Incentivizing Exploration with Selective Data Disclosure

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Abstract

We propose and design recommendation systems that incentivize efficient exploration. Agents arrive sequentially, choose actions and receive rewards, drawn from fixed but unknown action-specific distributions. The recommendation system presents each agent with actions and rewards from a subsequence of past agents, chosen ex ante. Thus, the agents engage in sequential social learning, moderated by these subsequences. We asymptotically attain first-best benchmarks for exploration, using a flexible frequentist behavioral model and mitigating rationality and commitment assumptions inherent in prior work. We suggest three components of effective recommendation systems: independent focus groups, group aggregators, and interlaced information structures.

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1 Introduction

A prominent feature of online platform markets is the pervasiveness of reviews and ratings. The review and rating ecosystem creates a deep dilemma for market design. On the one hand, platforms would like to allow each consumer to make an informed choice by presenting the most comprehensive and comprehensible information. On the other hand, platforms need to encourage consumers to explore infrequently-selected alternatives in order to learn more about them. Extensive exploration may be required in settings, like ours, where the reward of an alternative is stochastic. The said exploration, while beneficial for the common good, is often misaligned with incentives of individual consumers. Being short-lived, individuals prefer to exploit available information, selecting alternatives that look best based on this information. This behavior can cause herding in which all consumers take a sub-optimal alternative if, for example, all consumers see all prior ratings. Aside from such extreme behaviors, some alternatives may get explored at a very suboptimal rate, or suffer from selection bias.

Prior work leveraged information asymmetry to mitigate this tension between exploration and exploitation. The platform chooses a single recommendation for each consumer based on past ratings, and does not disclose any other information about the ratings. Assuming, as is standard, that consumers are Bayesian rational and the platform has the power to commit, platforms can incentivize sufficient exploration to enable asymptotically efficient social learning. However, these assumptions can be problematic in practice: consumers may hesitate to follow recommendations because of limited rationality, a preference for detailed and interpretable information, or a desire for transparency stemming from insufficient trust in the platform’s commitment power.

Our work also leverages information asymmetry to induce social learning, but does so with a restricted class of platform policies which enable a more permissive behavioral model. We restrict the platform to order-based disclosure policies which provide each consumer with a subhistory of past ratings. Specifically, a partial order on the arrivals is fixed ex-ante (and can be made public w.l.o.g.), and each consumer observes the ratings of everyone who precedes her in this partial order. Put differently, an order-based disclosure policy constructs a communication network for the consumers, and lets them engage in social learning on this
network. We assume consumers act like frequentists: to estimate the reward of a given alternative, they follow the empirical mean of past ratings and form a confidence interval. The actual estimate can lie in a wide range consistent with the confidence interval. This is justified because each provided subhistory is unbiased – it cannot be biased to make a particular action look good as it is chosen ex ante – and transitive – it contains the information sets of all consumers therein. The latter property (and the assumption that consumers do not receive private signals) implies that a consumer does not need to reason about the rationale for the observed prior choices.

Our framework provides several key benefits. First, our behavioral model only needs to define how consumers react when they observe the history of ratings for all previous consumers. This is because, due to the unbiasedness and transitivity properties mentioned above, the only ratings that can possibly influence a consumer’s beliefs are those included in her subhistory. Second, the consumers can deviate from exact utility-maximization, and do not need to define or justify their behavior via complex reasoning and detailed knowledge of the platform’s policy. Third, the platform does not require detailed information about consumers’ initial preferences. Fourth, order-based disclosure policies are arguably easier to audit than the complex code bases and behaviors of general data-dependent policies (see, e.g., Agarwal et al. (2017)), thereby weakening the need for commitment. In contrast to prior work, our framework therefore relaxes rationality and commitment assumptions without abusing them, and provides detailed interpretable information to consumers.

We design several order-based disclosure policies in the context of this framework, of increasing complexity and improving performance guarantees. Our policies intertwine subhistories in a certain way, provably providing consumers with enough information to converge on the optimal alternative. Our best policy matches the best possible convergence rates, even absent incentive constraints. This policy also ensures that each consumer sees a substantial fraction of the prior history.

