P) - \psi(P)]| = o(1) \).
provided that \( \liminf_{n \to \infty} || \ell[\varphi_k(n)(P)] ||_{L^2(P)} > 0 \). That is, the asymptotic distribution of 
\[ \frac{\sqrt{n} \ell[\varphi_k(n)(P)], P - \psi(P)}{|| \ell[\varphi_k(n)(P)] ||_{L^2(P)}} \]

is given that of \( n^{-1/2} \sum_{i=1}^{n} \frac{\ell[\varphi_k(n)(P)](Z_i)}{|| \ell[\varphi_k(n)(P)] ||_{L^2(P)}} \), which under Lindbergh-Feller type conditions, will be an standard Gaussian.

**Remark 5.2** (Discussion of the Assumption 5.3). From Theorem 5.1,
\[ \eta_{k, \ell}(P_n, P) = o_p(n^{-1/2} || \ell[\varphi_k(P)] ||_{L^2(P)}) . \]

Hence, the actual restriction implied by Assumption 5.3(2) is, first, uniformity over \( k \) of \( o_p(.) \) in the previous display, and second, existence of a bound for \( || \ell[\varphi_k(P)] ||_{L^2(P)} \) that does not depend on \( P \).

Assumption 5.3(1) essentially requires that elements in \((D\psi_k)_{k \in \mathbb{N}_0}\) are not “too far apart”. Since \( || nD\psi_k(P)[P_n - P] || = O_P(|| \ell[\varphi_k(P)] ||_{L^2(P)}) \), in view of the previous discussion, part (1) is implied by:

(1.a) \( || \ell[D\psi_k(n)(P)[P_n - P] - D\psi_k(n)(P)[P_n - P] || = O_P(|| \ell[D\psi_k(n)(P)[P_n - P] ||) \), and
(1.b) \( || \ell[\varphi_k(n)(P)] ||_{L^2(P)} || C_{k, \ell} = o(l_n^{-1}) . \)

Moreover, if \( k \mapsto || \ell[\varphi_k(P)] ||_{L^2(P)} \) is non-decreasing, then it is easy to see that the LHS in (1.a) is of order \( O_P(|| \ell[\varphi_k(n)(P)] ||_{L^2(P)}) \); hence (1.a) holds if \( \sqrt{n} \ell[D\psi_k(n)(P)[P_n - P] \) is of the same order as \( || \ell[\varphi_k(n)(P)] ||_{L^2(P)} \) wpa1-P. For instance, this is the case if \((\sqrt{n}D\psi_k(n)(P)[P_n - P])_{n \in \mathbb{N}}\) is Cauchy (e.g. Example 5.1).

Parts (3)-(4) of Assumption 5.3 are analogous to Conditions 1-2 in Proposition ??.

### 5.4.2 Sufficient Conditions to control the reminder term

The next proposition proposes several sufficient conditions for expression 10.

**Proposition 5.7.** For any \( k \in \mathbb{N}_0 \) and \( \ell \in \Theta^* \) such that \( 0 < || \ell[\varphi_k(P)] ||_{L^2(P)} \), the following statements are true:

1. If \( \sqrt{n} \eta_{k, \ell}(P_n, P) = o_P(1) \), then condition 10 holds.
2. If \( \eta_{k, \ell}(P_n, P) = o_P(|| \ell[D\psi_k(P)[P_n - P] ||) \), then condition 10 holds.

---

38 For the applications we have in mind, this restriction over the liminf is natural and non-binding. Our Proposition is not designed for cases where \( \lim_{n \to \infty} || \ell[\varphi_k(n)(P)] ||_{L^2(P)} = 0 \); this case can be handled separately — and rather easily — since both the approximation error and the rate of \( \eta_{k, \ell}(P_n, P) \) both decrease as \( k \) increases.

39 Again, the only real restriction of part (1.b) is to find a bound that does not depend on \( P \); the fact that \( C_{k, \ell} \) grows faster than \( || \ell[\varphi_k(n)(P)] ||_{L^2(P)} \) can always be achieved simply by “blowing up” such bound by, say, \( \log \log k(n) \).

40 As pointed out above, by the Markov inequality, \( \sqrt{n}D\psi_k(P)[P_n - P] = O_P(|| \ell[\varphi_k(P)] ||_{L^2(P)}) \), so the condition essentially requires \( || \ell[\varphi_k(P)] ||_{L^2(P)} \) to be the “sharp rate” for \( \sqrt{n}D\psi_k(P)[P_n - P] \).
3. Suppose the class $\mathcal{S}$ is P-Donsker. If $\limsup_{t \to 0} \sup_{Q \in K} t^{-1} \eta_{k, \ell}(P + tQ, P) = 0$ for any $K \subseteq \mathcal{T}_0(P)$ compact, then $\sqrt{n} \eta_{k, \ell}(P_n, P) = o_P(1)$.

Proof. See Appendix D.1.

The first and second statements are trivial. The last statement uses the fact that if $\mathcal{S}$ is P-Donsker, then, by an application of Skohorod Lemma (see Lemma D.1 in Appendix D), $(\sqrt{n}(P_n(\omega) - P))_n$ forms a compact set. Part (3) in the proposition essentially states that if $\psi_k$ is Hadamard differentiable for each $k$, then an ALR follows; this is akin to the standard results for $\psi$-based estimators (see Van der Vaart [2000]). For regularized estimators, however, it is important to have an explicit and tight bound for the reminder of the linear approximation as a function of the tuning parameter $k$, hence in some cases it is better to bound expression 10 in a case-by-case basis than to rely on general result like Hadamard differentiability.

6 Strong Asymptotic Linear Representations for Regularized Estimators

We now establish an analogous result to Theorem 5.1 but for a stronger notion than W-ALR($\Xi, k$), one that is better suited when the parameter of interest is infinite dimensional.

**Definition 6.1** (Strong Asymptotic Linear Representation: S-ALR($\Xi, k$)). We say that a regularization $\psi$ admits a strong asymptotic linear representation for $k \equiv (k(n))_n \in \mathbb{N}$ under $\Xi \subseteq \Theta^*$ at $P \in \mathcal{D}_\psi$ with influence $\nu$, if for all $(n, \ell) \in \mathbb{N} \times \Xi$, $\ell[\nu_{k(n)}] \in L_0^2(P)$ and

$$
\sup_{\ell \in \Xi} \left| \frac{\ell[\psi_{k(n)}(P_n) - \psi_{k(n)}(P)]}{\|\ell[\nu_{k(n)}]\|_{L^2(P)}} - n^{-1} \sum_{i=1}^n \frac{\ell[\nu_{k(n)}](Z_i)}{\|\ell[\nu_{k(n)}]\|_{L^2(P)}} \right| = o_P(n^{-1/2}).
$$

(15)

Clearly, this representation is stronger than W-ALR($\Xi, k$) and is designed to capture situations where W-ALR($\Xi, k$) is too weak; e.g. the parameter of interest is infinite-dimensional as in the examples below.

The next theorem shows that differentiable regularizations admits an S-ALR, provided that the reminder is appropriately controlled.

**Theorem 6.1.** Suppose all conditions in Theorem 5.1 hold, and

$$
\sup_{\ell \in \Xi} \eta_{k, \ell}(P_n, P) = o_P(1).
$$

(16)

Then there exists a diverging sequence $(k(n))_n$ for which $\psi$ admits a $S - ALR(\Xi, k)$. 

33
Theorem 6.1 (Density Estimation (cont.)). Consider the setup in Example 2.1 but now the parameter of interest is \( z \mapsto \psi(P)(z) = p(z) \) and the regularization is given by \( z \mapsto \psi_k(P)(z) = (\kappa_{1/k} \ast P)(z) \).\footnote{Since what matters is \( n \mapsto h(k(n)) \) the choice of \( h(k) = 1/k \) is actually WLOG.} The goal is to obtain an asymptotic linear representation \( \text{uniformly over} \ z \in \mathbb{R} \), specifically,

\[
\sup_{z \in \mathbb{R}} \left| \sqrt{n} \left( \frac{(\kappa_{1/k(n)} \ast P_n)(z) - p(z)}{\sqrt{Var_P(\kappa_{1/k(n)}(z - Z))}} \right) - n^{-1/2} \sum_{i=1}^{n} \frac{\kappa_{1/k(n)}(z - Z_i) - E_P[\kappa_{1/k(n)}(z - Z)]}{\sqrt{Var_P(\kappa_{1/k(n)}(z - Z))}} \right| = o_p(1),
\]

for certain diverging sequences \( (k(n))_{n \in \mathbb{N}} \). From this representation, by invoking known limit theorems results one can derive the asymptotic distribution of \( \sup_{z \in \mathbb{R}} \left| \sqrt{n} \left( \frac{(\kappa_{1/k(n)} \ast P_n)(z) - p(z)}{\sqrt{Var_P(\kappa_{1/k(n)}(z - Z))}} \right) \right| \); see Bickel and Rosenblatt [1973].

We now illustrate how Theorem 6.1 can be used to achieve the asymptotic representation in the previous display. Let \( \Xi = \{ \text{All Delta Dirac measures over} \ \mathbb{R} \} \). Suppose that \( \Theta \) as is in Example 3.1, then \( \Xi \subseteq \Theta^* \) and \( \ell[p] = p(z_\ell) \) for any \( \ell = \delta z_\ell \). Since \( \psi_k \) is linear, it also follows that \( \ell[D\psi_k(P)[Q]] = \ell[\kappa_{1/k} \ast Q] = (\kappa_{1/k} \ast Q)(z_\ell) \) for any \( Q \in \mathcal{T}_0(P) = ca(\mathbb{R}) \), and \( \eta_{k,\ell}(\ldots) = 0 \). By Theorem 5.1, the regularization is \( W-\text{ALR}(k, \Xi) \) for any diverging sequence \( k = (k(n))_{n \in \mathbb{N}} \) with influence function given by \( z \mapsto \ell[\varphi_k(P)](z) = \kappa_{1/k}(z_\ell - z) - E_P[\kappa_{1/k}(z_\ell - Z)] \).

Hence, by Theorem 6.1 it follows that

\[
\sup_{z \in \mathbb{R}} \left| \sqrt{n} \left( \frac{(\kappa_{1/k(n)} \ast P_n)(z) - p(z)}{\sqrt{Var_P(\kappa_{1/k(n)}(z - Z))}} \right) - n^{-1/2} \sum_{i=1}^{n} \frac{\kappa_{1/k(n)}(z - Z_i) - E_P[\kappa_{1/k(n)}(z - Z)]}{\sqrt{Var_P(\kappa_{1/k(n)}(z - Z))}} \right| = O \left( \sup_{z \in \mathbb{R}} \left| \sqrt{n} \left( \frac{(\kappa_{1/k(n)} \ast P)(z) - p(z)}{\sqrt{Var_P(\kappa_{1/k(n)}(z - Z))}} \right) \right| \right) + o_p(1)
\]

for any \( (k(n))_{n \in \mathbb{N}} \).

Under our conditions over \( p \), it follows that

\[
\left| \sqrt{n} \frac{(\kappa_{1/k} \ast P)(z) - p(z)}{\sqrt{Var_P(\kappa_{1/k}(z - Z))}} \right| \leq \frac{\sqrt{n}}{k^{1+q}} E_{\|\kappa\|_2^q} \left| U \right|^{1+q} \frac{C(z)}{\sqrt{Var_P(\kappa_{1/k}(z - Z))}}.
\]

To give a more precise uniform bound for this term we need to estimate a lower bound for \( \sqrt{Var_P(\kappa_{1/k}(z - Z))} \). If \( C > p \), it can be proved that \( Var_P(\kappa_{1/k}(z - Z)) \geq k p(z) \| \kappa \|_2^2 + |p'(z)| \int k(u)^2 u du \) at least for large \( k \).\footnote{\( C \gg p \) is imposed to simplify the derivations in \( Var_P(\kappa_{1/k}(z - Z)) \).} Hence, \( \sup_{z \in \mathbb{R}} \left| \sqrt{n} \frac{(\kappa_{1/k(n)} \ast P_n)(z) - p(z)}{\sqrt{Var_P(\kappa_{1/k(n)}(z - Z))}} \right| = O \left( \frac{\sqrt{n}}{k(n)^{1+q}} \right) \), and thus for any \( (k(n))_{n \in \mathbb{N}} \) such that \( n^{1+q} = o(k(n)) \) the desired representation follows. \( \triangle \)
Example 6.2 (Regularized M-Estimator (cont.)). The goal is to provide asymptotic confidence bands for the regularized M-estimator. To do so, we assume that $\Theta \subseteq L^q \cap L^\infty$ for $q \in [1, \infty]$, thereby ensuring that the function-evaluation operation is well-defined; in Appendix E.1 we discuss the role of this assumption and offer an alternative procedure when it does not hold. Since $\Theta \subseteq L^q \cap L^\infty$, the set $\Xi \equiv \{ \mu \in ca(\mathbb{R}) : \mu = \delta_z, \text{ some } z \in \mathbb{Z} \}$ is a valid subset of the dual $\Theta^*$. By showing that expression 16 holds and applying Theorem 6.1 under $\Xi_\infty$ one obtains the following representation.

Lemma 6.1. Suppose Assumptions 5.1 and 5.2 hold and $S$ is P-Donsker. Then, for any $q \in [1, \infty]$ and any $k \in \mathbb{N}_0$,

$$
\left\| \sqrt{n} \frac{\psi_k(P_n) - \psi_k(P)}{\sigma_k(P)} - (\kappa^k)^T \Delta_k(P)^{-1} n^{-1/2} \sum_{i=1}^{n} \frac{\nabla_k(P)(Z_i)}{\sigma_k(P)} \right\|_{L^q} = o_P(1),
$$

where $\sigma_k(P) : \mathbb{Z} \to \mathbb{R}_+$ given by

$$
z \mapsto \sigma^2_k(P)(z) = (\kappa^k(z))^T \Delta_k^{-1}(P) \Sigma_k(P) \Delta_k^{-1}(P)(\kappa^k(z)).
$$

Proof. See Appendix E.1. \hfill \square

Lemma 6.1 shows that in order to characterize the asymptotic distribution of $\left\| \sqrt{n} \frac{\psi_k(P_n) - \psi_k(P)}{\sigma_k(P)} \right\|_{L^q}$ it suffices to characterize the one of $\left\| (\kappa^k)^T \Delta_k(P)^{-1} n^{-1/2} \sum_{i=1}^{n} \frac{\nabla_k(P)(Z_i)}{\sigma_k(P)} \right\|_{L^q}$. The following proposition accomplishes this by showing that the latter quantity can be approximated by a simple Gaussian process; the proof relies on coupling results (e.g. Pollard [2002]).

Proposition 6.1. Suppose Assumptions 5.1 and 5.2 hold. Then, for any $q \in [1, \infty]$ and any $k \in \mathbb{N}_0$,

$$
\left\| \sqrt{n} \frac{\psi_k(P_n) - \psi_k(P)}{\sigma_k(P)} - (\kappa^k)^T \frac{Z_k}{\sigma_k(P)} \right\|_{L^q} = O_P \left( \frac{\beta_k}{\sqrt{n}} \left( 1 + \frac{\log \left( \frac{\sqrt{n}}{\beta_k} \right)}{k} \right) + r_{n,k}^{-1} \right),
$$

where $Z_k \sim N(0, \Delta_k(P)^{-1}\Sigma_k(P)\Delta_k(P)^{-1})$, $\beta_k \equiv E[\|\Delta_k(P)^{-1}\nabla_k(P)(Z)\|^q]$ and $r_{n,k}^{-1} \equiv n^{-1/2} \varepsilon_k(P)^{-1} C(l_n, P, k)$ where $\varepsilon_k(P) \equiv \varepsilon_{\min} \left( \Delta_k^{-1}(P) \Sigma_k(P) \Delta_k^{-1}(P) \right)$ and $(l_n)_n$ is any slowly diverging sequence.\footnote{We note that for each $z \in \mathbb{Z}$, $\frac{(\kappa^k)^T(z)Z_k}{\sigma_k(P)} \sim N(0, I_k)$.}

Proof. See Appendix E.1. \hfill \square

Proposition 6.1 provides the basis for constructing confidence bands under general $L^q$ norms, and it also illustrates the type of restrictions on the sequence of tuning parameters that are needed to obtain these results. On the one hand, the sequence $(k(n))_n$ has to be such
that \((\beta_{k(n)}k(n) + \varepsilon_{k(n)}(P)^{-1}C(l_n, P, k(n))) / \sqrt{n} = o(1)\); on the other hand, it has to be such that the approximation error is negligible, i.e., \(\|\sqrt{n}\psi_{k(n)}(P) - \psi(P) / \sigma_{k(n)}(P)\|_{L^q} = o(1)\). If this last requirement does not hold, then Proposition 6.1 only provides a result for obtaining confidence bands of the “pseudo-true” parameter \(\psi_{k(n)}(P)\), which may or may not be of interest in certain applications.

Provided the aforementioned conditions hold and one has consistent estimators of \(\sigma_{k(n)}(P)\), \(\Sigma_{k(n)}(P)\) and \(\Delta_{k(n)}(P)\), the asymptotic distribution of \(\sqrt{n} \psi_{k(n)}(P) / \sigma_{k(n)}(P)\) can be approximated by the distribution of \(\|\frac{(\kappa^h)^T \varepsilon_k}{\sigma_k(P)}\|_{L^q}\).

Belloni et al. [2015] obtained analogous results for the \(L^\infty\)-norm in a linear regression model. Our methodology extends these results in two directions: general \(M\)-estimation problems and general \(L^q\) norms.

7 Conclusion

Our results extend the scope of the existing large sample theory described in the introduction to regularized estimators for which “plug-in” is an special case, but, at the same time, they also suggest that the large sample theory for regularized estimators does not constitute a large departure from the existing large sample theory. In the sense that both are based on local properties of the mappings used for constructing the estimators. This last observation indicates that other large sample results developed for “plug-in” estimators can also be extended to the more general setting of regularized estimators; e.g., estimation of the asymptotic variance of the estimator and, more generally, inference procedure like the bootstrap. We view this as a potentially worthwhile avenue for future research.

References


A Extension to General Stationary Models

We now briefly discuss how to extend our theory to general stationary models. In this case a model is a family of stationary probability distributions over $\Omega^\infty$, i.e., a subset of $\mathcal{P}(\Omega^\infty)$ (the set of stationary Borel probability distributions over $\Omega^\ast$).

Let $P$ denote the marginal distribution over $Z_0$ corresponding to $P \in \mathcal{P}(\Omega^\infty)$ (by stationarity, the time dimension is irrelevant). For a given model $\mathcal{M}^\infty$, let $\mathcal{M}$ denote the set of marginal probability distribution over $Z_0$ corresponding to $\mathcal{M}^\infty$. A parameter on model $\mathcal{M}^\infty$ is a mapping from $\mathcal{M}$ to $2^\Theta$. That is, we restrict attention to mappings that depend only on the marginal distribution. Our theory can also be extended to cases where $\psi$ depends on the joint distribution of a finite sub-collections of $\Omega^\infty$. Allowing for the mapping to depend on the entire $P^\infty$ is mathematical possible, but such object is of little relevance since it cannot be estimated from the data.

A regularization of a parameter $\psi$ is defined analogously and the (relevant) empirical distribution is given, for each $\omega \in \Omega^\infty$, by $P_n(A) \equiv n^{-1} \sum_{i=1}^n 1\{\omega : Z_i(\omega) \in A\}$ for any Borel set $A \subseteq \Omega$. 

Notation: Recall that $ca(X)$ for some set $X$ is the space of all Borel measures over $X$ endowed with the total variation norm, $||\mu||_{TV} = |\mu|(X)$ where $|.|$ is the total variation; the space $ca(X)$ is a Banach space (see Stroock [2010]).
Theorem 4.1 can be applied to this setup essentially without change, the difference with the i.i.d. setup lies on how to establish converges of \( P_n \) to \( P \) under \( d \). Similarly, the notion of differentiability (Definition 5.2) can also be applied without change. The influence function will also be given by \( z \mapsto \delta_{\psi_k(P^\infty)} \). The scaling, however, will be different, since

\[
E_{P^\infty} \left[ (\sqrt{n}\ell D\psi_k(P^\infty)[P_n - P])^2 \right] = E_{P^\infty} \left[ \left( \frac{n}{2} \sum_{i=1}^{n} \ell D\psi_k(P^\infty)[\delta_{Z_i} - P] \right)^2 \right]
\]

\[
= ||\ell[\varphi_k(P^\infty)]||_{L^2(P)}^2 + 2n^{-1} \sum_{i < j} E_{P^\infty} \left[ (\ell D\psi_k(P^\infty)[\delta_{Z_i} - P])(\ell D\psi_k(P^\infty)[\delta_{Z_j} - P]) \right]
\]

\[
= ||\ell[\varphi_k(P^\infty)]||_{L^2(P)}^2 + 2n^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \gamma_{j-i,k,\ell}(P^\infty)
\]

\[
\equiv ||\ell[\varphi_k(P^\infty)]||_{L^2(P)}^2 (1 + 2\Phi_{n,k,\ell}(P^\infty))
\]

where \( \gamma_{j,i,k,\ell}(P^\infty) \equiv E_{P^\infty} \left[ (\ell D\psi_k(P^\infty)[\delta_{Z_0} - P])(\ell D\psi_k(P^\infty)[\delta_{Z_j} - P]) \right] \) and

\[
\Phi_{n,k,\ell}(P^\infty) \equiv \sum_{i=1}^{n-1} \left( 1 - \frac{i}{n} \right) \gamma_{i,k,\ell}(P^\infty)
\]

Hence, the natural scaling is \( ||\ell[\varphi_k(P^\infty)]||_{L^2(P)} \sqrt{(1 + 2\Phi_{n,k,\ell}(P^\infty))} \) and not \( ||\ell[\varphi_k(P^\infty)]||_{L^2(P)} \) as in the i.i.d. case. We note that our theory, a priori, does not require \( \limsup_{n \to \infty} \Phi_{n,k,\ell}(P^\infty) = \infty \).