Our work suggests the importance of several design considerations. First, independent focus groups provide natural exploration due to random fluctuations in observed rewards. These natural experiments can then be provided to future consumers. Second, improving beyond very suboptimal learning rates requires adaptive exploration which gradually zooms in on the better alternatives. For ex-
ample, if the focus groups learn the optimal alternative quickly, then this information should be propagated; otherwise additional exploration is required. This adaptivity can be achieved, even with subhistories chosen ex ante, by introducing group aggregators that see the subhistory of some, but not all, focus groups. Third, optimal learning rates require reusing observations; otherwise too many consumers make choices with limited information. The reused observations must be carefully interlaced to avoid contamination between experiments.

We start with a simple policy which runs a full-disclosure policy “in parallel” on several disjoint subsets of consumers (the “focus groups” mentioned above), collects all data from these runs, and discloses it to all remaining consumers.1 We think of this policy as having two “levels”: Level 1 contains the parallel runs, and Level 2 is everyone else (corresponding to exploration and exploitation, respectively). While this policy provably avoids herding on a suboptimal alternative, it over-explores bad alternatives and/or under-explores the good-but-suboptimal ones, which makes for very inefficient learning.

Our next step is a proof-of-concept implementation of adaptive exploration, achieving a proof-of-concept improvement over the previous construction. We focus on the case of two alternatives, and upgrade the simple two-level policy with a middle level. Each consumer in this new level is a “group aggregator” who receives the data collected by its respective “group”: a subset of parallel runs from the first level. These consumers explore only if the gap between the best and second-best alternative is sufficiently small, and exploit otherwise. When the gap is small, the parallel runs do not have sufficient time to distinguish the two alternatives before herding on one of them. However, a given alternative can, with some probability, empirically outperform the others within in a given “group”, inducing the “group aggregator” to explore it more. This provides the third-level consumers with enough samples to distinguish the two alternatives.

The main result essentially captures the full benefits of adaptive exploration. We extend the three-level construction to multiple “levels”, connected in fairly intricate ways, using “group aggregators” and reusing observations as discussed above. For each piece of our construction, we prove that consumers’ collective self-interested behavior guarantees a certain additional amount of exploration if,

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1The full-disclosure policy on a given subset $S$ of consumers reveals to each consumer in $S$ the history of ratings for all previous consumers in $S$. 

and only if, more exploration is needed at this point. The guarantee substantially
depends on the parameters of the problem instance (and on the level at which this
piece resides), and critically relies on how the pieces are wired together.

Our framework is directly linked to multi-armed bandits, a popular abstraction for
designing algorithms to balance exploration and exploitation. An order-based policy incentivizes consumers to implement some bandit algorithm, and consumers’ welfare is precisely the total reward of this algorithm. The two-level policy implements a well-known bandit algorithm called explore-then-exploit, which explores in a pre-defined way for a pre-set number of rounds, then picks one alternative for exploitation and stays with it for the remaining rounds. Our multi-level policy implements a bandit algorithm which can change its exploration schedule only a small number of times, each change-point corresponding to a level in our construction. (This is “adaptive exploration” with severely limited adaptivity, and not one of the standard bandit algorithms.)

We analyze our policies in terms of regret, a standard notion from the literature on multi-armed bandits, defined as the difference in total expected rewards between the best alternative and the algorithm. We obtain regret rates that are sublinear in the time horizon, implying that the average expected reward converges to that of the best alternative. The multi-level policy matches the optimal regret rates for bandits, for a constant number of alternatives. The two-level policy matches the standard (and very suboptimal) regret rates of bandit algorithms such as explore-then-exploit that do not use adaptive exploration. And the three-level policy admits an intermediate guarantee. Moreover, regret bounds for the multi-level policy decrease drastically for “easy” instances of multi-armed bandits where the marginal benefit of the best option is high, a qualitative improvement compared to the two- and three-level policies.

Our performance guarantees are robust in that they hold in the worst case over
a class of reward distributions, and do not rely on priors. Moreover, our constructions are robust to small amounts of misspecification. First, all parameters can be increased by at most a constant factor (and the two-level construction allows a much larger amount of tweaking). Second, we accommodate some “information leakage”, e.g., rounds that are observable by other focus groups.

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2Essentially, this is how much one regrets not knowing the best arm in advance.
Map of the paper. Section 2 surveys related work. Section 3 presents the model of incentivized exploration and our approach within this model (i.e., order-based policies with frequentist agents). The next three sections present our results on order-based policies, progressing from two to three to multiple levels as discussed above. Section 7 is on robustness of our policies. Section 8 contains additional discussion of our model and related work. All proofs are deferred to the appendix.