In view of the previous discussion, the relevant restriction in Theorem 5.1 is

\[
\sqrt{n} \frac{\eta_{k,\ell}(P_n, P)}{||\ell[\varphi_k(P^\infty)]||_{L^2(P)} \sqrt{(1 + 2\Phi_{n,k,\ell}(P^\infty))}} = o_{P^\infty}(1).
\]

An analogous amendment applies to Theorem 6.1.

B Appendix for Section 3

The next lemma formalize verify Claims 1-3 and 1'-3' in the text.

Lemma B.1. Claims 1-3 and 1'-3' in the text hold.

Proof. For each case 1-3 and 1'-3', we show that the \( \kappa \) yields the associated estimators and that \( \kappa \) is a valid choice in each case. Throughout, let \( \rho_h(.) = h^{-1}\rho(./h) \).

(1) Follows directly from the fact that \( \rho_h \ast P_n = \hat{p}_h \).
We first show note that, for all $t \in \mathbb{R},$

$$\rho_h \ast \rho_h(t) = \int \rho((t - x)/h)\rho(x/h)dx = h^{-1} \int \rho(u)\rho(t/h - u)du$$

$$= h^{-1} \rho \ast \rho(t/h)$$

where the last line follows from symmetry of $\rho.$ Hence, $\kappa = \rho \ast \rho.$ It follows that $\kappa$ is indeed a pdf, symmetric and continuously differentiable.

We now show the form of the implied estimator. We use the notation $\langle ., . \rangle$ to denote the dual inner product between $L^\infty(\mathbb{R})$ and $ca(\mathbb{R})$, so

$$\int (\rho_h \ast \rho_h \ast P)(x)P(dx) = \langle \rho_h \ast \rho_h \ast P, P \rangle = \int \int \rho_h(x - y)(\rho_h \ast P)(y)dyP(dx)$$

$$= \int (\rho_h \ast P)(y) \int \rho_h(y - x)P(dx)dy$$

$$= \langle \rho_h \ast P, \rho_h \ast P \rangle_{L^2}$$

where the second line follows by symmetry of $\rho.$ Since $\rho_h \ast P_n = \tilde{p}_h.$

(3) We already showed that $\rho_h \ast \rho_h(t) = h^{-1} \rho \ast \rho(t/h)$ for all $t.$ Thus, $\kappa_h(\cdot) = h^{-1} \kappa(\cdot/h)$ with $\kappa(\cdot) \equiv (-\rho \ast \rho(\cdot) + 2\rho(\cdot)).$ It follows that $\int \kappa(u)du = -\int \rho \ast \rho(u)du + 2\int \rho(u)du = 1.$ Smoothness follows from smoothness of $\rho.$ Finally, we note that one can write $\kappa(t)$ as $\{\rho \ast \rho(t) + 2(\rho(t) - \rho \ast \rho(t))\}.$ The expression of the estimator thus follows.

(1') Since $P$ does not have atoms, $Z_i = Z_j$ iff $i = j$ a.s.-$P^\infty.$ It follows that the estimator is given by $n^{-2} \sum_{i,j} \kappa_h(Z_i - Z_j) = n^{-1} \kappa_h(0) + n^{-2} \sum_{i\neq j} \kappa_h(Z_i - Z_j)$ a.s.-$P^\infty$ and the result follows since $\kappa(0) = 0.$

(2') The expression of the estimator follows from analogous calculations to those in 1'.

(3') By the calculations in (3)

$$\int (\tilde{p}_h(z))^2dz = \int (\rho_h \ast \rho_h \ast P_n)(x)P_n(dx) = n^{-2} \sum_{i \neq j} \rho_h \ast \rho_h(Z_i - Z_j) + n^{-1} \rho_h \ast \rho_h(0)$$

$$= n^{-2} \sum_{i \neq j} \rho_h \ast \rho_h(Z_i - Z_j) + n^{-1} \int (\rho_h(z))^2dz$$
where the last line follows by symmetry. Hence
\[
\int (\hat{p}_h(z))^2 dz - 2 \int (\hat{p}_h(z))^2 dz + n^{-1} \int (\rho_h(z))^2 dz = -n^{-2} \sum_{i \neq j} \rho_h \ast \rho_h (Z_i - Z_j)
\]
\[
= -n^{-2} \sum_{i,j} \rho_h (Z_i - Z_j) \times 1 \{Z_i - Z_j \neq 0\}
\]
where the last line follows because \( P \) does not have atoms, so \( Z_i = Z_j \iff i = j \ a.s.-P^\infty \).

Similarly,
\[
2n^{-1} \sum_{i=1}^n \hat{p}_h(Z_i) - 2\rho_h(0)/n = 2 \left( n^{-2} \sum_{i,j} \rho_h (Z_i - Z_j) - \rho_h(0)/n \right)
\]
\[
= 2n^{-2} \sum_{i \neq j} \rho_h (Z_i - Z_j)
\]
\[
= 2n^{-2} \sum_{i,j} \rho_h (Z_i - Z_j) \times 1 \{Z_i - Z_j \neq 0\}.
\]

Proof of Proposition 3.1. Since \( P \in M \) it admits a smooth pdf, \( p \), and thus
\[
\psi_k(P) - \psi(P) = \int \left( \int \kappa_{h(k)}(x - y)p(y)dy - p(x) \right) p(x)dx
\]
\[
= \int \left( \int \kappa(u)p(x - h(k)u)du - p(x) \right) p(x)dx.
\]
where the second line follows from a change in variables and the fact that \( \int \kappa_{h(k)}(x - y)p(y)dy = \int h(k)^{-1}\kappa((x - y)/h(k))p(y)dy \).

Under our assumptions, for any \( x, u \) reals, \( |p(x + u) - p(x) - p'(x)u| \leq C(x)|u|^\theta \). Thus
\[
|\psi_k(P) - \psi(P)| \leq \int \left| \int \kappa(u)(p(x - h(k)u)du - p(x) + p'(x)h(k)u)du \right| p(x)dx
\]
\[
+ \int \left| \int \kappa(u)(p'(x)h(k)u)du \right| p(x)dx
\]
\[
\leq h(k)^\theta \int |\kappa(u)||u|^\theta du \int C(x)p(x)dx
\]
because, by symmetry at 0, \( \int \kappa(u)u du = 0 \).
If $\kappa$ is a twicing kernel, $\kappa = -\rho \ast \rho + 2\rho$. Hence, since $P \in \mathcal{M}$ admits a pdf $p$,

$$\psi_k(P) - \psi(P) = -\langle \rho \ast \rho \ast p, p \rangle_{L^2} + 2 \langle \rho \ast p, p \rangle_{L^2} - \langle p, p \rangle_{L^2}$$

$$= -\langle \rho \ast p, p \rangle_{L^2} + 2 \langle \rho \ast p, p \rangle_{L^2} - \langle p, p \rangle_{L^2}$$

$$= -||\rho \ast p - p||_{L^2}^2.$$

By calculations analogous to the previous ones, it follows

$$||\rho \ast p - p||_{L^2}^2 = \int \left( \int \kappa(u)(p(x - h(k)u) - p(x))\,du \right)^2\,dx$$

$$\leq h(k)^2 \left( E_{|\kappa|}[|U|^2] \right)^2 ||C||_{L^2}^2.$$ 

\[\square\]

### B.1 Some Remarks on the Regularization Structure in the NPIV Example.

The general regularization structure, $(R_k, P, T_k, P, r_k, P)_{k \in \mathbb{N}_0}$, and conditions 1-2 are taken from Engl et al. [1996] Ch. 3-4. It is clear from the problem that

$$\mathbb{D}_\psi = \{ \mu \in ca(\mathbb{R} \times [0, 1]^2) : E_\mu[|Y|^2] < \infty \text{ and } E_\mu[|h(W)|^2] < \infty \forall h \in L^2([0, 1]) \}. \quad (17)$$

The next lemma presents useful properties of $\mathbb{D}_\psi$. The proof is straightforward and thus omitted.

**Lemma B.2.** (1) $\mathbb{D}_\psi \supseteq \mathcal{M} \cup \mathcal{D}$; (2) $\mathbb{D}_\psi$ is a linear subspace.

We now discuss canonical examples of regularizations methods for the first and second stage that we consider in this paper.

**First Stage Regularization.** For any $P \in \mathbb{D}_\psi$ and any $k \in \mathbb{N}_0$, we can generically write $r_{k,P}$ as

$$r_{k,P}(x) \equiv \int y \int U_k(x', x)P(dy, dx'), \quad \forall x \in [0, 1],$$

where $U_k \in L^\infty([0, 1]^2)$ symmetric. For instance, if

$$(x', x) \mapsto U_k(x', x) = h(k)^{-1}u((x - x')/h(k))$$

where $u$ is a symmetric around 0, smooth pdf and $h(k) = o(1)$, then $x \mapsto r_{k,P}(x) = \int y \int h(k)^{-1}u((x - x')/h(k))P(dy, dx')$, which is the the so-called kernel-based approach; e.g., for ill-posed inverse problems see Hall and Horowitz [2005] among others.
In the case one defined \( r_P \) using conditional probabilities, i.e., \( r_P(x) = \int y p(y|x) dy \). The kernel approach becomes

\[
x \mapsto r_{k,P}(x) = \int y \frac{h(k)^{-1} u((x - x')/h(k)) P(dy, dx')}{h(k)^{-1} u((x - x')/h(k)) P(dx')};
\]

(e.g. Darolles et al. [2011]). Observe that \( r_P \) is only defined for probability measures for which the pdf exists.

Another approach is to directly set

\[
(x', x) \mapsto U_k(x', x) = (u^k(x))^T Q_{uu}^{-1} u^k(x'),
\]

where \( (u_k)_{k \in \mathbb{N}_0} \) is some basis function in \( L^2([0, 1]) \) and \( Q_{uu} \equiv E_{Leb}[(u^k(X))(u^k(X))^T] \). In this case, \( x \mapsto r_{k,P}(x) = (u^k(x))^T Q_{uu}^{-1} E_P[u^k(X)Y], \) which is the so-called series-based approach; e.g., for ill-posed inverse problems see Ai and Chen [2003], Newey and Powell [2003] among others.

Analogously, one can define \( T_{k,P} \) as

\[
g \mapsto T_{k,P}[g](x) \equiv \int g(w) \int U_k(x', x) P(dw, dx'), \quad \forall x \in [0, 1],
\]

and the same observations above applied to this case.

The next lemma characterizes the adjoint for any \( P \in \mathcal{M} \) (i.e., \( P \) as a pdf \( p \)). In this case, we can view the regularization as an operator acting on \( T_P[g](x) = \int g(w)p(w, x)dw \), given by \( \mathcal{U}_k : L^2([0, 1]) \to L^2([0, 1]) \), where \( \mathcal{U}_k T_P[g](x) \equiv \int U_k(x', x) \int g(w)p(w, x')dwdx' \).

**Lemma B.3.** For any \( k \in \mathbb{N}_0 \) and any \( P \in \mathcal{M} \) (in particular, it admits a pdf \( p \)), the adjoint of \( T_{k,P} \) is \( T_{k,P}^* : L^2([0, 1]) \to L^2([0, 1]) \) and is given by

\[
f \mapsto T_{k,P}^*[f] = T^* \mathcal{U}_k[f].
\]

**Proof.** For any \( k \in \mathbb{N}_0 \) and any \( P \in \mathcal{M} \),

\[
\langle T_{k,P}[g], f \rangle_{L^2([0, 1])} = \int \langle \mathcal{U}_k T_P[g](x) \rangle f(x) dx
\]

\[
= \int g(w) \int \int U_k(x', x) f(x) dxp(w, x') dx' dw
\]

\[
= \langle g, T_P^* \mathcal{U}_k[f] \rangle_{L^2([0, 1])}
\]

for any \( g, f \in L^2([0, 1]) \). \( \square \)

If \( P \notin \mathcal{M} \), in particular if it does not have a pdf (with respect to Lebesgue), the adjoint operator is different; the reason being that \( T_P^* \) does not map onto a space of functions because \( P \) does not have a pdf. In this case, consider the operator \( A_P : L^2([0, 1]) \to ca([0, 1]) \) given by \( f \mapsto A_P[f](B) = \int_{w \in B} \int f(x)p(dw, dx) \) for any \( B \subseteq [0, 1] \) Borel. Note that
$|A_P[f](\cdot)| \leq \int |f(x)|P(dx) < \infty$ provided that $f \in L^2(P)$, which is the case for any $P \in \mathbb{D}_\psi$. The next lemma characterizes the adjoint in this case.

**Lemma B.4.** For any $k \in \mathbb{N}_0$ and any $P \in \mathbb{D}_\psi$, the adjoint of $T_{k,P}$ is given by

$$f \mapsto T_{k,P}^*[f] = A_P U_k[f].$$

Since $U_k \in L^2([0, 1]^2)$, $T_{k,P}^*[f]|([0, 1]) \lesssim ||f||_{L^2(P)}||U_k||_{L^2([0, 1]^2)}$ which is finite for $P \in \mathbb{D}_\psi$. So $T_{k,P}^*[f]$ in fact maps to $ca([0, 1])$.

**Proof.** For any $k \in \mathbb{N}_0$ and any $P \in \mathbb{D}_\psi$,

$$\langle T_{k,P}[g], f \rangle_{L^2([0, 1])} = \int \int g(w) \int U_k(x', x)P(dw, dx') f(x)dx$$

$$= \int g(w) \left( \int U_k(x', x)f(x)dx \right) P(dw, dx')$$

$$= \int g(w) \int U_k[f](x')P(dw, dx'),$$

for any $g, f \in L^2([0, 1])$.

One possibility to avoid the aforementioned technical issue with the adjoint operator is to define a regularization given by

$$g \mapsto T_{k,P}[g](x) \equiv \int g(w) \left\{ \int U_k(x', x)V_k(w', w)P(dw', dx') \right\} dw, \ \forall x \in [0, 1],$$

where $U_k \in L^\infty([0, 1]^2)$ symmetric. For example, if $V_k(w', w) = h(k)^{-1}v((w' - w)/h(k)$ (and $U_k$ is also given by the kernel-based approach), then in this case $(x, w) \mapsto W_k[P](x, w) \equiv \int U_k(x', x)V_k(w', w)P(dw', dx')$ is a pdf over $[0, 1]^2$ (regardless of whether $P$ has a pdf or not), and thus

$$f \mapsto T_{k,P}[f](w) = \int f(x)W_k[P](x, w)dx.$$

For instance, Hall and Horowitz [2005] considered a method akin to this.

In the case $T_P$ is defined as a conditional operator, one can consider the sieve-based approach for $U_k$ and $V_k(w', w) = (v^k(w))^T Q_{vv}^{-1} v^k(w')$ for some $(v_k)_{k \in \mathbb{N}_0}$ basis function in $L^2([0, 1])$. Then, in this case,

$$T_{k,P}[g](x) = (v^k(x))^T Q_{uu}^{-1} E_P[u^k(X)(v^k(W))^T Q_{vv}^{-1} E_{Leb}[v^k(W)g(W)]]$$

$$= (v^k(x))^T Q_{uu}^{-1} Q_{uv} Q_{vv}^{-1} E_{Leb}[v^k(W)g(W)]$$

$$= \int g(w) \left\{ \int (v^k(x))^T Q_{uu}^{-1} u^k(x')(v^k(w'))^T Q_{vv}^{-1} v^k(w)P(dw', dx') \right\} dw.$$
where \( Q_{vv} \equiv E_{Leb}[(v^k(W))(v^k(W))^T] \) and \( Q_{uv} \equiv E_P[u^k(X)(v^k(W))^T] \), and
\[
f \mapsto T^*_{k,P}[f](w) = (v^k(w))^T Q_{vv}^{-1} Q_{uv}^T E_{Leb}[u^k(X)f(X)]
\]
\[
= \int f(x) W_k[P](x, w) dx
\]
where \( W_k[P](x, w) = \int (u^k(x))^T Q_{uu}^{-1} u^k(x') (v^k(w'))^T Q_{vv}^{-1} v^k(w) P(dw', dx'). \)

**Second Stage Regularization.** For the second stage regularization, one widely used approach is the so-called Tikhonov- or Penalization-based approach, given by solving
\[
\arg\min_{\theta \in \Theta} \{ E_P[ (r_{k,P}(X) - T_{k,P}[\theta](X))^2 ] + \lambda_k ||\theta||^2_{L^2([0,1])} \}
\]
which is non-empty and a singleton. This specification implies that
\[
\mathcal{R}_{k,P} = (T^*_{k,P} T_{k,P} + \lambda_k I)^{-1},
\]
which is well-known to be well-defined, i.e., 1-to-1 and bounded for any \( \lambda_k > 0 \).

Another widely used approach is the sieve-based approach that consists on setting up
\[
\arg\min_{\theta \in \Theta_k} E_P[ (r_{k,P}(X) - T_{k,P}[\theta](X))^2 ]
\]
and specific the \( (\Theta_k)_k \) such that (1) \( \cup_k \Theta_k \) is dense in \( \Theta \) and \( \Theta_k \) has dimension \( k \), and (2) \( \arg\min \) exists and is a singleton. For instance if \( \Theta_k \) is convex, then a solution exists and is unique provided that \( \text{Kernel}(T_{k,P}|\Theta_k) = \{0\} \). In this case
\[
\mathcal{R}_{k,P} = (\Pi_k^* T^*_{k,P} T_{k,P} \Pi_k)^{-1},
\]
where \( \Pi_k \) is the projection onto \( \Theta_k \).

**Verification of Definition 3.1.** The next Lemma shows that given Conditions 1-2 listed in Example 3.2, \( (\psi_k(P))_{k \in \mathbb{N}_0} \) (and hence \( (\gamma_k(P))_{k \in \mathbb{N}_0} \)) is in fact a regularization.

**Lemma B.5.** Suppose Conditions 1-2 listed in Example 3.2 hold. Then \( (\psi_k(P))_{k \in \mathbb{N}_0} \) (and hence \( (\gamma_k(P))_{k \in \mathbb{N}_0} \)) is a regularization with \( \mathcal{D}_\psi \) given in 17.

**Proof.** Condition 1 in Definition 3.1 is satisfied by Lemma B.2. Regarding condition 2 in Definition 3.1, note that
\[
||\psi_k(P) - \psi(P)||_{L^2([0,1])} \leq ||\mathcal{R}_{k,P} T^*_{k,P} [r_{k,P} - r_P]||_{L^2([0,1])} + ||[(\mathcal{R}_{k,P} T^*_{k,P} - (T^*_{P} T_P)^{-1} T^*_{P})][r_P]||_{L^2([0,1])},
\]
which vanishes as \( k \) diverges by our conditions 1-2. 
\[\square\]
C Appendix for Section 4

The next lemma provides an useful “diagonalization argument” that is used throughout the paper.

Lemma C.1. Take a real-valued sequence \((x_{k,n})_{k \in \mathbb{N}_0, n \in \mathbb{N}_0}\) such that, for each \(k \in \mathbb{N}_0\), \(\lim_{n \to \infty} |x_{k,n}| = 0\). Then, there exists a mapping \(n \mapsto k(n)\) such that (a) \(\lim_{n \to \infty} |x_{k(n),n}| = 0\) and (b) \(k(n) \uparrow \infty\).

Proof. By pointwise convergence of the sequence \((x_{k,n})_n\), for any \(k \in \mathbb{N}_0\), there exists a \(n(k) \in \mathbb{N}_0\) such that \(|x_{k,n}| \leq 1/2^k\) for all \(n \geq n(k)\). WLOG we take \(n(k) + 1 > n(k)\).

We now construct the mapping \(n \mapsto k(n)\) as follows: For each \(l \in \mathbb{N}\), \(k(n) = l\) for all \(n \in \{n(l) + 1, \ldots, n(l+1)\}\), and for \(n \in \{0, \ldots, n(0)\}\) let \(k(n) = 0\). Since the cutoffs \(n(.)\) are increasing the set \(\{n(l) + 1, \ldots, n(l+1)\}\) is non-empty for each \(l\). For integer \(L > 0\), \(k(n) > L\) for all \(n \geq n(L) + 1\); so (b) follows.

To show (a), for any \(\epsilon > 0\) take \(l_\epsilon\) such that \(1/2^{l_\epsilon} \leq \epsilon\). Observe that for any \(n \geq n(l_\epsilon) + 1\), \(|x_{k(n),n}| \leq 1/2^{l_\epsilon} \leq \epsilon\) by construction of \((n, k(n))\). Thus, (a) follows.

Proof of Theorem 4.1. For any \(k \in \mathbb{N}_0\) and any \(n \in \mathbb{N}\), by triangle inequality and continuity it follows \(d_\Theta(\psi_k(P_n(\omega)), \psi(P)) \leq \delta_k(d(P_n(\omega), P)) + d_\Theta(\psi_k(P), \psi(P))\) a.s.-\(P^\infty\).

So it suffices to show that
\[
d_\Theta(\psi_{k(n)}(P), \psi(P)) = o(1) \text{ and } \delta_{k(n)}(d(P_n(\omega), P)) = o_P(1),
\]

Let \((\epsilon_n)_n\) be defined as
\[
n \mapsto \epsilon_n = \inf \left\{ \epsilon : \sup_{m \geq n} P^\infty(d(P_m(\omega), P) > \epsilon) \leq \epsilon \right\}.
\]

Since by assumption \(d(P_n(\omega), P) = o_P(1)\), it follows that \(\epsilon_n = o(1)\).

Then
\[
P^\infty(\delta_k(d(P_n(\omega), P)) \geq \delta) \leq 1 \{\delta_k(\epsilon_n) \geq \delta\} + \epsilon_n
\]
for any \(k \in \mathbb{N}_0\) and any \(\delta > 0\) and any \(n \in \mathbb{N}\). Since, for each \(k \in \mathbb{N}_0\), \(t \mapsto \delta_k(t)\) is continuous, it follows by Lemma C.1 that there exists a \(\langle k(n) \rangle_n\) such that \(\delta_{k(n)}(\epsilon_n) = o(1)\) so there exists a \(N_\delta\) such that
\[
P^\infty(\delta_{k(n)}(d(P_n(\omega), P)) \geq \delta) \leq \epsilon_n
\]
for any \(n \geq N_\delta\). Therefore, \(\delta_{k(n)}(d(P_n(\omega), P)) = o_P(1)\).

Since \(\langle k(n) \rangle_n\) diverges, it follows that
\[
d_\Theta(\psi_{k(n)}(P), \psi(P)) = o(1)
\]
as desired.
C.1 Proof of Theorem 4.2

In what follows and to simplify the notation, for any \( n \in \mathbb{N} \), let
\[
\tilde{k}(n) \equiv \tilde{k}_n(r_n) \quad \text{and} \quad F_n \equiv F_n(r_n)
\] (18)

For any \( n \in \mathbb{N} \), let
\[
k(n) = \min\{k \in \mathbb{R}^+: \delta_k(r_n^{-1}) \geq \bar{B}_k(P)\}.
\]

Lemma C.2. Suppose Assumption 4.2 holds. Then, for each \( n \in \mathbb{N} \), \( k(n) \) exists and solves
\[
delta_{k(n)}(r_n^{-1}) = \bar{B}_{k(n)}(P).
\]

Proof. For each \( n \) consider the set \( \{k \in \mathbb{R}^+: \delta_k(r_n^{-1}) \geq \bar{B}_k(P)\} \). The set is closed since \( k \mapsto \bar{B}_k(P) \) is continuous and \( k \mapsto \delta_k(r_n^{-1}) \) is lsc by Assumption 4.2. Since \( \delta_k(r_n^{-1}) \geq 0 \) and \( \bar{B}_k(P) = o(1) \), if follows that there exists a \( K(n) < \infty \) such that \( \delta_k(r_n^{-1}) \geq \bar{B}_k(P) \) for any \( k \geq K(n) \). Thus the set is non-empty and since we are minimizing the identity function, the minimizer exists and uniquely determined by
\[
delta_{k(n)}(r_n^{-1}) = \bar{B}_{k(n)}(P).
\]

The next two lemmas show some useful properties of the choice \( (k(n))_n \). The first one shows that balancing the sampling and approximation error yields the same rate as the “optimal” choice. The second lemma shows that the approximation term vanishes asymptotically under \( (k(n))_n \).

Lemma C.3. Suppose Assumption 4.2 holds. For any \( n \in \mathbb{N} \),
\[
delta_{k(n)}(r_n^{-1}) \leq \inf_{k \in \mathbb{R}^+} \{\delta_k(r_n^{-1}) + \bar{B}_k(P)\} \leq 2\delta_{k(n)}(r_n^{-1}).
\]

Proof. Observe that for any \( n \in \mathbb{N} \) and any \( \epsilon > 0 \), there exists \( k^*(n) \) such that
\[
\max\{\delta_{k^*(n)}(r_n^{-1}), \bar{B}_{k^*(n)}(P)\} - \epsilon \leq \inf_{k \in \mathbb{R}} \{\delta_k(r_n^{-1}) + \bar{B}_k(P)\} \leq 2\delta_{k(n)}(r_n^{-1})l_n.
\]

The upper bound follows from the fact that \( \inf_{k \in \mathbb{R}^+} \{\delta_k(r_n^{-1}) + \bar{B}_k(P)\} \leq \delta_{k(n)}(r_n^{-1}) + \bar{B}_{k(n)}(P) \) and Lemma C.2.

If \( k^*(n) \geq k(n) \), then \( \delta_{k^*(n)}(r_n^{-1}) \geq \delta_{k(n)}(r_n^{-1}) \) since \( k \mapsto \delta_k(t) \) is non-decreasing for any \( t \geq 0 \) by Assumption 4.2. On the other hand, if \( k^*(n) \leq k(n) \), then \( \bar{B}_{k^*(n)}(P) = \bar{B}_{k(n)}(P) = \delta_{k(n)}(r_n^{-1}) \) where the last equality follows from Lemma C.2. Therefore, for any \( n \in \mathbb{N} \) and any \( \epsilon > 0 \),
\[
\delta_{k(n)}(r_n^{-1}) - \epsilon \leq \inf_{k \in \mathbb{R}} \{\delta_k(r_n^{-1}) + \bar{B}_k(P)\} \leq 2\delta_{k(n)}(r_n^{-1}).
\]
Since $\epsilon > 0$ is arbitrary the result follows.

\textbf{Lemma C.4.} $\limsup_{n \to \infty} \delta_{k(n)}(r_n^{-1}) = 0$.

\textit{Proof.} Suppose not. Then there exists a sub-sequence $(n(j))_j$ and a $c > 0$ such that $\delta_{k(n(j))}(r_{n(j)}^{-1}) \geq c$ for all $j$. Clearly $(k(n(j)))_j$ must diverge, so $\bar{B}_{k(n(j))}(P) = o(1)$, but then $k(n(j))$ cannot be balancing both terms.

\textbf{Remark C.1.} Lemma C.4 provides a sufficient criteria for Assumption 4.1(iii). Let $(j(n))_n$ be such that $\liminf_{n \to \infty} \delta_{j(n)}(r_n^{-1}) > 0$. Since $\limsup_{n \to \infty} \delta_{k(n)}(r_n^{-1}) = 0$ (see Lemma C.4), it follows that $j(n) > k(n)$ for eventually. Thus $\mathcal{G}_n = \{0, \ldots, j(n)\}$ will satisfy Assumption 4.1(iii). △

The following lemma provides an upper bound for the quantity of interest in terms of an arbitrary element of the set $\mathcal{G}_n$ greater than $\tilde{k}(n)$.

\textbf{Lemma C.5.} For any $\epsilon > 0$, there exists a $N$ such that

$$P^\infty \left( d_\Theta \right( \psi_{k(n)}(\omega)(P_n(\omega)), \psi(P) \right) > 5\delta_k(r_n^{-1}) + \bar{B}_k(P) \right) \leq \epsilon$$

for all $k \in \mathcal{G}_n$, $k \geq \tilde{k}(n)$ and all $n \geq N$.

\textit{Proof.} Throughout the rest of the proof, we leave the dependence on $\omega$ implicit. For any $n \geq N$,

$$P^\infty \left( d_\Theta \left( \psi_{k(n)}(P_n), \psi(P) \right) > 5\delta_k(r_n^{-1}) + \bar{B}_k(P) \right) \\
\leq P^\infty \left( d_\Theta \left( \psi_k(P_n), \psi(P) \right) + ||\psi_{k(n)}(P_n) - \psi_k(P_n)||_\Theta > 5\delta_k(r_n^{-1}) + \bar{B}_k(P) \right) \\
\leq P^\infty \left( d_\Theta \left( \psi_k(P_n), \psi(P) \right) + ||\psi_{k(n)}(P_n) - \psi_k(P_n)||_\Theta > 5\delta_k(r_n^{-1}) + \bar{B}_k(P) \right) \\
\leq P^\infty \left( d_\Theta \left( \psi_k(P_n), \psi(P) \right) > \delta_k(r_n^{-1}) + \bar{B}_k(P) \right) \\
= T_n$$

where the second line follows by triangle inequality; the fourth one follows because $k \geq \tilde{k}(n)$, the definition of $\mathcal{F}_n$ implies that $||\psi_{k(n)}(P_n) - \psi_k(P_n)||_\Theta \leq 4\delta_k(r_n^{-1})$.

By triangle inequality, $d_\Theta(\psi_k(P_n), \psi(P)) \leq \delta_k(d(P_n, P)) + d_\Theta(\psi_k(P, P))$ for any $k \in \mathbb{N}_0$. This fact, the fact that $d(P_n, P) = o_P(r_n)$ and that $t \mapsto \delta_k(t)$ is non-increasing imply that there exists a threshold which WLOG we set to $N$ such that

$$P^\infty \left( d_\Theta \left( \psi_k(P_n), \psi(P) \right) \geq \delta_k(r_n^{-1}) + d_\Theta(\psi_k(P, P)) \right) \leq \epsilon$$

for all $n \geq N$. Hence, applying this to $T_n$ and using the fact that $\bar{B}_k(P) \geq d_\Theta(\psi_k(P), \psi(P))$ for all $k$, it follows that

$$P^\infty \left( d_\Theta \left( \psi_{k(n)}(P_n), \psi(P) \right) > 5\delta_k(r_n^{-1}l_n) + \bar{B}_k(P) \right) \leq 2\epsilon$$
for all \( n \geq N \). Since the \( \epsilon > 0 \) is arbitrary, the desired result is shown. \[
\]

For any \( n \in \mathbb{N} \), let \( [k(n)] \) be defined as the smallest element in \( \mathcal{G}_n \) greater than \( k(n) \), if such element exists. If not, i.e., if all elements of \( \mathcal{G}_n \) are smaller than \( k(n) \), let \([k(n)]\) be the largest element in \( \mathcal{G}_n \).

The following lemma bounds \( \delta_{[k(n)]}(r_n^{-1}) \). In particular, it implies \( \delta_{[k(n)]}(r_n^{-1}) \leq C \inf_{k \in \mathcal{G}_n} \{ \delta_k(r_n^{-1}) + \bar{B}_k(P) \} \).

**Lemma C.6.** Suppose Assumptions 4.1(i) and 4.2 hold.\(^{44}\) Then

1. If \([k(n)] \leq k(n)\), then \( \delta_{[k(n)]}(r_n^{-1}) \leq \inf_{k \in \mathbb{R}_+} \{ \delta_k(r_n^{-1}) + \bar{B}_k(P) \} \).

2. If \( [k(n)] \geq k(n) \), then \( \delta_{[k(n)]}(r_n^{-1}) \leq C \inf_{k \in \mathcal{G}_n} \{ \delta_k(r_n^{-1}) + \bar{B}_k(P) \} \).

3. Suppose there exists a \( j \in \mathcal{G}_n \) such that \( \delta_j(r_n^{-1}) \leq \bar{B}_j(P) \). Then \( \delta_{[k(n)]}(r_n^{-1}) \leq C \inf_{k \in \mathbb{R}_+} \{ \delta_k(r_n^{-1}) + \bar{B}_k(P) \} \).

**Proof.** (1) If \([k(n)] \leq k(n)\), then \( \delta_{[k(n)]}(r_n^{-1}) \leq \delta_k(n^{-1}) \) by monotonicity (Assumption 4.2). So the result follows from Lemma C.3.

(2) First observe that if \([k(n)] \geq k(n)\) then there exists at least one \( j \in \mathcal{G}_n \) such that \( j \geq k(n) \). Let \([k(n)]\) be the largest element in \( \mathcal{G}_n \) that is smaller than \( k(n) \); if all elements of \( \mathcal{G}_n \) are greater than \( k(n) \), let \([k(n)]\) be the smallest element.

We divide the proof of this part in two cases:

(2.a) \([k(n)] > k(n)\). This can only occur if \( \min \{ k : k \in \mathcal{G}_n \} > k(n) \), thus \([k(n)] = \min \{ k : k \in \mathcal{G}_n \} \geq [k(n)] = \min \{ k : k \in \mathcal{G}_n \} \). This necessarily implies that \([k(n)]\) is smaller than the (approximate) minimizer of \( k \mapsto \delta_k(r_n^{-1}) + \bar{B}_k(P) \) over \( \mathcal{G}_n \), denoted as \( m(n) \). Therefore by monotonicity (Assumption 4.2), \( \delta_{[k(n)]}(r_n^{-1}) \leq \delta_{m(n)}(r_n^{-1}) \) which in turn implies that \( \delta_{[k(n)]}(r_n^{-1}) \leq \inf_{k \in \mathcal{G}_n} \{ \delta_k(r_n^{-1}) + \bar{B}_k(P) \} \).

(2.b) \([k(n)] \leq k(n)\). Therefore by monotonicity (Assumption 4.2), \( \delta_{[k(n)]}(r_n^{-1}) \leq \delta_{k(n)}(r_n^{-1}) \). So by Lemma C.3, \( \delta_{[k(n)]}(r_n^{-1}) \leq \inf_{k \in \mathbb{R}_+} \{ \delta_k(r_n^{-1}) + \bar{B}_k(P) \} \leq \inf_{k \in \mathcal{G}_n} \{ \delta_k(r_n^{-1}) + \bar{B}_k(P) \} \). In addition, \([k(n)]\) and \([k(n)]\) are consecutive elements in \( \mathcal{G}_n \). To show this, note that if there are not, there exists a \( j \in \mathcal{G}_n \) such that \( [k(n)] < j < [k(n)] \). Either \( j \leq k(n) \) or \( k(n) \leq j \). The former violates the definition of \([k(n)]\) and the latter the definition of \([k(n)]\). So by contradiction, \([k(n)]\) and \([k(n)]\) are consecutive elements in \( \mathcal{G}_n \). By Assumption 4.1(i), \( C^{-1} \delta_{[k(n)]}(r_n^{-1}) \leq \delta_{[k(n)]}(r_n^{-1}) \) and thus the result follows.

(3) By our assumptions, if there exists a \( j \in \mathcal{G}_n \) such that \( \delta_j(r_n^{-1}) \leq \bar{B}_j(P) \), all \( k \leq j \) in \( \mathcal{G}_n \) also satisfy this property. Therefore, either all \( k \in \mathcal{G}_n \) are such that \( \delta_k(r_n^{-1}) \leq \bar{B}_k(P) \), which by monotonicity of the functions imply \( \max \{ k : k \in \mathcal{G}_n \} \geq k(n) \), or \( k(n) \in \mathcal{G}_n \). Either case rules out case 2.a above. However, cases 1 and (2.b) yield a bound of the form \( \delta_{[k(n)]}(r_n^{-1}) \leq C \inf_{k \in \mathbb{R}_+} \{ \delta_k(r_n^{-1}) + \bar{B}_k(P) \} \)

---

\(^{44}\)Assumption 4.1(i) is only used in the second statement.
Lemma C.6 suggests that is sufficient to use \( \delta_{[k(n)]}(r_n^{-1}) \) as a bound the distance between our estimator with \( \hat{k}(n) \) and the true parameter. To do this, by Lemma C.5, it suffices to show that \([k(n)]\) is less or equal than \( \hat{k}(n) \) and it “undersmooth” (at least wp1-\( P \)). For the latter condition we need to ensure that for all sufficiently large \( n \), there exists \( j \in \mathcal{G}_n \) such that \( \delta_j(r_n^{-1}) \geq \bar{B}_j(P) \), which is ensured by Assumption 4.1(ii). To show the former condition it suffices to show that \( [k(n)] \in \mathcal{F}_n \) wp1-\( P \) because \( \hat{k}(n) \) is minimal over the set. This is shown in the next Lemma which use the following property.

**Remark C.2.** Under Assumption 4.1(ii), by definition of \([k(n)]\) it follows \( \delta_{[k(n)]}(r_n^{-1}) \geq \bar{B}_{[k(n)]}(P) \). \( \triangle \)

**Lemma C.7.** Suppose Assumption 4.2 and 4.1 hold. Then, for any \( \epsilon > 0 \), there exists an \( N \) such that

\[
P^\infty([k(n)] \notin \mathcal{F}_n(\omega)) \leq \epsilon
\]

for any \( n \geq N \).

**Proof.** For any \( n \in \mathbb{N} \), let \( D_n \equiv \{\omega: d(P_n(\omega), P) \leq r_n^{-1}\} \). It follows that

\[
P^\infty([k(n)] \notin \mathcal{F}_n(\omega))
\leq P^\infty(\exists k \in \mathcal{G}_n: k > [k(n)] \cap ||\psi_k(P_n(\omega)) - \psi_{[k(n)]}(P_n(\omega))||_\Theta > 4\delta_k(r_n^{-1}))
\leq P^\infty(\{\exists k \in \mathcal{G}_n: k > [k(n)] \cap ||\psi_k(P_n(\omega)) - \psi_{[k(n)]}(P_n(\omega))||_\Theta > 4\delta_k(r_n^{-1})\} \cap D_n)
\]

By triangle inequality

\[
||\psi_k(P_n(\omega)) - \psi_{[k(n)]}(P_n(\omega))||_\Theta \leq \delta_k(d(P_n(\omega), P)) + \bar{B}_k(P) + \delta_{[k(n)]}(d(P_n(\omega), P)) + \bar{B}_{[k(n)]}(P)
\]

a.s.-\( P \). Under \( D_n \) and the fact that \( k \rightarrow \delta_k(t) \) is non-decreasing (Assumption 4.2), this inequality implies \( ||\psi_k(P_n(\omega)) - \psi_{[k(n)]}(P_n(\omega))||_\Theta \leq \delta_k(r_n^{-1}) + \bar{B}_k(P) + \delta_{[k(n)]}(r_n^{-1}) + \bar{B}_{[k(n)]}(P) \), so that

\[
I_n \leq P^\infty(\{\exists k \in \mathcal{G}_n: k > [k(n)] \cap \delta_k(r_n^{-1}) + \bar{B}_k(P) + \delta_{[k(n)]}(r_n^{-1}) + \bar{B}_{[k(n)]}(P) > 4\delta_k(r_n^{-1})\} \cap D_n)
\]

\[
\leq P^\infty(\{\exists k \in \mathcal{G}_n: k > [k(n)] \cap 3\delta_k(r_n^{-1}) + \bar{B}_k(P) > 4\delta_k(r_n^{-1})\} \cap D_n)
\]

where the second line follows because, by definition of \([k(n)]\), \( \delta_{[k(n)]}(r_n^{-1}) + \bar{B}_{[k(n)]}(P) \leq 2\delta_{[k(n)]}(r_n^{-1}) \), and since \( k > [k(n)] \), \( \delta_{[k(n)]}(r_n^{-1}) \leq \delta_k(r_n^{-1}) \) due to Assumption 4.2. Since \( k \mapsto \bar{B}_k(P) \) is non-increasing, it follows that \( \bar{B}_k(P) \leq \bar{B}_{[k(n)]}(P) \leq \delta_{[k(n)]}(r_n^{-1}) \leq \delta_k(r_n^{-1}) \), so that

\[
I_n \leq P^\infty(\{\exists k: k > k(n) \cap 4\delta_k(r_n^{-1}) > 4\delta_k(r_n^{-1})\} \cap D_n)
\]

which is clearly zero. Thus \( P^\infty([k(n)] \notin \mathcal{F}_n(\omega)) \leq II_n \). Under our assumptions, for any \( \epsilon > 0 \) there exists a \( N \) such that \( II_n \leq \epsilon \) for all \( n \geq N \) and this implies the desired result. \( \square \)
Remark C.3. As a corollary, this Lemma implies that $F_n$ is non-empty with wpa1-$P$. △

Proof of Theorem 4.2. Let $A_n \equiv \{\omega : [k(n)] \in F_n(\omega)\}$. By Lemma C.6 and Lemma C.7, for any $\epsilon > 0$, there exists a $N$ such that

$$P^\infty \left(d_\Theta(\psi_{\tilde{k}(n)}(\omega), \psi(P)) \geq 6C \inf_{k \in \mathbb{N}} \{\delta_k(r_n^{-1}) + B_k(P)\}\right)$$

$$\leq P^\infty \left(d_\Theta(\psi_{\tilde{k}(n)}(\omega), \psi(P)) \geq 6\delta_{[k(n)]}(r_n^{-1})\right)$$

$$\leq P^\infty \left(d_\Theta(\psi_{\tilde{k}(n)}(\omega), \psi(P)) \geq 6\delta_{[k(n)]}(r_n^{-1}) \cap A_n\right) + \epsilon$$

for all $n \geq N$. For any $\omega \in A_n$, $[k(n)] \geq \tilde{k}(n)$ (recall that $\tilde{k}(n)$ does depend on $\omega$). Moreover, $\delta_{[k(n)]}(r_n^{-1}) \geq B_{[k(n)]}(P)$ by Assumption 4.1(ii). Thus, by Lemma C.5 the term in the RHS of the display is less than $\epsilon$. □

Proof of Corollary 4.2. The proof is analogous to that of Theorem 4.2 but using Lemma C.6(3). □

C.2 Appendix for Section 4.1

For any probability $P$ over a set $A$, let $\bigotimes_{i=1}^k P$ be the product probability over $\prod_{i=1}^k A$ induced by $P$. Also, recall that the Wasserstein distance for $p \geq 1$ over $\mathcal{P}(\prod_{i=1}^k \Omega)$ for some $k \geq 1$ is defined as

$$W_p(P, Q) \equiv \left(\inf_{\gamma \in H(P, Q)} \int ||x - y||^p \gamma(dx, dy)\right)^{1/p}$$

for any $P, Q$ in $\mathcal{P}(\prod_{i=1}^k \Omega)$, where $H(P, Q)$ is the class of Borel probability measures over $\Omega^{2k}$ with marginals $P$ and $Q$. It is well-known that

$$(P, Q) \mapsto ||P - Q||_{Lip(\prod_{i=1}^k \Omega)} = W_1(P, Q)$$

where for any set $A$, let $Lip(A)$ to denote the Lipschitz (with constant 1) real-valued functions on $A$.

The following lemma is used in the proof

Lemma C.8. For any $P, Q$ in $\mathcal{P}(\Omega)$, and any $\gamma \in H(P, Q)$, $\bigotimes_{i=1}^k \gamma \in H\left(\bigotimes_{i=1}^k P, \bigotimes_{i=1}^k Q\right)$.