2 Related work

The problem of incentivized exploration, as studied in our paper, was introduced in (Kremer et al., 2014) and was motivated, as is our work, by recommendation systems. Similar to this literature, our work features short-lived agents whose actions are coordinated by a central principal with an eye towards maximizing long-run welfare. However, the models in prior work required strong assumptions of Bayesian rationality and the power to commit to arbitrary policies. Under these assumptions, a platform's policy can be reduced to a multi-armed bandit algorithm which recommends an action to each agent and satisfies Bayesian incentive-compatibility (BIC). The novelty in our work is twofold: 1) the principal is restricted to disclosing past reviews (whereas prior work permitted an arbitrary message space, and w.l.o.g. focused on direct recommendations), and 2) the agents are allowed a broad class of data-driven behaviors. We design policies that incorporate these realistic assumptions without sacrificing performance, and illuminate novel insights that go beyond prior work. As discussed in the Introduction, we find that recommendation systems benefit from partitioning early agents into “focus groups”, providing subsequent agents with information gathered by “independent” focus groups, so as to avoid herding; carefully “reuse” this information to speed up the learning process. A detailed comparison of our assumptions to those in prior work is deferred to Section 8.1.

Our restriction to disclosure policies has precedence in the literature on strate-

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3 In a simultaneous and independent work, Che and Hörner (2018) formalize a similar motivation using a very different model with continuous information flow and a continuum of agents.

4 We call this problem BIC incentivized exploration. Various aspects thereof have been investigated after (Kremer et al., 2014), e.g., in Mansour et al. (2020, 2021); Bimpikis et al. (2018); Bahar et al. (2016, 2019); Sellke and Slivkins (2021). Surveys of this work can be found in Slivkins (2019, Chapter 11) and Slivkins (2021). A version with monetary incentives but without information asymmetry was studied in Frazier et al. (2014); Chen et al. (2018).
gic disclosure initiated by Grossman (1981); Milgrom (1981); Milgrom and Roberts (1986). As in those models, our paper assumes only a selection of facts can be disclosed to the potential consumer, and those facts (other consumers’ reviews in our case) can not be manipulated by the platform. Those papers often exhibit unraveling due to the ability of the sender to select which facts to disclose based on the facts themselves. In contrast, our policies commit to a selection of reviews up front, before they are revealed, and so avoid unraveling. Prior work has also considered full disclosure policies, especially in the context of recommendation systems, in which every prior review is revealed to each consumer. A full-disclosure policy implements the “greedy” bandit algorithm which only exploits, and suffers from herding on a suboptimal alternative (see Section 8.2). However, under strong assumptions on the primitives of the economic environment, including the structure of rewards and diversity of agent types, full disclosure avoids herding and performs well for heterogenous agents (Kannan et al., 2018; Bastani et al., 2021; Raghavan et al., 2021; Acemoglu et al., 2021). Our work avoids herding through design by leveraging partial disclosure policies that guarantee a degree of “independence” between information flows. Vellodi (2018) similarly observes that suppressing reviews can improve recommendation systems relative to full disclosure policies, albeit due to endogenous entry of firms.

Our behavioral assumptions have roots in prior work on non-Bayesian models of behavior. In much of this literature, agents use a (often naive) variant of statistical inference to form beliefs in which they infer the state of the world from samples. Our frequentist agents similarly rely on simple forms of data aggregation to form beliefs, albeit with good justification as the subhistories they observe are unbiased and transitive. Such behavioral assumptions are most prominent in the literature on social learning, starting from DeGroot (1974) who posits that agents use “naive”, mechanical rules-of-thumb, forming beliefs and/or taking actions based on naive averaging of observations. These models are well-founded in practice: Chandrasekhar et al. (2020) show experimentally that such behavioral models are good predictors of human behavior in some social learning scenarios; Dasaratha and He (2019) show similar results for sequential social learning. Theoretically, Dasaratha and He (2020) use this model of naivety to study the question

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5 Most of these results use a mathematically equivalent framing in terms of multi-armed bandits.
6 Our model also admits Bayesian-rational agents with certain priors, as discussed in Section 3.3.
of how to design the social network in a sequential learning model so as to induce optimal learning rates. They observe that silo structures akin to our two-level policy improve learning rates. The class of behaviors we study are also reminiscent of the inference procedures studied in Salant and Cherry (2020), or the learning algorithms and rules analyzed in Cho and Libgober (2020); Liang (2019), albeit in different settings.

Our work can be interpreted as coordinating social learning (by designing a network on which the social learning happens). However, all prior work on social learning studies models that are very different from ours. We defer the detailed comparison to Section 8.3.