Proof of Lemma C.8. It is clear that the marginal of $\bigotimes_{i=1}^k \gamma$ of a pair $(x_i, y_i)$ is $\gamma$. Therefore, for any $A_1, \ldots, A_k$ Borel subsets on $\Omega$,

$$\bigotimes_{i=1}^k \gamma((A_1 \times \Omega, \ldots, (A_k \times \Omega)) = \prod_{i=1}^k \gamma(A_i \times \Omega) = \prod_{i=1}^k P(A_i)$$
where $\ast$ follows because $\gamma \in H(P,Q)$. Equivalently,

$$\int g(\bar{x}) \bigotimes_{i=1}^{k} \gamma(d\bar{x}, d\bar{y}) = \int g(\bar{x}) \bigotimes_{i=1}^{k} P(d\bar{x})$$

for any $g$ belonging to the class of “simple” functions on $\prod_{i=1}^{k} \Omega$: The class of functions of the form $g(\bar{x}) = \sum_{i=1}^{k} 1_{A_i}(x_i)$ for any $A_1, \ldots, A_k$ Borel subsets on $\Omega$. Since the class of “simple” functions is dense in $C(\Omega^k, \mathbb{R})$ (the class of continuous and bounded functions over $\Omega$), by taking limits and using the previous display one can show that

$$\int f(\bar{x}) \bigotimes_{i=1}^{k} \gamma(d\bar{x}, d\bar{y}) = \int f(\bar{x}) \bigotimes_{i=1}^{k} P(d\bar{x})$$

for any $f \in C(\prod_{i=1}^{k} \Omega, \mathbb{R})$. That is, the marginal probability of $\bigotimes_{i=1}^{k} \gamma$ for the first $k$ coordinates is $\bigotimes_{i=1}^{k} P$. A completely analogous argument shows that the marginal probability of $\bigotimes_{i=1}^{k} \gamma$ for the last $k$ coordinates is $\bigotimes_{i=1}^{k} Q$.

**Proof of Proposition 4.1.** First note that, for any $f \in Lip$, $E_{\psi_n(P)}[f(Z)] = \int \psi_n(P)(f(Z) \geq t) dt = \int P^\infty(f(T_n(\omega, P)) \geq t) dt = E_{P^\infty}[f(T_n(\omega, P))]$. Hence, it follows that

$$||\psi_k(P) - \psi_k(Q)||_\Theta \leq \sup_{f \in Lip} |E_{P^\infty}[f(T_k(\omega, P))] - E_{Q^\infty}[f(T_k(\omega, P))]|$$

$$+ \sup_{f \in Lip} |E_{Q^\infty}[f(T_k(\omega, P))] - f(T_k(\omega, Q))|$$

$$\equiv T_{1,k}(P, Q) + T_{2,k}(P, Q).$$

We now show that both terms, $T_{1,k}(P, Q)$ and $T_{2,k}(P, Q)$, are bounded by $\sqrt{k} W_1(P,Q)$. For any $k \in \mathbb{N}$, $f(T_k(\omega, P)) \equiv f_k(\sqrt{k} \max\{k^{-1} \sum_{i=1}^{k} Z_i(\omega), 0\})$ where $f_k \equiv f(\cdot - \sqrt{k} \max\{E_P[Z], 0\})$. It is easy to see that for any $k$, $f_k \in Lip$ (given that $f \in Lip$). Moreover, the mapping $t \mapsto g_k(t) \equiv f_k(\max\{t, 0\})$ is also in $Lip$ because

$$|g_k(t) - g_k(t')| \leq |\max\{t, 0\} - \max\{t', 0\}| \leq |t' - t|, \forall t, t'.$$

Finally, the mapping $t \mapsto g_k(\sqrt{k}t)/\sqrt{k}$ is also in $Lip$ since $g_k \in Lip$. Hence,

$$T_{1,k}(P, Q) \leq \sqrt{k} \sup_{f \in Lip} \left| E_{\bigotimes_{i=1}^{k} P} \left[ f\left( k^{-1} \sum_{i=1}^{k} Z_i \right) \right] - E_{\bigotimes_{i=1}^{k} Q} \left[ f\left( k^{-1} \sum_{i=1}^{k} Z_i \right) \right] \right|.$$  \hspace{1cm} (19)
For any \( g : \mathbb{R} \to \mathbb{R} \), let \( \bar{g} : \mathbb{R}^k \to \mathbb{R}^k \) be defined as
\[
\bar{z} \equiv (z_1, \ldots, z_k) \mapsto \bar{g}(\bar{z}) \equiv g \left( k^{-1} \sum_{i=1}^{k} z_i \right).
\]

We now show that for any \( g \in Lip(\mathbb{R}) \), \( k \bar{g} \in Lip(\mathbb{R}^k) \). This follows because
\[
|\bar{g}(\bar{z}) - \bar{g}(\bar{z}')| \leq |k^{-1} \sum_{i=1}^{k} \{z_i - z'_i\}| \leq k^{-1}||\bar{z} - \bar{z}'||_1.
\]

This result allow us to bound from above the LHS of the expression 19 so that
\[
\sqrt{k} \sup_{f \in Lip} \left| E_{\bigotimes_{i=1}^{k} P} \left[ f \left( k^{-1} \sum_{i=1}^{k} Z_i \right) \right] - E_{\bigotimes_{i=1}^{k} Q} \left[ f \left( k^{-1} \sum_{i=1}^{k} Z'_i \right) \right] \right|
\leq k^{-1/2} \sup_{f \in Lip(\mathbb{R}^k)} \left| E_{\bigotimes_{i=1}^{k} P} \left[ f(Z) \right] - E_{\bigotimes_{i=1}^{k} Q} \left[ f(Z') \right] \right|
= k^{-1/2} W_1 \left( \bigotimes_{i=1}^{k} P, \bigotimes_{i=1}^{k} Q \right).
\]

For any \( \gamma \in H(P,Q) \), \( \bigotimes_{i=1}^{k} \gamma \in \mathcal{P}(\Omega^k \times \Omega^k) \) where \( \Omega^k \equiv \prod_{i=1}^{k} \Omega \). Moreover, by Lemma C.8, \( \bigotimes_{i=1}^{k} \gamma \in H \left( \bigotimes_{i=1}^{k} P, \bigotimes_{i=1}^{k} Q \right) \).

For any \( \eta > 0 \), let \( \gamma^* \in H(P,Q) \) be the approximate minimizer of \( W_1(P,Q) \), i.e.,
\[
\inf_{\gamma \in H(P,Q)} \int |x - y| \gamma^*(dx, dy) \leq W_1(P,Q) + \eta,
\]
as \( \bigotimes_{i=1}^{k} \gamma^* \in H \left( \bigotimes_{i=1}^{k} P, \bigotimes_{i=1}^{k} Q \right) \), it follows that
\[
W_1 \left( \bigotimes_{i=1}^{k} P, \bigotimes_{i=1}^{k} Q \right) \leq \int_{\Omega^{2k}} ||\bar{x} - \bar{y}||_1 \bigotimes_{i=1}^{k} \gamma^*(dx_i, dy_i)
= \sum_{i=1}^{k} \int_{\Omega^2} |x_i - y_i| \bigotimes_{i=1}^{k} \gamma^*(dx_i, dy_i)
= \sum_{i=1}^{k} \int_{\Omega^2} |x_i - y_i| \gamma^*(dx_i, dy_i)
= k W_1(P,Q) + k \eta.
\]
Since $\eta > 0$ is arbitrary, it follows that $\mathcal{W}_1(P^k, Q^k) \leq k\mathcal{W}_1(P, Q)$. Thus implying
\[ T_{1,k}(P, Q) \leq \sqrt{k}\mathcal{W}_1(P, Q). \]

Regarding the term $T_{2,k}(P, Q)$, observe that
\[ T_{2,k}(P, Q) \leq |T_k(\omega, P) - T_k(\omega, Q)| \]
\[ \leq \sqrt{k}\max\{E_P[Z], 0\} - \max\{E_Q[Z'], 0\} \]
\[ \leq \sqrt{k}|E_P[Z] - E_Q[Z']|. \]

Since $E_P[Z] = \int_{\Omega} z\gamma(dz, dz')$ for any $\gamma \in H(P, Q)$,
\[ T_{2,k}(P, Q) \leq \sqrt{k}|E_\gamma[Z] - E_\gamma[Z']| \leq \sqrt{k}\int |z - z'|\gamma(dz, dz'). \]

Choosing $\gamma$ as the (approximate) minimizer of $\mathcal{W}_1(P, Q)$ it follows that
\[ T_{2,k}(P, Q) \leq \sqrt{k}\mathcal{W}_1(P, Q). \]

\[ \square \]

**Proof of Proposition 4.2.** Since $P \in \mathcal{M}$, $T_k(\omega, P) = \max\{k^{-1/2} \sum_{i=1}^k (Z_i(\omega) - E_P[Z]), -\sqrt{k}E_P[Z]\}$ for any $k \in \mathbb{N}$. By triangle inequality and definition of $||.||_\Theta$,
\[ ||\psi_k(n)(P) - \psi_n(P)||_\Theta \leq \sup_{f \in Lip} E \left[ \left| f(T_{k,n}(\omega, P)) - f\left( \max\{\zeta, -\sqrt{k(n)}E_P[Z]\} \right) \right| \right] \]
\[ + \sup_{f \in Lip} E \left[ \left| f\left( \max\{\zeta, -\sqrt{k(n)}E_P[Z]\}\right) - f\left( \max\{\zeta, -\sqrt{n}E_P[Z]\}\right) \right| \right] \]
\[ + \sup_{f \in Lip} E \left[ \left| f\left( \max\{\zeta, -\sqrt{n}E_P[Z]\}\right) - f(T_n(\omega, P)) \right| \right] \]
\[ \equiv \text{Term}_1(k(n)) + \text{Term}_2(n) + \text{Term}_3(n), \]

where $\zeta \sim N(0, 1)$.

We now provide a bound for terms $\text{Term}_1(k(n))$ and $\text{Term}_3(n)$. For any $f \in Lip$ and any $k \in \mathbb{N}$, the mapping $t \mapsto f_k(t) \equiv f(\max\{t, -\sqrt{k}E_P[Z]\})$ satisfies, for any $t \leq t'$,
\[ |f_k(t') - f_k(t)| \leq |\max\{t', -\sqrt{k}E_P[Z]\} - \max\{t, -\sqrt{k}E_P[Z]\}| \]

where the RHS is equal to 0 if $t \leq t' \leq -kE_P[Z]$, $t' - (-kE_P[Z]) \leq t' - t$; if $t \leq -kE_P[Z] \leq t'$; and $t' - t$ if $-kE_P[Z] \leq t \leq t'$. Hence $|f_k(t') - f_k(t)| \leq |t' - t|$. The same inequality holds
when \( t' \leq t \), so \( f_k \) is in `Lip`. Therefore,

\[
Term_1(k(n)) \leq \sup_{f \in Lip} E_{P^n} \left[ f \left( (k(n))^{-1/2} \sum_{i=1}^{k(n)} (Z_i - E_P[Z]) \right) - f(\zeta) \right] \leq 6k(n)^{-1/2}E_P[|Z|^3]
\]

where the last line follows from Berry-Esseen Inequality for Lipschitz functions (see Barbour and Chen [2005] Thm. 3.2 in Ch. 1). Analogously, \( Term_3(n) \leq 6n^{-1/2}E_P[|Z|^3] \); and since \( k(n) \leq n \), it follows that \( Term_1(k(n)) + Term_3(n) \leq 12k(n)^{-1/2}E_P[|Z|^3] \).

Regarding \( Term_2(n) \), we note that if \( E_P[Z] = 0 \), then \( Term_2(n) = 0 \), so we only need a bound for \( E_P[Z] > 0 \). Under this condition,

\[
Term_2(n) \leq \sup_{f \in Lip} E \left[ 1\{ \zeta \leq -\sqrt{k(n)}E_P[Z] \} \left( f \left( \max\{\zeta, -\sqrt{k(n)}E_P[Z]\} \right) - f \left( \max\{\zeta, -\sqrt{n}E_P[Z]\} \right) \right) \right]
\]

because \( k(n) \leq n \). Since \( ||f||_{L^\infty} \leq 1 \), the inequality further implies that \( Term_2(n) \leq 2E \left[ 1\{ \zeta \leq -\sqrt{k(n)}E_P[Z] \} \right] \).

Since \( \Phi(-\sqrt{k(n)}E_P[Z]) \) converges faster than \( k(n)^{-1/2} \), this implies an estimate of \( 14k(n)^{-1/2}E_P[|Z|^3] \).

\[ \square \]

**Proof of Proposition 4.3.** Throughout, fix \( k \) and \( P, P' \) and let \( ||.||_{\Theta} \equiv ||.||_{L^\Theta} \). Let \( \Theta_k(M) \equiv \{ \theta \in \Theta_k : ||\theta - \psi_k(P)||_{\Theta} \geq M \} \). And, let

\[
t \mapsto U_k(t) \equiv \inf_{\theta \in \Theta_k(t)} \frac{Q_k(P, \theta) - Q_k(P, \psi_k(P))}{t}.
\]

Towards the end of the proof we show that \( U_k \) is continuous. Let \( t \mapsto \Gamma_k(t) \equiv \inf_{s \geq t} U_k(s) \); it follows that \( \Gamma_k \leq U_k \), \( \Gamma_k \) is non-decreasing and by the Theorem of the Maximum \( \Gamma_k \) is continuous.

We show that \( ||\psi_k(P) - \psi_k(P')||_{\Theta} \geq M \equiv \Gamma_k^{-1}(d(P, P')) \) cannot occur.\(^{45}\) To do this, we show that \( 1\{ ||\psi_k(P) - \psi_k(P')||_{\Theta} \geq M \} = 0 \). Observe that

\[
1\{ ||\psi_k(P) - \psi_k(P')||_{\Theta} \geq M \} = \bigcup_{j \in \mathbb{N}} \{ 2^j M \geq ||\psi_k(P) - \psi_k(P')||_{\Theta} \geq 2^{j-1} M \}
\]

\[
\leq \max_{j \in \mathbb{N}} \{ 2^j M \geq ||\psi_k(P) - \psi_k(P')||_{\Theta} \geq 2^{j-1} M \}.
\]

For each \( (j, k) \in \mathbb{N}^2 \), let \( S_{j,k} \equiv \{ \theta \in \Theta_k : 2^j M \geq ||\psi_k(P) - \theta||_{\Theta} \geq 2^{j-1} M \} \). It follows that, for any \( j \in \mathbb{N} \),

\[
1\{ \psi_k(P') \in S_{j,k} \} \leq 1 \left\{ \inf_{\theta \in S_{j,k}} Q(P', \theta) \leq Q(P', \psi_k(P)) \right\}
\]

\(^{45}\)Note that \( \Gamma_k^{-1}(t) \equiv \inf \{ s : U_k(s) \geq t \} \).
because \( \psi_k(P) \in \Theta_k \setminus S_{j,k} \). Observe that, for any \( \theta \in S_{j,k} \cup \{ \psi_k(P) \} \subseteq \{ \theta \in \Theta_k : ||\theta - \psi_k(P)||_\Theta \leq 2^j M \} \)

\[
Q(P', \theta) - Q(P', \psi_k(P)) \geq Q(P, \theta) - Q(P, \psi_k(P))
- |Q(P', \theta) - Q(P', \psi_k(P)) - \{Q(P, \theta) - Q(P, \psi_k(P))\}| \\
\geq Q(P, \theta) - Q(P, \psi_k(P)) - 2^j M \Delta_{j,k}(P, P')
\]

where

\[
\Delta_{j,k}(P, P') \equiv \sup_{\theta \in \Theta_k : ||\theta - \psi_k(P)||_\Theta \leq 2^j M} \frac{|Q(P', \theta) - Q(P', \psi_k(P)) - \{Q(P, \theta) - Q(P, \psi_k(P))\}|}{||\theta - \psi_k(P)||_\Theta}.
\]

Hence,

\[
1\{\psi_k(P') \in S_{j,k}\} \leq 1 \left\{ \inf_{\theta \in \Theta_k : ||\theta - \psi_k(P)||_\Theta \geq 2^j M} \frac{Q(P, \theta) - Q(P, \psi_k(P))}{2^j M} \leq 0.5 \Delta_{j,k}(P, P') \right\}
\]

\[
\leq 1 \left\{ \inf_{\theta \in \Theta_k : ||\theta - \psi_k(P)||_\Theta \geq 2^j M} \frac{Q(P, \theta) - Q(P, \psi_k(P))}{2^j M} \leq 0.5 \max_{k \in \mathbb{N}} \Delta_{\infty,k}(P, P') \right\}
\]

\[
\leq 1 \left\{ \Gamma_k(2^j M) \leq 0.5 \max_{k \in \mathbb{N}} \Delta_{\infty,k}(P, P') \right\}.
\]

Since \( \bar{U}_k \) is non-decreasing, the previous display implies that

\[
1\{\psi_k(P') \in S_{j,k}\} \leq 1 \left\{ 2^{j-1} M \leq \Gamma_k^{-1}(0.5 \max_{k \in \mathbb{N}} \Delta_{\infty,k}(P, P')) \right\}
\]

which equals zero by the definition of \( M \), the fact that \( \Gamma_k^{-1} \) is non-decreasing and \( 2^{j-1} \geq 1 \).

We now show that \( t \mapsto U_k(t) \) (and thus \( \Gamma_k \)) is continuous. Consider the problem \( \inf_{\theta \in \Theta_k(M)} Q_k(P, \theta) \), and consider the set \( L_k(M) \equiv \{ \theta \in \Theta_k(M) : \text{Pen}(\theta) \leq \lambda_k^{-1} Q(P, \theta_k) \} \) for some (any) \( \theta_k \in \Theta_k(M) \) which is non-empty and close. To solve the former minimization problem it suffices to solve \( \inf_{\theta \in L_k(M)} Q_k(P, \theta) \), because the minimum value cannot be outside \( L_k(M) \). Because \( \text{Pen} \) is lower-semi-compact, \( L_k(M) \) is compact (a closed subset of a compact set) so this and lower-semi-continuity of \( Q_k(P, \cdot) \) ensures that \( \inf_{\theta \in L_k(M)} Q_k(P, \theta) \) is achieved by an element in \( L_k(M) \) and the same is true for the original \( V_k(M) \equiv \inf_{\theta \in \Theta_k(M)} Q_k(P, \theta) \).

We just showed that the correspondence \( M \mapsto L_k(M) \) is compact-valued, it is also continuous. By virtue of the Theorem of the Maximum, \( V_k \) is continuous; it is also non-decreasing. The function \( t \mapsto U_k(t) = V_k(t)/t \) is also continuous in \( t > 0 \).

□
D Appendix for Section 5

Proof of Lemma 5.1. Let $||\ell||_* \equiv \sup_{a \in \Theta: a \neq 0} \frac{||\ell||_a}{||a||_a}$ for any $\ell \in \Theta^*$. Since $\ell \in \Theta^*$,

$$||\ell[\varphi_k(P)]||_{L^2(P)}^2 \leq ||\ell||_*^2 \int \left( \frac{||d\psi_k(P)[\delta_z - P]||}{||\delta_z - P||} \right)^2 P(dz).$$

If $\frac{d\psi_k(P)}{dP}[\delta_z - P] \leq K||\delta_z - P||_S$, we are done since $||\delta_z - P||_S \leq 2$ (the class $S$ is uniformly bounded by 1), and also $||\ell||_* < \infty$.

To show the inequality (recall that $\frac{d\psi_k(P)}{dP}$ is not linear, otherwise it would be trivial) note that $\frac{d\psi_k(P)}{dP}$ is continuous, so there exists a $\delta > 0$ such that $\frac{d\psi_k(P)}{dP}[Q] \leq 1$ for any $Q$ such that $||Q||_S \leq \delta$. By homogeneity,

$$\frac{d\psi_k(P)}{dP}[\delta_z - P] = \frac{||\delta_z - P||_S}{\delta} \frac{d\psi_k(P)}{dP}[\delta^{-1} (\delta_z - P)] \leq \frac{||\delta_z - P||_S}{\delta},$$

where the last inequality follows from our previous observation and the fact that $||\delta \frac{\delta_z - P}{||\delta_z - P||_S}||_S \leq \delta$. \hfill \Box

Proof of Theorem 5.1. We first show that the result holds for constant sequences $k(n) = k$ for all $n$. By Lemma 5.1, for each $k \in \mathbb{N}_0$, $\ell[\varphi_k(P)] \in L^2(P)$, so it satisfies the condition in the definition of ALR.

By applying the definition 5.2 to $t \mapsto P(t) = P + tG_n(\omega)$ where $G_n(\omega) \equiv \sqrt{n}(P_n(\omega) - P)$ which belongs to $\mathcal{T}_0(P)$ and $P(.) \in \mathcal{L}(P, \mathcal{T}_0(P))$, it follows that for $t = \sqrt{n}$,

$$\left| \ell \left[ \frac{\psi_k(P_n(\omega))}{t} - \psi_k(P) - D\psi_k(P)[G_n(\omega)] \right] \right| \leq \eta_{k,\ell}(P + tG_n(\omega), P)/t.$$

By dividing by $||\ell[\varphi_k(P)]||_{L^2(P)}$, the result then follows from condition 10.

We now shows existence of a diverging sequence by using the first part and the diagonalization lemma C.1.

For any $\epsilon > 0$, $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let $T(\epsilon, k, n) \equiv P^\infty \left( \frac{\eta_{k,\ell}(P_n(\omega), P)}{||\ell[\varphi_k(P)]||_{L^2(P)}} \geq \epsilon \right)$. To show the desired result it suffices to show that there exists a non-decreasing diverging sequence $(j(n))_n$ such that for all $\epsilon > 0$, there exists a $N$ such that

$$T(\epsilon, j(n), n) \leq \epsilon,$$

for all $n \geq N$.

Under condition 10, for any $k \in \mathbb{N}_0$, $\lim_{n \to \infty} T(2^{-k}, k, n) = 0$. By Lemma C.1, there exists a non-decreasing diverging sequence $(j(n))_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} T(2^{-j(n)}, j(n), n) = 0$; i.e., for any $\epsilon > 0$, there exists a $N(\epsilon)$ such that $T(2^{-j(n)}, j(n), n) \leq \epsilon$ for all $n \geq N(\epsilon)$.
Since \( j(.) \) diverges, there exists a \( N_\epsilon \) such that \( 1/2^{j(n)} \leq \epsilon \) for all \( n \geq N_\epsilon \). By these observations and the fact that \( \epsilon \mapsto T(\epsilon, k, n) \) is non-increasing,

\[
T(\epsilon, j(n), n) \leq T(2^{-j(n)}, j(n), n) \leq \epsilon
\]

for all \( n \geq \tilde{N}_\epsilon \equiv \max\{N_\epsilon, N(\epsilon)\} \), and we thus showed the desired result. \(\square\)

### D.1 Appendix for Section 5.4

**Proof of Proposition 5.6.** We divide the proof into several steps. Throughout we fix a \( \ell \in \Xi \).

**Step 1.** We first show that \( k(n) \in F_{n,\ell} \) wpa1-\( P \). To do this, we have to check that for any \( k' \geq k(n) \), then \( |\ell[\psi_{k'}(P_n) - \psi_{k(n)}(P_n)]| \leq 5\frac{C_{k',\ell}}{t_n \sqrt{n}} \) wpa1-\( P \), i.e., for any \( \epsilon > 0 \), there exists a \( N \) such that

\[
P^\infty \left( |\ell[\psi_{k'}(P_n) - \psi_{k(n)}(P_n)]| \leq 5\frac{C_{k',\ell}}{t_n \sqrt{n}}, \forall k' \geq k(n) \right) \geq 1 - \epsilon.
\]

for all \( n \geq N \).

By triangle inequality

\[
|\ell[\psi_{k'}(P_n) - \psi_{k(n)}(P_n)]| \leq |\ell[\psi_{k'}(P_n) - \psi_{k'}(P) - D\psi_{k'}(P)[P_n - P]]| + |\ell[D\psi_{k(n)}(P)[P_n - P] - D\psi_{k'}(P)[P_n - P]]| + |\ell[D\psi_{k(n)}(P)P - P] - \psi_{k'}(P)| + |\ell[\psi_{k'}(P) - \psi(P)]|
\]

\[\equiv \text{Term}_{1,n} + \text{Term}_{2,n} + \text{Term}_{3,n} + \text{Term}_{4,n} + \text{Term}_{5,n},\]

a.s.-\( P \).

By Theorem 5.1, \( \text{Term}_{1,n} \leq \eta_{k',\ell}(P_n, P) \) and \( \text{Term}_{2,n} \leq \eta_{k(n),\ell}(P_n, P) \) a.s.-\( P \). By Assumption 5.3(1) — since it holds uniformly over \( k \) — and \( \ell \mapsto C_{k,\ell} \) is non-decreasing (Assumption 5.3(3)), it follows that: For any \( \epsilon > 0 \), there exists a \( N \) such that

\[
P^\infty \left( \text{Term}_{1,n} + \text{Term}_{2,n} \leq 2\frac{C_{k',\ell}}{\sqrt{nl_n}}, \forall k' \geq k(n) \right) \geq 1 - \epsilon.
\]

for all \( n \geq N \).

By Assumption 5.3(2) an analogous display holds for \( \text{Term}_{3,n} \). Assumption 5.3(3), the fact that \( k' \geq k(n) \) and the definition of \( k(n) \), imply \( \text{Term}_{5,n} \leq C_{k(n),\ell} \frac{C_{k(n),\ell}}{\sqrt{nl_n}} \leq C_{k',\ell} \frac{C_{k',\ell}}{\sqrt{nl_n}} \) and \( \text{Term}_{4,n} \leq B_{k',\ell}(P) \leq B_{k(n),\ell}(P) \leq C_{k(n),\ell} \frac{C_{k(n),\ell}}{\sqrt{nl_n}} \leq C_{k',\ell} \frac{C_{k',\ell}}{\sqrt{nl_n}} \). Therefore

\[
|\ell[\psi_{k'}(P_n) - \psi_{k(n)}(P_n)]| \leq 5\frac{C_{k',\ell}}{\sqrt{nl_n}}
\]

Since this occurs wpa1-\( P \), then the desired result follows.
STEP 2. We now show Statement 1 in the proposition. By triangle inequality and the fact that \( n^{-1/2} \sum_{i=1}^{n} \ell[\varphi_{k(h)}(P)](Z_i) = \sqrt{n} \ell[D\psi_{k(n)}(P)(P_n - P)], \)

\[
\sqrt{n}[\ell[\psi_{k(n)}(P_n) - \psi(P) - D\psi_{k(n)}(P)[P_n - P]]] \\
\leq \sqrt{n}[\ell[\psi_{k(n)}(P_n) - \psi_{k(n)}(P_n)]] \\
+ \sqrt{n}[\ell[\psi_{k(n)}(P_n) - \psi_{k(n)}(P) - D\psi_{k(n)}(P)[P_n - P]]] \\
+ \sqrt{n}[\ell[D\psi_{k(n)}(P)[P_n - P] - D\psi_{k(n)}(P)[P_n - P]]] \\
+ \sqrt{n}[\ell[D\psi_{k(n)}(P) - \psi(P)]] \\
= Term_{6,n} + Term_{7,n} + Term_{8,n} + Term_{9,n},
\]
a.s.-\( P. \)

Since \( k(n) \in F_{n,\ell} \) wpal-\( P \) (by Step 1), then by definition of \( \tilde{k}(n) \), \( k(n) \leq \tilde{k}(n) \) wpal-\( P \). This fact and the definition of \( F_{n,\ell} \) imply \( Term_{6,n} \leq 5 \frac{C_{k(n),\ell}}{\sqrt{n}l_n} wpal-P^\infty. \) By Theorem 5.1 and Assumption 5.3(1), \( Term_{7,n} = \text{op}\left( \frac{C_{k(n),\ell}}{\sqrt{n}l_n} \right) \). By Assumption 5.3(2), \( Term_{8,n} = \text{op}\left( \frac{C_{k(n),\ell}}{\sqrt{n}l_n} \right) \). Finally, by definition of \( k(n) \), \( Term_{9,n} \leq \frac{C_{k(n),\ell}}{\sqrt{n}l_n} \) wpal-\( P \). Therefore,

\[
|\ell[\psi_{k(n)}(P_n) - \psi(P) - D\psi_{k(n)}(P)[P_n - P]]| \leq 8 \frac{C_{k(n),\ell}}{\sqrt{n}l_n},
\]
wpal-\( P. \)

STEP 3. We now show Statement 2 in the proposition. We do this by contradiction, that is, \( \liminf_{n \to \infty} \frac{C_{k(n),\ell}}{\sqrt{n}l_n} > 0 \). Then, for all \( n \in \mathbb{N} \) large enough, \( j(n) < k(n) \). Because, if not, since \( k \mapsto C_{k,\ell} \) is non-decreasing (Assumption 5.3(3)), \( o(1) = \frac{C_{j(n),\ell}}{\sqrt{n}l_n} \geq \frac{C_{k(n),\ell}}{\sqrt{n}l_n} \) which contradicts the hypothesis. Since \( j(n) < k(n) \), it follows that \( B_{k(n),\ell}(P) \leq B_{j(n),\ell}(P) \). Since \( \lim_{n \to \infty} \sqrt{n}[\ell[\psi_{j(n)}(P) - \psi(P)]] = 0 \), the same holds for the “lim sup” and so \( \limsup_{n \to \infty} \sqrt{n}B_{k(n)}(P) = 0 \).

Thus, \( \liminf_{n \to \infty} \frac{C_{k(n),\ell}}{\sqrt{n}l_n} > 0 \) but \( \limsup_{n \to \infty} \sqrt{n}B_{k(n)}(P) = 0 \). By the growth condition (Assumption 5.3(4)), \( C_{k-1,\ell} = C_{k,\ell} \frac{C_{k-1,\ell}}{C_{k,\ell}} \geq C_{k,\ell}(1 + C)^{-1} \). Therefore, there exists a \( n \) sufficiently large such that \( \frac{C_{k(n),\ell}}{\sqrt{n}l_n} > B_{k(n)-1,\ell}(P) \) but this contradicts the fact that \( k(n) \) is minimal. \( \square \)

To show Proposition 5.7, we need the following result which is a well-known representation result (see van der Vaart and Wellner [1996]) and is stated merely for convenience.

**Lemma D.1.** Let \( \omega \mapsto \mathcal{G}_n(\omega) \equiv \sqrt{n}[\mathcal{P}_n(\omega) - P] \in \mathcal{T}_0(P) \). There exists a tight Borel measurable \( \mathcal{G} \in L^\infty(\mathcal{S}) \) such that for any \( \epsilon > 0 \), there exists a Borel set \( A \subseteq \Omega^\infty \) such that \( P^\infty(A) \geq 1 - \epsilon \) and \( \|\mathcal{G}_n(\omega) - \mathcal{G}\|_{\mathcal{S}} = o(1) \) for all \( \omega \in A \).

**Proof of Lemma D.1.** It is well-known that the following representation is also valid: \( \mathcal{G}_n : \Omega^\infty \to L^\infty(\mathcal{S}) \). Since \( \mathcal{S} \) is a Donsker Class, \( \mathcal{G}_n \) converges weakly to some \( \mathcal{G} \) tight Borel measurable element in \( L^\infty(\mathcal{S}) \) (e.g. see van der Vaart and Wellner [1996] Ch. 2.1). By
Theorem 1.10.3 in van der Vaart and Wellner [1996] there exists a probability space $(\Omega^\infty, \bar{P}^\infty)$ and a sequence of maps $\bar{G}_n : \Omega^\infty \to L^\infty(S)$ for all $n \in \mathbb{N}$ and $\bar{G} : \Omega^\infty \to L^\infty(S)$ such that (i) $\|\bar{G}_n - \bar{G}\|_{L^\infty(S)} = o(1)$ almost uniformly;\footnote{By almost uniformly it means that for any $\epsilon > 0$, there exists a Borel set $A \subseteq \Omega^\infty$ such that $\bar{P}^\infty(A) \geq 1 - \epsilon$ and $\sup_{\tau \in A} \|\bar{G}_n(\tau) - \bar{G}(\tau)\|_{L^\infty(S)} = o(1)$.} and (ii) $\bar{G}_n$ and $\bar{G}$ have the same law as $G_n$ and $G$ resp.

Proof of Proposition 5.7. (1) Trivial.

(2) Follows from the fact that, by the Markov inequality, $\sqrt{n} \ell[D\psi_k(P)|P_n - P|] = O_{P^\infty} \left( ||\ell[\varphi_k(P)]||_{L^2(P)} \right)$.

(3) By Lemma D.1, for any $\epsilon > 0$, there exists a set $A \subseteq \Omega^\infty$ such that $P^\infty(A) \geq 1 - \epsilon$ and $||G_n(\omega) - G||_S = o(1)$ for all $\omega \in A$

For each $\omega \in A$ the set $H(\omega) \equiv (G_n(\omega))_{n \in \mathbb{N}} \subseteq \mathcal{T}_0(P)$ is a compact set in $ca(\Omega)$ under $||.||_S$. Hence, the assumption in (2) implies that, for each $\omega \in A$,

$$\limsup_{n \to \infty} \sqrt{n} \eta_{k,\ell}(P + n^{-1/2}G_n(\omega), P) \leq \limsup_{n \to \infty} \sqrt{n} \sup_{Q \in H(\omega)} \eta_{k,\ell}(P + n^{-1/2}Q, P) = 0.$$ 

Since this occurs for each $\omega \in A$ and $A$ occurs with probability larger than $1 - \epsilon$, the result in (2) follows.

D.2 Appendix for Example 5.1

Lemma D.2. For all $k \in \mathbb{N}$ and $P \in \mathcal{M}$,

$$||\varphi_k(P)||_{L^2(P)} \leq ||p||^2_{L^\infty(R)} ||\kappa||^2_{L^1(R)}.$$ 

Proof. It suffices to show that $E_P[||((\kappa h_k) \ast P)(Z)||^2] \leq ||p||_{L^\infty}^2 \left( \int |\kappa(u)|^2 du \right)^2$. To do this, note that

$$E_P[||((\kappa h_k) \ast P)(Z)||^2] = \int \left( \int h(k)^{-1} \kappa((x - z)/h(k))p(x)dx \right)^2 p(z)dz$$

$$= \int \left( \int \kappa(u)p(z + h(k)u)du \right)^2 p(z)dz$$

$$\leq ||p||^2_{L^\infty} \left( \int |\kappa(u)|^2 du \right)^2$$

where the first line follows from the fact that $\kappa_h(\cdot) = h^{-1}\kappa(\cdot/h)$ a.s.-$P$.\qed
Proof of Proposition 5.1. Consider the curve \( t \mapsto P + tQ \). It is a valid curve because \( \mathbb{D}_\psi = ca(\mathbb{R}) \). Therefore

\[
\psi_k(P + tQ) - \psi_k(P) = t \left\{ \int (\kappa_{h(k)} \ast Q)(x)P(dx) + \int (\kappa_{h(k)} \ast P)(x)Q(dx) \right\} + t^2 \int (\kappa_{h(k)} \ast Q)(x)Q(dx).
\]

Since \( \kappa \) is symmetric, \( \int (\kappa_{h(k)} \ast P)(x)Q(dx) = \int (\kappa_{h(k)} \ast Q)(x)P(dx) \). From this display, \( \eta_k(P + tQ, Q) = t^2 \int (\kappa_{h(k)} \ast Q)(x)Q(dx) \) and \( D\psi_k(P)[Q] = 2 \int (\kappa_{h(k)} \ast P)(x)Q(dx) \).

The mapping \( Q \mapsto 2 \int (\kappa_{h(k)} \ast P)(x)Q(dx) \) is homogeneous of degree 1. Also, note that \( (\kappa_{h(k)} \ast P)(x) = \int \kappa(u)p(x + h(k)u)du \). Hence, for any reals \( x \) and \( x' \)

\[
| (\kappa_{h(k)} \ast P)(x) - (\kappa_{h(k)} \ast P)(x') | = \int \kappa(u)\{p(x + h(k)u) - p(x' + h(k)u)\}du \leq C|x - x'|
\]

for some constant \( C < \infty \). Thus, \( \kappa_{h(k)} \ast P \) is Lipschitz (uniformly on \( k \)), so the mapping \( Q \mapsto 2 \int (\kappa_{h(k)} \ast P)(x)Q(dx) \) is continuous with respect to the \( \| \cdot \|_{Lip} \).

To establish the rate result, we use the Markov inequality. We also introduce the following notation \( \int (\kappa_{h(k)} \ast Q)(x)Q(dx) = \langle \kappa_{h(k)} \ast Q, Q \rangle \) where \( \langle \cdot, \cdot \rangle \) is the inner product of the dual \( (L^\infty(\mathbb{R}), ca(\mathbb{R})) \).

It follows that

\[
\sqrt{E[(\eta_k(P_n, P))^2]} = \sqrt{E[\langle \kappa_{h(k)} \ast (P_n - P), P_n - P \rangle^2]}
\]

\[
= \sqrt{E[(\langle \kappa_{h(k)} \ast P_n, P_n \rangle - 2\langle \kappa_{h(k)} \ast P, P_n \rangle + \langle \kappa_{h(k)} \ast P, P \rangle )^2]}
\]

where the second line follows from symmetry of \( \kappa \) which implies \( \langle \kappa_{h(k)} \ast P_n, P \rangle = \langle P_n, \kappa_{h(k)} \ast P \rangle \).

We note that

\[
\langle \kappa_{h(k)} \ast P_n, P_n \rangle = \frac{\kappa_{h(k)}(0)}{n} + \frac{1}{n^2} \sum_{i \neq j} \kappa_{h(k)}(Z_i - Z_j)
\]

\[
= \frac{\kappa_{h(k)}(0)}{n} + \frac{1}{n^2} \left( \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \kappa_{h(k)}(Z_i - Z_j) + \sum_{i=2}^{n} \sum_{j=1}^{i-1} \kappa_{h(k)}(Z_i - Z_j) \right),
\]

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also
\[
\langle \kappa_{h(k)} \ast P, P_n \rangle = \frac{1}{n} \sum_{i=1}^{n} (\kappa_{h(k)} \ast P)(Z_i) = \frac{1}{n} \sum_{i=1}^{n} EP[\kappa_{h(k)}(Z_i - Z)]
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} EP[\kappa_{h(k)}(Z_i - Z)] \frac{i}{n} + \frac{1}{n} \sum_{i=1}^{n} EP[\kappa_{h(k)}(Z_i - Z)] \frac{n - i}{n}
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} EP[\kappa_{h(k)}(Z_i - Z)] \frac{i}{n} + \frac{1}{n} \sum_{i=1}^{n-1} EP[\kappa_{h(k)}(Z_i - Z)] \frac{n - i}{n}
\]
\[
= \frac{1}{n^2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} EP[\kappa_{h(k)}(Z_i - Z_j)] + \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} EP[\kappa_{h(k)}(Z_i - Z_j)]
\]
\[
+ \frac{1}{n^2} EP[\kappa_{h(k)}(Z_1 - Z)]
\]

where the third line follows because \( EP[\kappa_{h(k)}(Z_n - Z)] \frac{n-n}{n} = 0 \), and the fourth one follows from the fact that by iid-ness, \( EP[\kappa_{h(k)}(Z_i - Z_j)] = EP[\kappa_{h(k)}(Z_i - Z)] \) for all \( j \).

Therefore,
\[
\langle \kappa_{h(k)} \ast P_n, P_n \rangle - 2 \langle \kappa_{h(k)} \ast P, P_n \rangle = \frac{\kappa_{h(k)}(0)}{n} + \frac{1}{n^2} \left( \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \kappa_{h(k)}(Z_i - Z_j) + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \kappa_{h(k)}(Z_i - Z_j) \right)
\]
\[
- \frac{2}{n^2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} EP[\kappa_{h(k)}(Z_i - Z_j)] + \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} EP[\kappa_{h(k)}(Z_i - Z_j)]
\]
\[
- \frac{2}{n^2} EP[\kappa_{h(k)}(Z_1 - Z)]
\]
\[
= \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \{ \kappa_{h(k)}(Z_i - Z_j) - 2EP[\kappa_{h(k)}(Z_i - Z_j)] \}
\]
\[
+ \frac{1}{n^2} \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \{ \kappa_{h(k)}(Z_i - Z_j) - 2EP[\kappa_{h(k)}(Z_i - Z_j)] \}
\]
\[
+ \frac{\kappa_{h(k)}(0)}{n} - \frac{2}{n^2} EP[\kappa_{h(k)}(Z_1 - Z)]
\]
\[
= \frac{2}{n^2} \sum_{i<j} \{ \kappa_{h(k)}(Z_i - Z_j) - 2EP[\kappa_{h(k)}(Z_i - Z_j)] \}
\]
\[
+ \frac{\kappa_{h(k)}(0)}{n} - \frac{2}{n^2} EP[\kappa_{h(k)}(Z_1 - Z)]
\]

where the last line follows by symmetry of \( \kappa \) since \( \kappa(Z_i - Z_j) = \kappa(Z_j - Z_i) \) for all \( i, j \).
Since \( \langle \kappa_h(k) \times P, P \rangle = E_{P \cdot P}[\kappa_h(k)(Z - Z')] = \frac{1}{n^2} \sum_{i,j} E_{P \cdot P}[\kappa_h(k)(Z - Z')] \), it follows that

\[
\langle \kappa_h(k) \times P_n, P_n \rangle - 2\langle \kappa_h(k) \times P, P_n \rangle + \langle \kappa_h(k) \times P, P \rangle = \frac{2}{n^2} \sum_{i<j} \tilde{\kappa}_h(k)(Z_i - Z_j) + \frac{\kappa_h(k)(0)}{n} - \frac{2}{n^2} E_P[\kappa_h(k)(Z_1 - Z)] + \frac{1}{n} E[\kappa_h(k)(Z - Z')]
\]

where \((z, z') \mapsto \tilde{\kappa}_h(z - z') \equiv \kappa_h(z - z') - E_P[\kappa_h(z - Z)] - E_P[\kappa_h(z' - Z)] + E_{P \cdot P}[\kappa_h(z - z')].\) Therefore,

\[
\sqrt{E \left[ (\eta_h(P_n, P))^2 \right]} \leq 2 \sqrt{E \left[ \left( \frac{1}{n^2} \sum_{i<j} \tilde{\kappa}_h(k)(Z_i - Z_j) \right)^2 \right]} + \frac{\kappa_h(k)(0)}{n} + \frac{2}{n^2} \sqrt{E \left[ (E[\kappa_h(k)(Z_1 - Z)])^2 \right]}
\]

\[
+ \frac{1}{n} E[\kappa_h(k)(Z - Z')].
\]

We now bound each term on the RHS. First note that

\[
\frac{1}{n} E[\kappa_h(Z - Z')] = \frac{1}{n} \int h(k)^{-1} \kappa((z - z')/h(k)) p(z)p(z')dzdz' = \frac{1}{n} \int \kappa(u)p(z' + h(k)u)p(z')dz'du \leq n^{-1}\|p\|_{L^\infty},
\]

and

\[
\sqrt{E \left[ (E[\kappa_h(k)(Z_1 - Z)])^2 \right]} \leq \sqrt{E \left[ (\kappa_h(k)(Z' - Z))^2 \right]}
\]

\[
= \sqrt{\int (h(k)^{-1} \kappa((z' - z)/h(k)))^2 p(z)p(z')dzdz'}
\]

\[
= \sqrt{h(k) \int (\kappa(u))^2 p(z + h(k)u)p(u)du} \leq h(k)^{-1/2} \sqrt{\|p\|_{L^\infty} \|\kappa\|_{L^2}}.
\]

where the first line follows by Jensen inequality. Finally, by Gine and Nickl [2008] Sec. 2

\[
\sqrt{E \left[ \left( \frac{1}{n^2} \sum_{i<j} \tilde{\kappa}_h(k)(Z_i - Z_j) \right)^2 \right]} \leq \frac{2}{\sqrt{n^2}} \sqrt{E[(\tilde{\kappa}_h(Z - Z'))^2]} \leq \frac{2\|\kappa\|_{L^2} \sqrt{\|p\|_{L^2}}}{n\sqrt{h(k)}}.
\]
D.3 Appendix for Example 5.2

Proof of Proposition 5.2. Under our assumptions, \( Q_k(\cdot, P) \) is strictly convex and smooth over \( \Theta_k \), so the following FOC holds:

\[
\frac{dQ_k(\psi_k(P), P)}{d\theta} [\pi^T \kappa^k] = E_P \left[ \frac{d\phi(Z, \psi_k(P))}{d\theta} [\pi^T \kappa^k] \right] + \lambda_k \frac{d\text{Pen}(\psi_k(P))}{d\theta} [\pi^T \kappa^k] = 0
\]

for any \( \pi \in \mathbb{R}^k \). Let \( H_k : \Theta_k(\delta_k, n) \times T_0 \subseteq \mathbb{R}^k \times \text{ca}(Z) \rightarrow \mathbb{R}^k \) where \( H_k(\pi, P) = E_P \left[ \frac{d\phi(Z, \pi^T \kappa^k)}{d\theta} [\kappa^k] \right] + \lambda_k \frac{d\text{Pen}(\pi^T \kappa^k)}{d\theta} [\kappa^k] \). Letting \( \pi_k(P) \in \mathbb{R}^k \) be the vector that \( \psi_k(P) = \pi_k(P)T \kappa^k \), it follows that \( H_k(\pi_k(P), P) = 0 \). Also, observe that

\[
\frac{dH_k(\pi, P)}{dP}[Q] = E_Q \left[ \frac{d\phi(Z, \pi^T \kappa^k)}{d\theta} [\kappa^k] \right] \in \mathbb{R}^k,
\]

for any \( Q \in T_0 \) and

\[
\frac{dH_k(\pi, P)}{d\pi}[a] = \left( E_P \left[ \frac{d^2\phi(Z, \pi^T \kappa^k)}{d\theta^2} [\kappa^k, \kappa^k] \right] + \lambda_k \frac{d^2\text{Pen}(\pi^T \kappa^k)}{d\theta^2} [\kappa^k, \kappa^k] \right) a
\]

for any \( a \in \mathbb{R}^k \), where \( \Delta_k(P) = E_P \left[ \frac{d^2\phi(Z, \pi^T \kappa^k)}{d\theta^2} [\kappa^k, \kappa^k] \right] + \lambda_k \frac{d^2\text{Pen}(\pi^T \kappa^k)}{d\theta^2} [\kappa^k, \kappa^k] \in \mathbb{R}^{k \times k} \) and since \( \text{Pen} \) is strictly convex and \( \phi \) convex under our assumptions, the matrix is positive-definite for all \( k \).