On the machine learning side, we focus on regret-minimization, following the vast literature on multi-armed bandits over the past 2-3 decades (see books by Bubeck and Cesa-Bianchi (2012), Slivkins (2019), and Lattimore and Szepesvári (2020)). The social-planner version of our model corresponds to stochastic bandits, a standard, basic version of multi-armed bandits with i.i.d. rewards and no auxiliary structure. Markovian bandit formulations with time-discounting (Gittins et al., 2011; Bergemann and Välimäki, 2006) and various other connections between bandits and mechanism design (surveyed, e.g., in Slivkins (2019, Chapter 11.7)) are less relevant to our model.

3 Our model and approach

We study the incentivized exploration problem, in which a platform (principal) faces a sequence of T myopic consumers (agents). There is a set \( \mathcal{A} \) of possible actions (arms). At each round \( t \in [T] := \{1, 2, \ldots, T\} \), a new agent \( t \) arrives, receives a message \( m_t \) from the principal, chooses an arm \( a_t \in \mathcal{A} \), and collects a binary reward \( r_t \in \{0, 1\} \). The message \( m_t \) could be arbitrary, e.g., a recommended action or, in our case, a subset of past reviews. The principal chooses messages according to a decision rule called the disclosure policy. The agents’ response \( a_t \) is defined as a function of round \( t \) and message \( m_t \). Both the disclosure policy and agent behavior are constrained to a subset of well-motivated choices, as outlined in Sections 3.2 and 3.3. The reward from pulling an arm \( a \in \mathcal{A} \) is drawn independently from Bernoulli distribution \( D_a \) with an unknown mean reward \( \mu_a \). A problem instance is defined by (known) parameters \( |\mathcal{A}|, T \) and (unknown) mean rewards \( \mu_a : a \in \mathcal{A} \).
The information structure is as follows. Each agent $t$ does not observe anything from the previous rounds, other than the message $m_t$. The chosen arm $a_t$ and reward $r_t$ are observed by the principal (which corresponds, e.g., to the consumer leaving a rating or review on the platform).

We assume that mean rewards are bounded away from 0 and 1, to ensure sufficient entropy in rewards. For concreteness, we posit $\mu_a \in \left[ \frac{1}{3}, \frac{2}{3} \right]$. While we focus on the paradigmatic case of Bernoulli rewards, we can handle arbitrary rewards $r_t \in [0,1]$ with only minor modifications to the analysis. In essence, the range of rewards is small compared to the number of samples, like in all prior work on incentivized exploration. This is a very standard assumption throughout machine learning, and it is justified in small-stakes applications such as recommendation systems for movies, restaurants, etc.

### 3.1 Objective: regret

The principal’s objective is to maximize agents’ rewards. The social-planner version, when the principal can directly choose actions $a_t$ without any restrictions, is precisely the basic version of multi-armed bandits termed stochastic bandits. We characterize the principal’s performance using the notion of regret, a standard objective in stochastic bandits. Formally, regret is defined as

$$
\text{Reg}(T) = T \max_{a \in A} \mu_a - \sum_{t \in [T]} E[\mu_{a_t}],
$$

where the expectation is over the randomness in rewards and the messaging policy. Thus, regret is the difference, in terms of the total expected reward, between the principal’s policy and the first-best policy which knows the mean rewards a priori. Notably, the rewards in this objective are not discounted over time.

Following the bandit literature, we focus on the dependence on $T$, the number
of agents (which is, effectively, the time horizon). Assuming regret is sublinear in \( T \), the average expected reward converges to that of the best arm at rate \( \text{Reg}(T)/T \).

We are mainly interested in robust upper bounds on regret that hold in the worst case over all (valid) mean rewards. This provides guarantees (even) for a principal that has no access to a prior or simply does not make use of one due to extreme risk aversion. We are also interested in performance of a policy at a given round \( t \), as measured by instantaneous regret \( \max_{a \in A} \mu_a - \mathbb{E}[\mu_{a_t}] \), also known as simple regret.\(^{10}\)

Regret in our model can be directly compared to regret in the stochastic bandit problem with the same mean rewards. Following the literature, we define the gap parameter \( \Delta \) as the difference between the largest and second largest mean rewards (informally, the difference in quality between the top two options). The gap parameter is not known (to the platform in incentivized exploration, or to the algorithm in bandits). Large \( \Delta \), i.e., the best option being far better than the second-best, naturally corresponds to “easy” problem instances. The literature is mainly concerned with asymptotic upper bounds.\(^{11}\) on regret in terms of the time horizon \( T \), as well as parameters \( \Delta \) and the number of arms \( K \). Throughout, we assume that the number of arms \( K = |A| \) is constant. However, we explicitly note the dependence on \( K \) when appropriate, e.g., we use \( O_K(\cdot) \) notation to note that the “constant” in \( O() \) can depend on \( K \) (and nothing else).