We now show that the partial derivatives are continuous and by Theorem 1.1.9 in Ambrosetti and Prodi [1995] this implies that the mapping \( H_k \) is Frechet differentiable at \( \psi_k(P) \). To do this, consider a sequence \((\pi_n, P_n)\) that converges to \((\pi \equiv \pi_k(P), P)\) under \( ||.|| + ||.||_S \), moreover \( \pi_m^T \kappa^k \in \Theta_k(\delta_k, n) \) for all \( m \). Then, letting \( ||A||_S = \sup_{Q \in \text{ca}(Z)} \frac{||A(Q)||}{||Q||_S} \),

\[
\left\| \frac{dH_k(\pi_n, P_n)}{dP}[Q] - \frac{dH_k(\pi, P)}{dP}[Q] \right\|_S = \left\| E_Q \left[ \frac{d\phi(Z, \pi_n^T \kappa^k)}{d\theta} [\kappa^k] - \frac{d\phi(Z, \pi^T \kappa^k)}{d\theta} [\kappa^k] \right] \right\|_S.
\]

Observe that, for each \( z \in Z \),

\[
\sup_{a \in \mathbb{R}^k} \frac{\left| \frac{d\phi(z, \pi_n^T \kappa^k)}{d\theta} [a \kappa^k] - \frac{d\phi(z, \pi^T \kappa^k)}{d\theta} [a \kappa^k] \right|}{||a||} \leq ||(\pi_n^T - \pi^T) \kappa^k||_{L^q} \sup_{a \in \mathbb{R}^k} \frac{||a \kappa^k||_{L^q}}{||a||},
\]

where the first inequality follows from the assumption of twice differentiability and the Mean Value Theorem (Theorem 1.8 in Ambrosetti and Prodi [1995]). By Assumption 5.2, \( \Phi_{2,k}(z) \equiv \)
\[
\sup_{k,v_1,v_2 \in \Theta_k(\delta_{k,n}) \times \Theta_k} \left| \frac{d^2 \phi(a_k)}{d\theta^2} \right|_{[v_1,v_2]} [v_1,v_2] \quad \text{and} \quad \Phi_{2,k} \in \mathcal{S}. \] 

Hence, given that \( \|x\| = \sup_{a \in \mathbb{R}^k} |ax|/\|a\|, \)

\[
\left\| \frac{dH_k(\pi_n, P_n)}{d\pi}[Q] - \frac{dH_k(\pi, P)}{d\pi}[Q] \right\| \leq \left\| (\pi_n^T - \pi^T) \kappa_k \right\|_{L^q} \sup_{a \in \mathbb{R}^k} \left\| a^T \kappa_k \right\|_{L^q} E_Q[\Phi_{2,k}(Z)].
\]

Since \( \frac{E_Q[\Phi_{2,k}(Z)]}{||Q||_S} \leq 1 \) and \( \sup_{a \in \mathbb{R}^k} \left\| a^T \kappa_k \right\|_{L^q} < \infty, \) this display implies that \( (a, P) \mapsto \frac{dH_k(a,P)}{d\pi} \) is continuous.

Also,

\[
\left\| \frac{dH_k(\pi_n, P_n)}{d\pi}[a] - \frac{dH_k(\pi, P)}{d\pi}[a] \right\| \leq \left( E_{P_n} \left[ \frac{d^2 \phi(Z, \pi_n^T \kappa_k)^{[\kappa_k, \kappa_k]}}{d\theta^2} \right] - E_{P} \left[ \frac{d^2 \phi(Z, \pi^T \kappa_k)^{[\kappa_k, \kappa_k]}}{d\theta^2} \right] \right) a \\
+ \lambda_k \left( \frac{d^2 P e_n(\pi_n^T \kappa_k)^{[\kappa_k, \kappa_k]}}{d\theta^2} - \frac{d^2 P e_n(\pi^T \kappa_k)^{[\kappa_k, \kappa_k]}}{d\theta^2} \right) a \\
\equiv \text{Term}_{1,n} + \text{Term}_{2,n}.
\]

The first term in the RHS is bounded by two terms: \( \text{Term}_{1,1,n} \equiv \left\| E_{P_n-P} \left[ \frac{d^2 \phi(Z, \pi_n^T \kappa_k)^{[\kappa_k, \kappa_k]}}{d\theta^2} \right] \right\|_{\|a\|} \)

and \( \text{Term}_{1,2,n} \equiv \sup_{\{b: \|b\| = 1\}} \left| E_{P_n-P} \left[ \frac{d^2 \phi(Z, \pi_n^T \kappa_k)^{[\kappa_k, \kappa_k]}}{d\theta^2} \right] b \right|. \) By assumption 5.2, for each \( j, l, z \mapsto \frac{d^2 \phi(z, \kappa_j, \kappa_l)}{d\theta^2} \) does, to \( \text{Term}_{1,1,n} \) does too. By Assumption 5.2, \( h \mapsto \frac{d^2 \phi(z, h, \kappa_j, \kappa_l)}{d\theta^2} \) is continuous at \( \psi_k(P) \) uniformly on \( z \in \mathbb{Z}, \) so \( \text{Term}_{1,2,n} \) vanishes as \( \|P_n - P\|_S \) does, to \( \text{Term}_{1,1,n} \) does too. By Assumption 5.1, \( a \mapsto \frac{d^2 P e_n(\kappa_k)}{d\theta^2} \) is continuous and thus \( \text{Term}_{2,n} \) vanishes as \( \pi_n \) converges to \( \pi. \)

By continuity of both derivatives of \( H_k, \) \( H_k \) is Frechet differentiable (Theorem 1.9 in Ambrosetti and Prodi [1995]). Hence, this and the fact that \( \Delta_k(P) \) is non-singular imply by the implicit function theorem (see Theorem 2.3 in Ambrosetti and Prodi [1995]) that, in a neighborhood of \( P, \) \( \pi_k \) is Frechet differentiable with the derivative given by

\[
Q \mapsto \left( \Delta_k(P)^{-1} \frac{dH_k(\pi, P)}{d\pi}[Q] \right)^T.
\]

Since \( \psi_k(P) = \pi_k(P)^T \kappa_k, \) this implies the result. \( \square \)

**Proof of Proposition 5.3.** By the calculations in p. 14 in Ambrosetti and Prodi [1995], it follows that

\[
\|\psi_k(P + tQ) - \psi_k(P) - tD\psi_k(P)[Q]\|_{\Theta} \leq t \|Q\|_S \sup_{t \in [0,1]} \|D\psi_k(P + tQ) - D\psi_k(P)\|_s,
\]

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and, for any \( \ell \in \Theta^* \),
\[
||\ell[\psi_k(P + tQ) - \psi_k(P) - tD\psi_k(P)[Q]]||_\Theta \leq t||Q||_S \sup_{t \in [0,1]} ||\ell[D\psi_k(P + tQ) - D\psi_k(P)]||_* .
\]

Thus, we can take \( \eta_k, \ell(P + tQ, P) \equiv t||Q||_S \sup_{t \in [0,1]} ||\ell[D\psi_k(P + tQ) - D\psi_k(P)]||_* . \) Observe that
\[
\ell[D\psi_k(P + tQ)[Q] - D\psi_k(P)[Q]] = (E_Q[\nabla_k(P + tQ)(Z)]^T (\Delta_k(P + tQ)^{-1} - \Delta_k(P)^{-1}) \ell[\kappa^k] \\
+ (E_Q[\nabla_k(P + tQ)(Z) - \nabla_k(P)(Z)]^T (\Delta_k(P)^{-1} - \Delta_k(P)^{-1}) \ell[\kappa^k] \\
= (E_Q[\nabla_k(P + tQ)(Z) - \nabla_k(P)(Z)]^T (\Delta_k(P + tQ)^{-1} - \Delta_k(P)^{-1}) \\
\times \ell[\kappa^k] \\
+ (E_Q[\nabla_k(P)(Z)]^T (\Delta_k(P + tQ)^{-1} - \Delta_k(P)^{-1}) \ell[\kappa^k] \\
+ (E_Q[\nabla_k(P + tQ)(Z) - \nabla_k(P)(Z)]^T \Delta_k(P)^{-1} \ell[\kappa^k] \\
\equiv Term_{1,k,n} + Term_{2,k,n} + Term_{3,k,n}.
\]

In Step 1 below we show that
\[
||Term_{1,k,n}|| \leq t^{1+\theta} \max\{||Q||^2_{S^6}, ||Q||^3_{S^3}\} \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_* \sup_{a \in \mathbb{R}^k} \frac{||a^T \kappa^k||_{L^q}}{||a||} \\
\times \left( (\lambda_k C_0 + C_0) \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_* + 1 \right) \sup_{(a,b) \in \mathbb{R}^{2k}} \frac{||a^T \kappa^k||_{L^q} ||b^T \kappa^k||_{L^q}}{||a|| ||b||} \\
\times ||\ell[\kappa^k]|| \\
||Term_{2,k,n}|| \leq t^q \max\{||Q||^1_{S^6}, ||Q||^2_{S^3}\} ||\Delta_k(P)^{-1}|| \\
\times \left( (\lambda_k C_0 + C_0) \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_* + 1 \right) \sup_{(a,b) \in \mathbb{R}^{2k}} \frac{||a^T \kappa^k||_{L^q} ||b^T \kappa^k||_{L^q}}{||a|| ||b||} \\
||\Delta_k(P + tQ)^{-1}|| \times ||\ell[\kappa^k]|| \\
||Term_{3,k,n}|| \leq t ||Q||^2_{S^3} \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_* \sup_{a \in \mathbb{R}^k} \frac{||a^T \kappa^k||_{L^q} ||\Delta_k(P)^{-1}|| \times ||\ell[\kappa^k]||}{||a||} .
\]

Observe that \( ||a^T \kappa^k||_{L^q} = (\int ||a^T \kappa^k(z)||^q \mu(dz))^{1/q} \leq ||a|| (\int ||\kappa^k(z)||^q \mu(dz))^{1/q} = ||a|| \times \)
\[ |||\kappa^k|||_{L^3} \]. Thus
\[
|||\epsilon_{T_{1,k,n}}||| \leq t^{1 + \theta} \max\{||Q||^{2 + \theta}_S, ||Q||^{3}_S\} \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_{\epsilon} \times |||\kappa^k|||_{L^3}^3 \\
\times \left(\lambda_k C_0 + C_0\right) \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_{\epsilon} + 1 \right) \\
\times ||[\kappa^k]|| \}
\]
\[
|||\epsilon_{T_{2,k,n}}||| \leq t^\theta \max\{||Q||^{1 + \theta}_S, ||Q||^{3}_S\} ||\Delta_k(P)^{-1}|| \\
\times \left(\lambda_k C_0 + C_0\right) \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_{\epsilon} + 1 \right) |||\kappa^k|||_{L^3}^2 \\
\times ||[\kappa^k]|| \}
\]
\[
|||\epsilon_{T_{3,k,n}}||| \leq t||Q||^2_3 \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_{\epsilon} \times |||\kappa^k|||_{L^3} \times ||\Delta_k(P)^{-1}|| \times ||[\kappa^k]|| \).
\]
By step 2, \[ ||\Delta_k(P + tQ)^{-1}|| \leq 2||\Delta_k(P)^{-1}|| \] for all \( t \) less or equal than some \( T_{B_0,P,k} \) (specified in Step 2). Therefore
\[
|||\epsilon_{T_{2,k,n}}||| \leq 2t^\theta \max\{||Q||^{1 + \theta}_S, ||Q||^{3}_S\} ||\Delta_k(P)^{-1}||^2 \\
\times \left(\lambda_k C_0 + C_0\right) \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_{\epsilon} + 1 \right) |||\kappa^k|||_{L^3}^2 \\
\times ||[\kappa^k]|| \}
\]
So, after some simple algebra and the fact that \( \lambda_k \leq 1 \),
\[
||[D\psi_k(P + tQ)[Q] - D\psi_k(P)[Q]]|| \\
\leq 3t \max\{||Q||^3_3, ||Q||^{1 + \theta}_S\} ||[\kappa^k]|| \\
\times \max\{1, |||\kappa^k|||_{L^3}^3\} \times \left(\max\left\{1, \sup_{t \in [0,1], Q \in \mathcal{Q}} ||D\psi_k(P + tQ)||_{\epsilon} + 2||\Delta_k^{-1}(P)||\right\}\right)^2 \\
\times \left(2C_0 \sup_{t \in [0,1], Q \in \mathcal{Q}} \sup_{t \in [0,1], Q \in \mathcal{Q}} ||D\psi_k(P + tQ)||_{\epsilon} + 1 \right).
\]
By letting
\[
C(B_0, k, P) \equiv 3 \max\{1, |||\kappa^k|||_{L^3}^3\} \times \left(\max\left\{1, \sup_{t \in [0,1], Q \in \mathcal{Q}} ||D\psi_k(P + tQ)||_{\epsilon} + 2||\Delta_k^{-1}(P)||\right\}\right)^2 \\
\times \left(2C_0 \sup_{t \in [0,1], Q \in \mathcal{Q}} ||D\psi_k(P + tQ)||_{\epsilon} + 1 \right)
\] (20)

the result follows.

**STEP 1.** By the calculations in the proof of Proposition 5.2 and the Mean Value Theorem,

\[
|E Q[\nabla_k(P + tQ) - \nabla_k(P)]| \leq \|\psi_k(P + tQ) - \psi_k(P)\|_{L^q} \sup_{a \in \mathbb{R}^k} \frac{|a^T \kappa^k|}{|a|} \|Q\|_s
\]

\[
\leq t\|Q\|_s^2 \sup_{t \in [0,1]} \|D\psi_k(P + tQ)\| \sup_{a \in \mathbb{R}^k} \frac{|a^T \kappa^k|}{|a|} \|\Delta_k(P)^{-1}\| \times \|\kappa^k\|.
\]

Hence

\[
|\text{Term}_{3,k,n}| \leq t\|Q\|_s^2 \sup_{t \in [0,1]} \|D\psi_k(P + tQ)\| \sup_{a \in \mathbb{R}^k} \frac{|a^T \kappa^k|}{|a|} \|\Delta_k(P)^{-1}\| \times \|\kappa^k\|.
\]

Regarding Term\textsubscript{2,k,n}, it follows that

\[
|\text{Term}_{2,k,n}| = |E Q[\nabla_k(P)(Z)]\Delta_k(P)^{-1}(\Delta_k(P) - \Delta_k(P + tQ))\Delta_k(P + tQ)^{-1}\ell[\kappa^k]|
\]

\[
\leq |E Q[\nabla_k(P)(Z)]\Delta_k(P)^{-1}| \times \|\Delta_k(P) - \Delta_k(P + tQ)\| \times \|\Delta_k(P + tQ)^{-1}\ell[\kappa^k]|
\]

By Assumption 5.2(ii), \(\frac{d\phi(\psi_k(P))}{d\theta}[a^T \kappa]/\|a^T \kappa\|_{L^2} \in S\) for any \(a \in \mathbb{R}^k\), so

\[
|E Q[\nabla_k(P)(Z)]\Delta_k(P)^{-1}| \leq |E Q[\nabla_k(P)(Z)]| \times \|\Delta_k(P)^{-1}\| \leq \|Q\|_s \sup_{a \in \mathbb{R}^k} \frac{|a^T \kappa|}{|a|} \|\Delta_k(P)^{-1}\|.
\]

Also

\[
|\Delta_k(P) - \Delta_k(P + tQ)| \leq \lambda_k \left\| \frac{d^2 \text{Pen}(\psi_k(P))}{d\theta^2} \kappa^k, \kappa^k - \frac{d^2 \text{Pen}(\psi_k(P + tQ))}{d\theta^2} \kappa^k, \kappa^k \right\|
\]

\[
+ \left\| \left[ \frac{d^2 \phi(Z, \psi_k(P))}{d\theta^2} \kappa^k, \kappa^k \right] - \left[ \frac{d^2 \phi(Z, \psi_k(P + tQ))}{d\theta^2} \kappa^k, \kappa^k \right] \right\|
\]

\[
\equiv \text{Term}_{4,k,n} + \text{Term}_{5,k,n},
\]

and in turn

\[
\text{Term}_{5,k,n} \leq \left\| \left[ \frac{d^2 \phi(Z, \psi_k(P))}{d\theta^2} \kappa^k, \kappa^k \right] - \left[ \frac{d^2 \phi(Z, \psi_k(P + tQ))}{d\theta^2} \kappa^k, \kappa^k \right] \right\|
\]

\[
+ t \left\| \left[ \frac{d^2 \phi(Z, \psi_k(P + tQ))}{d\theta^2} \kappa^k, \kappa^k \right] \right\|
\]

\[
\equiv \text{Term}_{6,k,n} + \text{Term}_{7,k,n}.
\]
Observe that

\[ Term_{4,k,n} = \lambda_k \sup_{(a,b) \in \mathbb{R}^{2k}} \frac{d^2 Pen(\psi_k(P))}{d\theta^2} [a^T \kappa^k, b^T \kappa^k] - \frac{d^2 Pen(\psi_k(P+tQ))}{d\theta^2} [a^T \kappa^k, b^T \kappa^k]. \]

By assumption 5.1,

\[ \left| \frac{d^2 Pen(\psi_k(P))}{d\theta^2} [a^T \kappa^k, b^T \kappa^k] - \frac{d^2 Pen(\psi_k(P+tQ))}{d\theta^2} [a^T \kappa^k, b^T \kappa^k] \right| \leq C_0 \|\psi_k(P) - \psi_k(P+tQ)\|_{\mathcal{L}_q}^6 \|a^T \kappa^k\|_{\mathcal{L}_q} \|b^T \kappa^k\|_{\mathcal{L}_q}, \]

and thus

\[ Term_{4,k,n} \leq \lambda_k \sup_{(a,b) \in \mathbb{R}^{2k}} \frac{\|D\psi_k(P+tQ)\|_{\mathcal{L}_q}^6}{\|a\| \times \|b\|} \sup_{(a,b) \in \mathbb{R}^{2k}} \frac{\|a^T \kappa^k\|_{\mathcal{L}_q} \|b^T \kappa^k\|_{\mathcal{L}_q}}{\|a\| \times \|b\|}. \]

By Assumption 5.2,

\[ \left| \frac{d^2 \phi(z, \psi_k(P))}{d\theta^2} [a^T \kappa^k, b^T \kappa^k] - \frac{d^2 \phi(z, \psi_k(P+tQ))}{d\theta^2} [a^T \kappa^k, b^T \kappa^k] \right| \leq C_0 \|\psi_k(P) - \psi_k(P+tQ)\|_{\mathcal{L}_q} \|a^T \kappa^k\|_{\mathcal{L}_q} \|b^T \kappa^k\|_{\mathcal{L}_q}, \]

so

\[ Term_{6,k,n} \leq C_0 \|tQ\|_{\mathcal{L}_q} \sup_{t \in [0,1]} \|D\psi_k(P+tQ)\|_{\mathcal{L}_q}^6 \sup_{(a,b) \in \mathbb{R}^{2k}} \frac{\|a^T \kappa^k\|_{\mathcal{L}_q} \|b^T \kappa^k\|_{\mathcal{L}_q}}{\|a\| \times \|b\|}. \]

Also, by Assumption 5.2(ii)

\[ Term_{7,k,n} \leq t \|Q\|_{\mathcal{L}_q} \sup_{t \in [0,1]} \sup_{(a,b) \in \mathbb{R}^{2k}} \frac{\|a^T \kappa^k\|_{\mathcal{L}_q} \|b^T \kappa^k\|_{\mathcal{L}_q}}{\|a\| \times \|b\|}. \]

Hence

\[ \|Term_{2,k,n}\| \leq \|Q\|_{\mathcal{L}_q} \Delta_k(P)^{-1} \times \left( \lambda_k C_0 + C_0 \right)^t \|Q\|_{\mathcal{L}_q} \sup_{t \in [0,1]} \|D\psi_k(P+tQ)\|_{\mathcal{L}_q}^6 + t \|Q\|_{\mathcal{L}_q} \right) \frac{\|a^T \kappa^k\|_{\mathcal{L}_q} \|b^T \kappa^k\|_{\mathcal{L}_q}}{\|a\| \times \|b\|} \|\Delta_k(P+tQ)^{-1}\| \|\ell[k]\|. \]
Finally, by the previous calculations,

\[
||Term_{1,k,n}|| \leq t||Q||_S^2 \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_S \frac{||a^T\kappa^k||_{L^q}}{||a||} \\
\times \left( (\lambda_k C_0 + C_0) t^q ||Q||_S^2 \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_S^2 + t||Q||_S \right) \frac{||a^T\kappa^k||_{L^q}||b^T\kappa^k||_{L^q}}{||a||||b||} \\
\times ||\ell[\kappa^k]||.
\]

**Step 2.** We now show that there exists a \(T_{B_0,P,k}\) such that for all \(t \leq T_{B_0,P,k}\), \(||\Delta_k(P + tQ)^{-1}|| \leq 2||\Delta_k(P)^{-1}||\). To do this, note that

\[
||\Delta_k(P + tQ)^{-1}|| = \frac{1}{e_{\min}(\Delta_k(P + tQ))}.
\]

If \(e_{\min}(\Delta_k(P + tQ)) \geq e_{\min}(\Delta_k(P))\) then \(||\Delta_k(P + tQ)^{-1}|| \leq ||\Delta_k(P)^{-1}||\). If \(e_{\min}(\Delta_k(P + tQ)) \leq e_{\min}(\Delta_k(P))\), then, by Weyl inequality, \(e_{\min}(\Delta_k(P + tQ)) \geq e_{\min}(\Delta_k(P)) - ||\Delta_k(P + tQ) - \Delta_k(P)||\). By our previous calculations for terms 4 and 5,

\[
||\Delta_k(P + tQ) - \Delta_k(P + tQ)|| \leq \left( (\lambda_k C_0 + C_0) t^q ||Q||_S^2 \sup_{t \in [0,1]} ||D\psi_k(P + tQ)||_S^2 + t||Q||_S \right) \\
\times \sup_{a,b \in \mathbb{R}^{2k}} \frac{||a^T\kappa^k||_{L^q}||b^T\kappa^k||_{L^q}}{||a||||b||} \\
\leq 2B_0 \left( C_0 t^q \sup_{t \in [0,1], Q \in \mathcal{Q}} ||D\psi_k(P + tQ)||_S^2 + t \right) \\
\times \sup_{a,b \in \mathbb{R}^{2k}} \frac{||a^T\kappa^k||_{L^q}||b^T\kappa^k||_{L^q}}{||a||||b||}
\]

because \(||Q||_S \leq B_0\) and \(\lambda_k \leq 1\). Hence, there exists a \(T = T_{B_0,P,k}\) such that for all \(t \leq T\)

\(e_{\min}(\Delta_k(P + tQ)) \geq 0.5e_{\min}(\Delta_k(P))\) and thus \(||\Delta_k(P + tQ)^{-1}|| \leq 2||\Delta_k(P)^{-1}||\) for any \(t \leq T_{B_0,P,k}\).

\[\square\]

**D.4 Appendix for Examples 5.3 and 5.4**

First, note that \(\mathbb{D}_\psi \subseteq ca(\mathbb{R} \times [0,1]^2)\) (defined in Appendix B.1).

For any \(k \in \mathbb{N}_0\), let \(F_k : L^2([0,1]) \times \mathbb{D}_\psi \to L^2([0,1])\) be such that

\[
F_k(\theta, Q) \equiv \begin{cases} 
(T^*_{k,Q}T_{k,Q} + \lambda_k I)[\theta] - T^*_{k,Q}[r_{k,Q}] & \text{for Penalization-Based} \\
(\Pi^*_k T^*_{k,Q}T_{k,Q} \Pi_k)[\theta] - \Pi^*_k T^*_{k,Q}[r_{k,Q}] & \text{for Sieve-Based}
\end{cases}
\]

for any \((\theta, Q) \in L^2([0,1]) \times \mathbb{D}_\psi\). Note that for any \(Q \in \mathbb{D}_\psi\) the integrals defining the operators
are well-defined by assumptions and so is $T_{k,Q}^*$; see Appendix B.1 for a discussion.

Let $\varepsilon_p(Y,W) \equiv Y - \psi_{id}(P)(W)$ and $\varepsilon_{k,p}(Y,W) \equiv Y - \psi_k(P)(W)$. Also, throughout this section we use the notation introduced in Appendix B.1 to denote $T_{k,P}$ and other quantities.

**Lemma D.3.** For any $P \in \mathbb{D}_\psi$ and any $k \in \mathbb{N}_0$, $\psi_k$ is $F$-differentiable (over $\mathbb{D}_\psi \subseteq (ca(\mathbb{R} \times [0,1]^2), ||.||_{TV})$) at $P$ with derivative given by:

1. For the Penalization-Based: \[ D\psi_k(P)[Q] = (T_{k,P}^* T_{k,P} + \lambda_k I)^{-1} T_{k,P}^* T_{k,Q} [\varepsilon_{k,P}] = - (T_{k,P}^* T_{k,P} + \lambda_k I)^{-1} T_{k,Q}^* T_{k,P} [\psi_k(P) - \psi_{id}(P)], \quad \forall Q \in \mathbb{D}_\psi \]

where $x \mapsto T_{k,P}[g](x) \equiv \int \kappa_k(x' - \cdot) \int (\psi_k(P)(w) - y)Q(dy,dw,dx')$.

2. For the Sieve-Based:

\[ D\psi_k(P)[Q] = (\Pi_k^* T_{k,P}^* P \Pi_k)^{-1} \Pi_k^* T_{k,P}^* T_{k,Q} [\varepsilon_{k,P}] = - (\Pi_k^* T_{k,P}^* P \Pi_k)^{-1} \Pi_k^* T_{k,Q}^* T_{k,P} [\psi_k(P) - \psi_{id}(P)], \quad \forall Q \in \mathbb{D}_\psi \]

where $x \mapsto T_{k,P}[g](x) \equiv (u^{J(k)}(x))^T Q^{-1} E_P [u^{J(k)}(X)g(Y,W)]$ for any $g \in L^2(P)$.

**Proof.** See the end of this Section. \[ \square \]

The following corollary trivially follows (the proof is omitted).

**Corollary D.1.** For any $P \in \mathbb{D}_\psi$ and any $k \in \mathbb{N}_0$, $\gamma_k$ is $F$-differentiable (over $\mathbb{D}_\psi \subseteq (ca(\mathbb{R} \times [0,1]^2), ||.||_{TV})$) at $P$.

**Lemma D.4.** For the sieve-based and the penalization-based: For any $P \in \mathbb{D}_\psi$

1. The regularization $\psi$ is $DIFF(P(\cdot), \mathbb{D}_\psi)$ under all $P(\cdot) \in \mathcal{L}(P, \mathbb{D}_\psi)$.

2. For each $k \in \mathbb{N}_0$, $\sup_{t \in L^2([0,1])}: ||t||_{L^2([0,1])} = 1 \eta_k,t(P + t\zeta, P) \leq t o(||\zeta||_{TV})$ and \[ \sqrt{n} \sup_{t \in L^2([0,1])}: ||t||_{L^2([0,1])} = 1 \eta_k,t(P_n, P) = o_P(1). \]

**Proof.** See the end of this Section. \[ \square \]

The following corollary trivially follows (the proof is omitted).

**Corollary D.2.** For the sieve-based and the penalization-based: For any $P \in \mathbb{D}_\psi$

1. The regularization $\gamma$ is $DIFF(P(\cdot), \mathbb{D}_\psi)$ under all $P(\cdot) \in \mathcal{L}(P, \mathbb{D}_\psi)$.

---

\[ ^{47} \text{The operator } \mathcal{U}_k \text{ was defined in Appendix B.1.} \]

\[ ^{48} \text{The “} o \text{“ function may depend on } k. \]
(2) For each $k \in \mathbb{N}_0$, the reminder of $\gamma_k$, $\eta_{k,\pi}$, is such that $\eta_{k,\pi}(P + t\zeta, P) \leq o(||\zeta||_{TV})$ and $\sqrt{n\eta_{k,\pi}(P_n, P)} = o_P(1)$.

Proof of Proposition 5.4. The result follows from Lemmas D.3(2) and D.4 and the subsequent corollaries. We now expand the expression in Lemma D.3 in terms of the basis functions.

For any $g, f \in L^2([0, 1])$,

$$T_{k,P}^*\Pi_{k}[g](x) = T_{k,P} \left[ (v^{L(k)}(.))^T Q_{vv}^{-1} \ell_{Leb}[v^{L(k)}(W)] g(W) \right](x) = (u^{j(k)}(x))^T Q_{uu}^{-1} Q_{uv}^{-1} \ell_{Leb}[v^{L(k)}(W)] g(W),$$

and

$$\langle T_{k,P}^*\Pi_{k}[g], f \rangle_{L^2([0,1])} = \int (u^{j(k)}(x))^T Q_{uu}^{-1} Q_{uv}^{-1} \ell_{Leb}[v^{L(k)}(W)] g(W) f(x) dx = \int \ell_{Leb}[(u^{j(k)}(X))^T f(x)] Q_{uu}^{-1} Q_{uv}^{-1} v^{L(k)}(w) g(w) dw$$

so $T_{k,P}^* \Pi_{k} : L^2([0, 1], \ell_{Leb}) \rightarrow L^2([0, 1], \ell_{Leb})$ and is given by

$$f \mapsto T_{k,P}^* \Pi_{k} [f](.) = \ell_{Leb}[(u^{j(k)}(X))^T f(X)] Q_{uu}^{-1} Q_{uv}^{-1} v^{L(k)}(.) .$$

Hence

$$\Pi_{k}^* T_{k,P}^* \Pi_{k}[g](.) = (v^{L(k)}(.))^T Q_{vv}^{-1} Q_{uu}^{-1} Q_{uv}^{-1} \ell_{Leb}[v^{L(k)}(W)] g(W) .$$

We now compute the inverse of this operator. Consider solving for $g(\cdot) = (v^{L(k)}(.))^T Q_{vv}^{-1} b$ for some $b \in \mathbb{R}^k$ such that

$$\Pi_{k}^* T_{k,P}^* \Pi_{k}[g](\cdot) = (v^{L(k)}(.))^T Q_{vv}^{-1} b$$

$$\iff Q_{vv}^T Q_{uu}^{-1} Q_{uv}^{-1} \ell_{Leb}[v^{L(k)}(W)] g(W) = b$$

$$\iff \ell_{Leb}[v^{L(k)}(W)] g(W) = Q_{vv}(Q_{vv}^T Q_{uu}^{-1} Q_{uv}^{-1})^{-1} b.$$}

Hence, $(\Pi_{k}^* T_{k,P}^* T_{k,P} \Pi_{k})^{-1}[g](\cdot) = (v^{L(k)}(.))^T (Q_{vv}^T Q_{uu}^{-1} Q_{uv}^{-1})^{-1} b$. Therefore

$$(\Pi_{k}^* T_{k,P}^* T_{k,P} \Pi_{k})^{-1}[g](\cdot) = (v^{L(k)}(.))^T (Q_{vv}^T Q_{uu}^{-1} Q_{uv}^{-1})^{-1} \ell_{Leb}[u^{j(k)}(X) T_{k,Q} \varepsilon_{k,P}] = (v^{L(k)}(.))^T (Q_{vv}^T Q_{uu}^{-1} Q_{uv}^{-1})^{-1} \ell_{Leb}[u^{j(k)}(X) T_{k,Q} \varepsilon_{k,P}(Y, W)].$$

And

$$(\Pi_{k}^* T_{k,P}^* T_{k,P} \Pi_{k})^{-1}[g](\cdot) = (v^{L(k)}(.))^T (Q_{vv}^T Q_{uu}^{-1} Q_{uv}^{-1})^{-1} \ell_{Leb}[v^{L(k)}(W) T_{k,Q} \varepsilon_{k,P}(Y, W)].$$

$\text{The } \circ \text{ function may depend on } k.$
It is easy to see that for any $Q$, $D\gamma_k(P)[Q] = \int \pi(w) D\psi_k(P)(w) [Q] dw$, the goal is to cast this as $\int D\psi^*_k(P)[\pi](z)Q(dz)$. To this end, note that

$$\int \pi(w) D\psi_k(P)(w)[Q] dw$$

$$= \int \pi(w) (v^{L(k)}(w))^T (Q^{T}_{uv} Q^{-1}_{uu} Q_{uw}) Q^{-1}_{uw} Q^{-1}_{uu} E_Q[u^{J(k)}(X) \varepsilon_{k,P}(Y, W)] dw$$

$$- \int \pi(w) (v^{L(k)}(w))^T (Q^{T}_{uv} Q^{-1}_{uu} Q_{uw}) E_{Leb}[v^{L(k)}(W) \Pi_k^* T_{k,Q}^* \Pi_k[P(\psi_k(P) - \psi_{id}(P))[W)] dw$$

$$\equiv Term_{1,k} + Term_{2,k}.$$ 

Regarding the first term, note that

$$Term_{1,k} = \int E_{Leb}[\pi(W) (v^{L(k)}(W))^T (Q^{T}_{uv} Q^{-1}_{uu} Q_{uw}) Q^{-1}_{uw} u^{J(k)}(x) \varepsilon_{k,P}(y, w) Q(dy, dw, dx).$$

We can cast $Term_{2,k} = -\langle \Pi_k[\pi], (\Pi_k^* T_{k,P}^* T_{k,P} \Pi_k)^{-1} [\Pi_k^* T_{k,Q}^* \Pi_k[P(\psi_k(P) - \psi_{id}(P))] \rangle_{L^2}$, and thus

$$Term_{2,k} = -\langle T_{k,Q}(\Pi_k^* T_{k,P}^* T_{k,P} \Pi_k)^{-1} \Pi_k[\pi], T_{k,P}[\psi_k(P) - \psi_{id}(P)] \rangle_{L^2}$$

$$= -\int (u^{J(k)}(x))^T Q^{-1}_{uu} E_Q[u^{J(k)}(X) (\Pi_k^* T_{k,P}^* T_{k,P} \Pi_k)^{-1} \Pi_k[\pi](W)] T_{k,P}[\psi_k(P) - \psi_{id}(P)](x) dx$$

$$= \int E_P[(\psi_{id}(P)(W) - \psi_k(P)(W))(u^{J(k)}(X))^T Q^{-1}_{uu} u^{J(k)}(x)(\Pi_k^* T_{k,P}^* T_{k,P} \Pi_k)^{-1} \Pi_k[\pi](w)$$

$$\times Q(dw, dx)$$

where the second line follows from definition of $T_{k,P}$.

Therefore,

$$D\psi^*_k(P)[\pi](y, w, x) = E_{Leb}[\pi(W) (v^{L(k)}(W))^T (Q^{T}_{uv} Q^{-1}_{uu} Q_{uw}) Q^{-1}_{uw} u^{J(k)}(x) \varepsilon_{k,P}(y, w)$$

$$+ E_P[(\psi_{id}(P)(W) - \psi_k(P)(W))(u^{J(k)}(X))^T Q^{-1}_{uu} u^{J(k)}(x)$$

$$\times (v^{L(k)}(w))^T (Q^{T}_{uv} Q^{-1}_{uu} Q_{uw})^{-1} E_{Leb}[v^{L(k)}(W) \pi(W)].$$

In the operator notation this expression equals

$$D\psi^*_k(P)[\pi](y, w, x) = T_{k,P}(\Pi_k^* T_{k,P}^* T_{k,P} \Pi_k)^{-1} \Pi_k[\pi](x) \varepsilon_{k,P}(y, w)$$

$$+ T_{k,P}[\psi_{id}(P) - \psi_k(P)](x) \times (\Pi_k^* T_{k,P}^* T_{k,P} \Pi_k)^{-1} \Pi_k[\pi](w).$$

(21)

Proof of Proposition 5.5. The result follows from Lemmas D.3(2) and D.4 and the subsequent corollaries. We now expand the expression in Lemma D.3.

Note that $\psi_k(P) = (T_{k,P}^* T_{k,P} + \lambda_k I)^{-1} T_{k,P}^* r_{k,P}$ and $r_{k,P}(\cdot) = \int \kappa_k(x' - \cdot) \int \psi_{id}(P)(w)p(w, x') dw dx'$.  

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so that \( \psi_k(P) = (T^*_{k,P}T_{k,P} + \lambda_k I)^{-1}T^*_{k,P}T_{k,P}[\psi_{id}(P)] \). Hence

\[
\begin{align*}
(T^*_{k,P}T_{k,P} + \lambda_k I)^{-1}T^*_{k,Q}T_{k,P}[\psi_k(P) - \psi_{id}(P)] \\
= (T^*_{k,P}T_{k,P} + \lambda_k I)^{-1}T^*_{k,Q}T_{k,P}[(T^*_{k,P}T_{k,P} + \lambda_k I)^{-1}T^*_{k,P}T_{k,P} - I]\psi_{id}(P) \\
= - \lambda_k(T^*_{k,P}T_{k,P} + \lambda_k I)^{-1}T^*_{k,Q}T_{k,P}(T^*_{k,P}T_{k,P} + \lambda_k I)^{-1}[\psi_{id}(P)].
\end{align*}
\]

Thus

\[
D\gamma_k(P)[Q] = \langle \pi, R_{k,P}T^*_{k,P}T_{k,Q}[\epsilon_{k,P}] \rangle_{L^2} + \lambda_k \langle \pi, R_{k,P}T^*_{k,Q}T_{k,P}R_{k,P}[\psi_{id}(P)] \rangle_{L^2} \\
= \langle T_{k,P}R_{k,P}[\pi], T_{k,Q}[\epsilon_{k,P}] \rangle_{L^2} + \lambda_k \langle T_{k,Q}R_{k,P}[\pi], T_{k,P}R_{k,P}[\psi_{id}(P)] \rangle_{L^2}.
\]

Note that

\[
T_{k,P}[g](x) = \int k_k(x' - x) \int g(w)p(w, x') dw, dx' = \int k_k(x' - x)T_{P}[g](x') dx' = K_k[T_{P}[g]](x)
\]

and by symmetry of \( \kappa, T^*_{k,P} = T_{P}[K_k] \), where \( K_k \) is simply the convolution operator. Therefore,

\[
\begin{align*}
\langle T_{k,P}R_{k,P}[\pi], T_{k,Q}[\epsilon_{k,P}] \rangle_{L^2} &= \int T_{k,P}R_{k,P}[\pi](x) \int k_k(x' - x) \int \epsilon_{k,P}(y, w)Q(\epsilon, dw, d\epsilon') dx' \\
&= \int \left( \int k_k(x - x')T_{k,P}R_{k,P}[\pi](x) dx \right) \epsilon_{k,P}(y, w)Q(dy, dw, dx') \\
&= \int K^2_kT_{P}R_{k,P}[\pi](x)\epsilon_{k,P}(y, w)Q(dy, dw, dx).
\end{align*}
\]

and

\[
\begin{align*}
\langle T_{k,Q}R_{k,P}[\pi], T_{k,P}R_{k,P}[\psi_{id}(P)] \rangle_{L^2} &= \int \int k_k(x' - x) \int R_{k,P}[\pi](w)Q(dw, dx')T_{k,P}R_{k,P}[\psi_{id}(P)](x) dx \\
&= \int R_{k,P}[\pi](w) \left( \int k_k(x' - x)T_{k,P}R_{k,P}[\psi_{id}(P)](x) dx \right) Q(dw, dx') \\
&= \int R_{k,P}[\pi](w)K^2_kT_{P}R_{k,P}[\psi_{id}(P)](x)Q(dw, dx').
\end{align*}
\]

Therefore

\[
D\psi^*_k(P)[\pi](y, w, x) = K^2_kT_{P}R_{k,P}[\pi](x)\epsilon_{k,P}(y, w) + \lambda_k R_{k,P}[\pi](w)K^2_kT_{P}R_{k,P}[\psi_{id}(P)](x).
\]

\[
\square
\]
D.4.1 Proofs of Supplementary Lemmas.

Proof of Lemma D.3. The proof follows from the Implicit Function Theorem for Banach spaces (see Ambrosetti and Prodi [1995]). Note that $L^2([0,1]) \times ca(\Omega)$ is a Banach space with norm $|| \cdot ||_{L^2} + || \cdot ||_{TV}$.

(1) Observe that $\theta \mapsto F_k(\theta, Q)$ is linear, so
$$\frac{dF_k(\psi_k(P), P)}{dP}[Q] = (T_{k,P}^*T_{k,P} + \lambda_k I) : L^2([0,1]) \rightarrow L^2([0,1]).$$
By our conditions (1)-(2) stated in Example 3.2 Kernel($((T_{k,P}^*T_{k,P} + \lambda_k I)) = \{0\}$ and $(T_{k,P}^*T_{k,P} + \lambda_k I)$ has closed range — the range of an operator $A$ is closed iff 0 is not an accumulation point of the spectrum of $A^*A$. Thus $\frac{dF_k(\psi_k(P), P)}{dP}$ is 1-to-1 and onto.

Also, for any $k \in \mathbb{N}_0$,
$$\frac{dF_k(\psi_k(P), P)}{dP}[Q] = (T_{k,Q}^*T_{k,P} + T_{k,P}^*T_{k,Q})[\psi_k(P)] - (T_{k,Q}^*r_{k,P} + T_{k,P}^*r_{k,Q})$$
$$= - T_{k,Q}^*[r_{k,P} - T_{k,P}[\psi_k(P)]] + T_{k,P}[T_{k,Q}[\psi_k(P)] - r_{k,Q}]$$
$$\equiv \text{Term}_1 + \text{Term}_2.$$

Note that
$$r_{k,P} - T_{k,P}[\psi_k(P)][.] = \int \kappa_k(x' - \cdot)(y - \psi_k(P)(w))p(y, w, x')dydwdx'$$
$$= T_{k,P}[\psi_{id}(P) - \psi_k(P)][.]$$
where the last equality follows because $\int (y - \psi_{id}(P)(w))p(y, w, X)dydw = 0$. Thus
$$\text{Term}_1 = - T_{k,Q}^*T_{k,P}[\psi_{id}(P) - \psi_{id}(P)].$$

Also
$$T_{k,Q}[\psi_k(P)][.] - r_{k,Q}(.) = \int \kappa_k(x' - \cdot) \int (\psi_k(P)(w) - y)Q(dy, dw, dx')$$
so
$$\text{Term}_3 = T_{k,P}T_{k,Q}[\varepsilon_{k,P}]$$
Thus, the result follows.