Optimal regret rates in stochastic bandits are as follows (Auer et al., 2002a,b; Lai and Robbins, 1985):

\[
\text{Reg}(T) \leq O\left( \min\left( \sqrt{KT \log T}, \frac{K}{\Delta} \log T \right) \right).
\]

This includes a worst-case regret rate \( O(\sqrt{KT \log T}) \) which applies to all problem instances, and a gap-dependent regret rate of \( O(\frac{K}{\Delta} \log T) \). We match both regret rates for a constant number of arms. Either regret rate can only be achieved via adaptive exploration: i.e., when the exploration schedule is adapted to the observations. It is particularly notable that we implement adaptive exploration in our environment, even though the platform must commit to the subhistories ex ante. This constraint requires a substantial number of agents to naturally serve an exploration role in

\(^{10}\)Note that summing up the simple regret over all rounds \( t \in [T] \) gives \( \text{Reg}(T) \).

\(^{11}\)We use standard asymptotic notation to characterize regret rates: \( O(f(T)) \) and \( \Omega(f(T)) \) mean, resp., at most and at least \( f(T) \), up to constant factors, starting from large enough \( T \). Similarly, \( \widetilde{O}(f(T)) \) notation suppresses polylog\( (T) \) factors.
some outcomes and an exploitation role in others.

A simple example of non-adaptive exploration is the explore-then-exploit algorithm, which samples arms uniformly at random for the first $N$ rounds, for some pre-set number $N$, then chooses one arm and sticks with it till the end. We implement this algorithm in our economic environment with our 2-level policy outlined in Section 4. Such algorithms suffer from $\Omega(T^{2/3})$ regret, both in the worst case and for each problem instance.\footnote{More precisely, there is a tradeoff between the worst-case and per-instance performance: if the algorithm achieves regret $O(T^\gamma)$ for all instances, for some $\gamma \in [2/3, 1)$, then its regret for each instance can be no better than $\Omega\left(T^{2(1-\gamma)}\right)$. The latter is $\Omega(T^{2/3})$ when $\gamma = 2/3$. This result extends to a more general model of non-adaptive exploration, where each round either gives up on exploitation (namely: the chosen arm does not depend on the previous observations), or does not contribute to exploration (namely: its reward cannot be used in the future). This result is from Babaioff et al. (2014), but the worst-case lower bound has been “folklore knowledge” in the community.}

Regret in incentivized exploration is subject to two important limitations, even under the BIC model from prior work. Mansour et al. (2020) matches (2) for a constant number of arms $K$, but with (i) a multiplicative factor that can get arbitrarily large depending on the prior, and (ii) an exponential dependence on $K$. Both limitations are essentially inevitable (Sellke and Slivkins, 2021).\footnote{Subsequently to our work, Sellke and Slivkins (2021) achieve poly($K$) scaling in regret, albeit only when the Bayesian prior is independent across arms and only in expectation over the prior.} Our result matches (2) in a similar way (dependence on the prior is replaced with that on a parameter in the agents’ choice model). We also note that much of the prior work on incentivized exploration targets $K = 2$ actions (e.g., Kremer et al., 2014; Che and Hörner, 2018; Bimpikis et al., 2018; Bahar et al., 2016).

### 3.2 Disclosure policies: order-based

We focus on messaging policies of a particular form. First, we use disclosure policies, where the message $m_t$ in each round $t$ discloses the subhistory for some subset $S = S_t$ of past rounds. Formally, the subhistory is defined as

$$H_S = \{(s, a_s, r_s) : s \in S\}, \quad (3)$$

where the tuple $(s, a_s, r_s)$ is the outcome for a given agent $s \in S$. The subhistory can correspond, for example, to a subset of past reviews. Second, we assume that the subset $S_t$ is chosen ex ante, before round 1, and therefore does not depend on the
previous observations. Such a message is called *unbiased subhistory*; it means the platform can not bias the set of reviews it shows a consumer, e.g., by selecting only those in which a particular arm has positive ratings. Third, we fix a partial order on the rounds, and define each $S_t$ as the set of all rounds that precede $t$ in the partial order. The resulting disclosure policy is called *order-based*.