(2) The proof is analogous to the one for part (1), so we only present an sketch. By assumption, $\Pi_k^*T_{k,P}T_{k,P}\Pi_k$ is 1-to-1 for each $P$.

Also, for any $k \in \mathbb{N}_0$,
$$\frac{dF_k(\psi_k(P), P)}{dP}[Q] = (\Pi_k^*T_{k,Q}^*T_{k,P}\Pi_k + \Pi_k^*T_{k,P}^*T_{k,Q}\Pi_k)[\psi_k(P)]$$
$$- \Pi_k^* (T_{k,Q}^*[r_{k,P}] + T_{k,P}^*[r_{k,Q}])$$
$$= \Pi_k^* [(T_{k,Q}^*T_{k,P} + T_{k,P}^*T_{k,Q})[\psi_k(P)] - (T_{k,Q}^*[r_{k,P}] + T_{k,P}^*[r_{k,Q}])]$$
where the second line follows because $\Pi_k[\psi_k(P)] = \psi_k(P)$.

We note that

$$T_{k,Q}[\psi_k(P)] - r_{k,Q} = \left(u^{J(k)}(X)\right)^T Q^{-1}_{uu} \int u^{J(k)}(x)(\psi_k(P)(w) - y)Q(dy, dw)$$

$$= - T_{k,Q}[\varepsilon_k,P]$$

and

$$T_{k,P}[\psi_k(P)] - r_{k,P} = \left(u^{J(k)}(X)\right)^T Q^{-1}_{uu} \int u^{J(k)}(x)(\psi_k(P)(w) - \psi_{id}(P)(w) - \varepsilon_P(y, w))P(dy, dw)$$

$$= T_{k,P}[\psi_k(P) - \psi_{id}(P)]$$

since $\int \varepsilon_P(y, w)P(dy, dw) = 0$. \qed

**Proof of Lemma D.4.** (1) By lemma D.3 $\psi_k$ is F-differentiable, i.e., for any $Q \in D_\psi$,

$$\left\| \psi_k(Q) - \psi_k(P) - D\psi_k(P)(Q - P) \right\|_{L^2([0,1])} = o(\|P - Q\|_{TV}).$$

Since $D_\psi$ is linear and $D_\psi \supseteq \text{lin}(D - \{P\})$ (see Lemma B.2), the curve $t \mapsto P + t\zeta$ with $\zeta \in D_\psi$ maps into $D_\psi$. Therefore, $\psi$ is DIFF($P(\cdot), D_\psi$) under all linear curves, $P(\cdot)$, at $P$ tangent to $D_\psi$.

(2) Part (1) and duality imply that

$$\sup_{\ell \in L^2([0,1]), \|\ell\|_{L^2([0,1])} = 1} |\ell[\psi_k(P + t\zeta) - \psi_k(P) - tD\psi_k(P)[\zeta]]| = o(|\zeta|_{TV})$$

(here we are abusing notation by using $\ell$ as both an element of $L^2([0,1])$ and as the functional). Therefore, one can set

$$\sup_{\ell \in L^2([0,1]), \|\ell\|_{L^2([0,1])} = 1} \eta_{k,\ell}(P + t\zeta, P) \leq o(|\zeta|_{TV})$$

(observe that the “o” may depend on $k$ but not on $\ell$).

In particular,

$$\sup_{\ell \in L^2([0,1]), \|\ell\|_{L^2([0,1])} = 1} \eta_{k,\ell}(P_n, P) \leq n^{-1/2}o(\|G_n\|_{TV})$$

where $G_n \equiv \sqrt{n}(P_n - P)$. By Lemma D.1 ($G_n)_{n \in \mathbb{N}}$ is weak* compact a.s.-$P^\infty$. This follows because, $||.||_S$ metrizes the weak* topology over the set of probability measures. Since weak* compact sets are norm bounded (see Lax [2002] p. 106 Thm 11), it follows that

$$\sup_{\ell \in L^2([0,1]), \|\ell\|_{L^2([0,1])} = 1} \eta_{k,\ell}(P_n, P) \leq o(n^{-1/2}).$$

\qed
E Appendix for Section 6

E.1 Appendix for Example 6.2

Proof of Lemma 6.1. Since \( \mu \) is a finite measure, it suffices to show the result for \( q = \infty \). Since \( \Theta \subseteq L^\infty \), evaluation functionals are linear and bounded and thus \( \Xi \subseteq \Theta^* \).

Note that for any \( \ell \in \Xi \) denoted by \( \ell = \delta_z \), \( t \mapsto \ell[\varphi_k(P)](t) = (\kappa^k(z))^{T}\Delta_k(P)^{-1}\nabla_k(P)(t) \) and \( ||\ell[\varphi_k(P)]||_{L^2(P)} = \sigma^2_k(P)(z) \). Therefore, if

\[
\sup_{\ell \in \Xi} \sqrt{n} \frac{\eta_{k,\ell}(P_n, P)}{||\ell[\varphi_k(P)]||_{L^2(P)}} = \sup_{z \in \mathbb{Z}} \sqrt{n} \frac{\eta_{k,\ell}(P_n, P)}{\sigma_k(P)} = o_P(1),
\]
then by Theorem 6.1,

\[
\sup_{z \in \mathbb{Z}} \left| \sqrt{n} \frac{\psi_k(P_n)(z) - \psi_k(P)(z)}{\sigma_k(P)} - (\kappa^k(z))^{T}\Delta_k(P)^{-1}n^{-1/2} \sum_{i=1}^{n} \frac{\nabla_k(P)(Z_i)}{\sigma_k(P)} \right| = o_P(1),
\]
as desired.

We now show that expression 22 in fact holds. For this we rely on Proposition 5.3. First, we note that since \( S \) is P-Donsker, \( G_n = \sqrt{n}(P_n - P) \) is \( \| \cdot \|_S \)-bounded a.s.-\( P \), hence we can set \( B_0 = \log n \) and this ensures that, for any \( \epsilon > 0 \), exists an \( M_{\epsilon} \geq 1 \) and a \( N_{\epsilon} \) such that the set \( A_\epsilon \equiv \{ \omega : \| G_n(\omega) \|_S \leq M_{\epsilon} \} \) has probability (under \( P \)) bigger than \( 1 - 0.5\epsilon \) for all \( n \geq N_{\epsilon} \).

Hence, applying Proposition 5.3 for any \( \ell = \delta_z \) (and with \( Q = G_n \), \( t = n^{-1/2} \) and \( B_0 = M_{\epsilon} \), over this set, it follows that

\[
P^\infty \left( \sup_{z \in \mathbb{Z}} \sqrt{n} \frac{\eta_k(P_n, P)}{\sigma_k(P)} \geq \epsilon \cap A_\epsilon \right) \\
\leq P^\infty \left( n^{-1/2} \max \left\{ \|G_n\|_S^3, \|G_n\|_S^{3+\theta} \right\} \|k^k(z)\| \times C(M, P, k) \geq \epsilon \cap A_\epsilon \right) \\
\leq 1 \left\{ M_{\epsilon}^3 \times n^{-1/2} \xi_k(P)^{-1} C(M, P, k) \geq \epsilon \right\}
\]

where the second line follows because \( \|G_n\|_S \) is a.s.-\( P \)-bounded and \( \sigma_k(P)(z) \geq \|k^k(z)\| \)

where \( \xi_k(P) \equiv e_{\min} \left( \Delta_k^{-1}(P)\Sigma_k(P)\Delta_k^{-1}(P) \right) \) (where \( e_{\min} \) is the minimal Eigenvalue).

Since, for any \( M \) and any \( k \in \mathbb{N}_0 \), \( n^{-1/2} \xi_k(P)^{-1} C(M, P, k) = o(1) \), expression 22 indeed follows.

The following Lemma is used in the proof of Proposition 6.1.

Lemma E.1. For any \((n, k) \in \mathbb{N}^2 \) and any \( q \in [1, \infty] \),

\[
\left\| \frac{(\kappa^k)^T\Delta_k(P)^{-1}}{\sigma_k(P)} \left( n^{-1/2} \sum_{i=1}^{n} \nabla_k(P)(Z_i) - \Delta_k(P)Z_k \right) \right\|_{L^q} = O_{Pr} \left( \frac{\beta_kk}{\sqrt{n}} \left( 1 + \frac{\log \left( \frac{\sqrt{n}}{\beta_k} \right)}{k} \right) \right)
\]

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where $\beta_k \equiv E_P[|| \Sigma_k(P)^{-1/2} \nabla_k(P)(Z)||^3]$, and the $Pr$ is the product measure between $P^\infty$ and standard Gaussian, and the “$O$” does not depend on $n$ or $k$.

**Proof.**

By Lemma 6.1 and the fact that $\mu$ is a finite measure, it is sufficient to show that

$$\left\| \frac{(k^k)^T \Delta_k(P)^{-1}}{\sigma_k(P)} \left( n^{-1/2} \sum_{i=1}^n \nabla_k(P)(Z_i) - \Delta_k(P)Z_k \right) \right\|_{L^\infty} = O_{Pr} \left( \beta_k \frac{\log \left( \frac{\sqrt{n}}{\beta k} \right)}{\sqrt{n}} \right),$$

where $Pr$ is the product measure between $P^\infty$ and the measure of $Z_k$. By letting $T_k \sim N(0, I_k)$ such that $\Delta_k(P)Z_k = \Sigma_k(P)^{1/2}T_k$, the LHS equals

$$\sup_{z \in Z} \left| \frac{(k^k(z))^T \Delta_k(P)^{-1} \Sigma_k(P)^{1/2}}{\sigma_k(P)(z)} \left( n^{-1/2} \sum_{i=1}^n \Psi_k(P)(Z_i) - T_k \right) \right|,$$

where $\Psi_k(P) \equiv \Sigma_k(P)^{-1/2} \nabla_k(P)$. By Cauchy-Swarchz inequality,

$$\left\| \frac{(k^k(z))^T \Delta_k(P)^{-1}}{\sigma_k(P)(z)} \left( n^{-1/2} \sum_{i=1}^n \nabla_k(P)(Z_i) - \Delta_k(P)Z_k \right) \right\|_{L^\infty} \leq \sup_{z \in Z} \left| \frac{(k^k(z))^T \Delta_k(P)^{-1} \Sigma_k(P)^{1/2}}{\sigma_k(P)(z)} \right| \left\| n^{-1/2} \sum_{i=1}^n \Psi_k(P)(Z_i) - T_k \right\| \leq \sqrt{\frac{(k^k(z))^T \Delta_k(P)^{-1} \Sigma_k(P) \Delta_k(P)^{-1}(k^k(z))}{\sigma_k^2(P)(z)}} \leq 1.$$

By Pollard [2002] Thm. 10, for any $\delta > 0,$

$$\Pr \left( \left\| n^{-1/2} \sum_{i=1}^n \Psi_k(P)(Z_i) - T_k \right\| \geq 3\delta \right) \leq C_0 \frac{\beta k}{n^{1/2}} \delta^{-3} \left( 1 + \frac{\log(n^{1/2} \delta^3/(\beta k))}{k} \right)$$

where $\beta \equiv E_P[|| \Psi_k(P)(Z)||^3]$ and $C_0$ is some universal constant.  

\[ \square \]
Proof of Proposition 6.1. By the triangle inequality

\[ \left\| \sqrt{n} \frac{\psi_k(P_n) - \psi_k(P)}{\sigma_k(P)} - \frac{(\kappa^k)^T Z_k}{\sigma_k(P)} \right\|_{L^q} \leq \left\| \frac{(\kappa^k)^T \Delta_k(P)^{-1}}{\sigma_k(P)} \left( n^{-1/2} \sum_{i=1}^n \nabla_k(P_i)Z_k - \Delta_k(P)Z_k \right) \right\|_{L^q} \]

Lemma 6.1 bounds the first term in the RHS. The second term in the RHS is bounded by \( \sup_{z \in Z} \sqrt{n} \frac{\eta_k,\delta_z(P_n, P)}{\sigma_k(P)(z)} \), so it remains to derive the rate of convergence of this term. That is, we want to find a \((r_{n,k})_{n,k}\) such that any \( \delta > 0 \) there exists a \( C_\delta \) and a \( N_\delta \) such that

\[ P^\infty \left( \sup_{z \in Z} r_{n,k} \sqrt{n} \frac{\eta_k,\delta_z(P_n, P)}{\sigma_k(P)(z)} \geq C_\delta \right) \leq \delta, \]

for all \( n \geq N_\delta \).

By the proof of Lemma 6.1, for any \( \epsilon > 0 \), there exists a \( M_\epsilon \geq 1 \) and a \( N_\epsilon \) such that for all \( n \geq N_\epsilon \) and all \( k \in \mathbb{N}_0 \),

\[ P^\infty \left( \sup_{z \in Z} r_{n,k} \sqrt{n} \frac{\eta_k,\delta_z(P_n, P)}{\sigma_k(P)(z)} \geq \epsilon \right) \]

\[ \leq P^\infty \left( \sup_{z \in Z} r_{n,k} \sqrt{n} \frac{\eta_k,\delta_z(P_n, P)}{\sigma_k(P)(z)} \geq \epsilon \cap A_{\epsilon} \right) \]

\[ \leq P^\infty \left( r_{n,k} n^{-1/2} \sup_{z \in Z} \frac{\max\{||G_n||_{\frac{3}{\delta}}, ||G_n||_{S_{\delta}}^{1+\epsilon}\}||k^N(z)|| \times C(M_\epsilon, P, k)}{\sigma_k(P)(z)} \geq \epsilon \cap A_{\epsilon} \right) + \epsilon \]

\[ \leq 1 \{ r_{n,k} M_3^3 \times n^{-1/2} E_k(P)^{-1} C(M_\epsilon, P, k) \geq \epsilon \} + \epsilon, \]

for all \( n \geq N_\epsilon \).

Therefore, setting \( \delta = \epsilon, C_\delta = 2M_3^3 \) and \( r_{n,k}^{-1} \equiv n^{-1/2} E_k(P)^{-1} C(l_n, P, k) \), it follows that

\[ P^\infty \left( \sup_{z \in Z} r_{n,k} \sqrt{n} \frac{\eta_k,\delta_z(P_n, P)}{\sigma_k(P)(z)} \geq C_\delta \right) \]

\[ \leq P^\infty \left( \sup_{z \in Z} r_{n,k} \sqrt{n} \frac{\eta_k,\delta_z(P_n, P)}{\sigma_k(P)(z)} \geq C_\delta \cap A_{\delta} \right) + \delta \]

\[ \leq P^\infty \left( r_{n,k} n^{-1/2} \sup_{z \in Z} \frac{\max\{||G_n||_{\frac{3}{\delta}}, ||G_n||_{S_{\delta}}^{1+\epsilon}\}||k^N(z)|| \times C(M_\delta, P, k)}{\sigma_k(P)(z)} \geq C_\delta \cap A_{\delta} \right) + \delta \]

\[ \leq 1 \{ r_{n,k} M_3^3 \times n^{-1/2} E_k(P)^{-1} C(M_\delta, P, k) \geq M_\delta \} + \delta \]

\[ \leq 1 \{ M_3^3 \geq C_\delta \} + \delta = \delta \]

where the last line follows because \( C(M, P, k)/C(l_n, P, k) < 1 \) for any constant \( M \) and \( n \) sufficiently large. 

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E.1.1 The Role of the Choice of \( \Theta \)

In the text we assumed that \( \Theta \subseteq L^q \cap L^\infty \), this assumption ensured that the evaluation functional is well-defined for elements of \( \Theta \), formally, it ensure that the evaluation functional belongs to \( \Theta^\ast \). If this is not the case, i.e., \( \Theta \subseteq L^q \) but not necessarily \( \Theta \subseteq L^\infty \), then \( z \mapsto \sqrt{n} \psi_k(P_n(z)) - \psi_k(P)(z) \) may not even be well-defined and the approach developed in the text is not valid. However, we now show that by exploiting results for duals of \( L^q \) (see Lax [2002]), Theorem 6.1 still can be used as the basis of inferential results for \( L^q \)-confidence bands; what changes is the scaling.

For all \( q < \infty \), let \( \Xi_q = \left\{ f \in L^q : ||f||_{q^*} \leq 1 \right\} \) with \( q^* = \frac{q}{q-1} \). We note that by defining \( \Xi_q \) in this way, we are abusing notation, because the mapping \( \ell \) is a linear bounded functional operating over \( L^q \), the fact that we view this mapping as an element of \( \Xi_q \) is due to dual representation results; see Lax [2002].

Also, let

\[
\bar{\sigma}^2_{k,q}(P) = \sup_{\ell \in \Xi_q} (\ell[k])^T \Delta_k^{-1}(P) \Sigma_k(P) \Delta_k^{-1}(P)(\ell[k]).
\]

Observe that for \( q = \infty \), \( \bar{\sigma}^2_{k,\infty}(P) = \sup_z \sigma^2_k(P)(z) \). But this relationship only holds in \( q = \infty \), for \( q < \infty \), \( \sigma_k(P)(z) \) may not be even well-defined.

For these choices, it follows that

**Lemma E.2.** Suppose Assumptions 5.1 and 5.2 hold. Then, for any \( q \in [1, \infty] \) and any \( k \in \mathbb{N}_0 \),

\[
\left\| \sqrt{n} \frac{\psi_k(P_n) - \psi_k(P)}{\bar{\sigma}_{k,q}(P)} - (\kappa^k)^T \Delta_k^{-1}(P) \Sigma_k(P) \Delta_k^{-1}(P)(\kappa^k) \right\|_{L^q} = o_P(1).
\]

**Proof.** To ease the notational burden, we use \( ||.||_q \) to denote \( ||.||_{L^q} \). Also, recall that \( \Xi_q = \left\{ f \in L^q : ||f||_{q^*} \leq 1 \right\} \) and \( \Xi_\infty = \left\{ \text{All Delta Dirac measures over } \mathbb{Z} \right\} \).

Let, for any \( q \in [1, \infty) \), \( (f, \eta) \mapsto \langle f, \eta \rangle_q \equiv \int f(z') \eta(z) \mu(dz) \) and \( (f, \eta) \mapsto \langle f, \eta \rangle_\infty \equiv \int f(z) \eta(dz) \). By duality \( ||.||_q = \sup_{\ell \in \Xi_q} ||\langle \cdot, \ell \rangle_q|| \) (see Lax [2002]), and

\[
||f||_q = \sup_{z \in \mathbb{Z}} |f(z)| = \sup_{z \in \mathbb{Z}} \int f(z') \delta_z(dz') = \sup_{\ell \in \Xi_\infty} \int f(z') \ell(dz'),
\]

so that \( ||.||_q = \sup_{\ell \in \Xi_q} ||\langle \cdot, \ell \rangle_q|| \).
By duality and straightforward algebra,

\[
\left\| \sqrt{n} \frac{\psi_k(P_n) - \psi_k(P)}{\bar{\sigma}_{k,q}(P)} - (\kappa^k)^T \Delta_k(P)^{-1/2} \sum_{i=1}^n \nabla_k(P)(Z_i) \right\|_q \\
= \sup_{\ell \in \Xi_q} \left| \ell(\sqrt{n}(\psi_k(P_n) - \psi_k(P)) - (\kappa^k)^T \Delta_k(P)^{-1/2} \sum_{i=1}^n \nabla_k(P)(Z_i)) \right| \\
\leq \sup_{\ell \in \Xi_q} \left| \ell(\sqrt{n}(\psi_k(P_n) - \psi_k(P))) - (\ell[\kappa^k])^T \Delta_k(P)^{-1/2} \sum_{i=1}^n \nabla_k(P)(Z_i)) \right| \\
\times \sup_{\ell \in \Xi_q} \left| \sqrt{(\ell[\kappa^k])^T \Delta_k^{-1}(P) \Sigma_k(P) \Delta_k^{-1}(P)(\ell[\kappa^k])} \right|.
\]

The second term equals one by definition of $\bar{\sigma}_{k,q}(P)$. The first term can be bounded by analogous arguments to those employed on the proof of Lemma 6.1 and they will be omitted.

Lemma 6.1 shows that in order to characterize the asymptotic distribution of $\left\| \sqrt{n} \frac{\psi_k(P_n) - \psi_k(P)}{\bar{\sigma}_{k,q}(P)} \right\|_{L^q}$ it suffices to characterize the one of $\left\| n^{-1/2} \sum_{i=1}^n \frac{\nabla_k(P)(Z_i)}{\bar{\sigma}_{k,q}(P)} \right\|_{L^q}$. In the following proposition we accomplish by employing coupling results (e.g. Pollard [2002]).

**Proposition E.1.** Suppose Assumptions 5.1 and 5.2 hold. Then, for any $q \in [1, \infty]$ and any $k \in \mathbb{N}_0$,

\[
\left\| \sqrt{n} \frac{\psi_k(P_n) - \psi_k(P)}{\bar{\sigma}_{k,q}(P)} \right\|_{L^q} = O_P \left( \frac{\beta_k}{\sqrt{n}} \left( 1 + \frac{\log \left( \frac{\sqrt{n}}{\beta_k} \right)}{k} \right)^{r_{k,n}^{-1}} \right),
\]

where $Z_k \sim N(0, \Delta_k(P)^{-1} \Sigma_k(P) \Delta_k(P)^{-1})$ and $\beta_k \equiv E[\| \Delta_k(P)^{-1} \nabla_k(P)(Z) \|^3]$.

**Proof.** By noting that

\[
\left\| \sqrt{n} \frac{\psi_k(P_n) - \psi_k(P)}{\bar{\sigma}_{k,q}(P)} \right\|_{L^q} \leq \left\| \sqrt{n} \frac{\psi_k(P_n) - \psi_k(P) - (\kappa^k)^T Z_k}{\bar{\sigma}_{k,q}(P)} \right\|_{L^q},
\]

the proof is analogous to that of Proposition 6.1 and thus omitted. □

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