Order-based disclosure policies are *transitive*, in the following sense:

$$t \in S_{t'} \Rightarrow S_t \subset S_{t'}$$ for all rounds $t, t' \in [T]$. \hspace{1cm} (4)

In words, if agent $t'$ observes the outcome for some previous agent $t$, then she observes the entire message revealed to that agent. In particular, agent $t'$ does not need to second-guess which message has caused agent $t$ to choose action $a_t$.

For convenience, we will represent an order-based policy as an undirected graph, where nodes correspond to rounds, and any two rounds $t < t'$ are connected if and only if $t \in S_{t'}$ and there is no intermediate round $t''$ with $t \in S_{t''}$ and $t'' \in S_{t'}$. This graph is henceforth called the *information flow graph* of the policy, or *info-graph* for short. (As an illustration, see Figure 1 below.) We assume that this graph is common knowledge.

Figure 1: The information flow graph for a full disclosure policy.

While detail-oriented agents may prefer to observe full data, our policies show all but a few past datapoints to all but a few agents, and our main result shows a certain fraction of the full history to *all* agents. Besides, even a small fraction of the full history would typically contain a large number of observations (preselected in an unbiased way), probably more than a typical agent ever needs.

One key benefit of order-based policies is that we only need to define how agents $t$ react to the full history $\mathcal{H}[t-1]$ collected by some such policy. Indeed, each agent $t$ observes the full history collected by the disclosure policy restricted to rounds in $S_t$. This policy can be interpreted as a standalone order-based disclosure policy, since (by unbiasedness and transitivity) it cannot be affected by any agents that are not in $S_t$. In particular, there is no reason for the agent to second-guess what data has been seen by previous agents when they chose their actions, because all this data is observed.
3.3 Agents’ behavior: frequentist

We assume agents behave as frequentists in response to order-based policies. In light of the discussion above, how would a frequentist agent choose an action given the full history of observations? She would construct a confidence interval on the expected reward of each action, taking into account the average reward of this action and the number of observations, and place the action’s estimate somewhere in this confidence interval. The system can provide summary statistics, so that agents would not even need to look at the raw data.

We formalize this behavior as follows. Each agent \( t \) uses its observed subhistory \( m_t \) to form a reward estimate \( \hat{\mu}_{t,a} \in [0,1] \) for each arm \( a \in A \), and chooses an arm with a maximal estimate.\(^{14}\) A simple instantiation is that \( \hat{\mu}_{t,a} \) is the sample average for arm \( a \) over the subhistory \( m_t \), as long as it includes at least one sample for \( a \); else, \( \hat{\mu}_{t,a} = \frac{1}{2} \). We allow a much more permissive model, where agents can form arbitrary reward estimates as long as they lie within some “confidence range” of the sample average. Moreover, we allow agents to have strong initial beliefs, whose effect is eventually drowned out. Formally, we make the following assumptions.

**Assumption 3.1.** Let \( N_{t,a} \) and \( \bar{\mu}_{t,a} \) denote the number of pulls and the empirical mean reward of arm \( a \) in subhistory \( m_t \). Then for some absolute constant \( N_{est} \in \mathbb{N} \) and \( C_{est} = \frac{1}{16} \), and for all agents \( t \in [T] \) and arms \( a \in A \) it holds that

\[
\text{if } N_{t,a} \geq N_{est} \quad \text{then} \quad |\hat{\mu}_{t,a} - \bar{\mu}_{t,a}| < \frac{C_{est}}{\sqrt{N_{t,a}}} \quad \text{(5)}
\]

\[
\text{if } N_{t,a} = 0 \quad \text{then} \quad \hat{\mu}_{t,a} \geq \frac{1}{3}. \quad \text{(6)}
\]

We make no assumptions if \( 1 \leq N_{t,a} < N_{est} \). The \( \frac{1}{3} \) threshold in Eq. (6) can be replaced with an arbitrary strictly positive constant, with very minor changes.

**Assumption 3.2.** In each round \( t \), the estimates \( (\hat{\mu}_{t,a} : a \in A) \) depend only on the multiset \( m'_t = \{(a_s,r_s) : s \in S_t\} \), called anonymized subhistory. Each agent \( t \) forms its estimates according to some function \( f_t \) from anonymized subhistories to \([0,1]|A|\), so that \( (\hat{\mu}_{t,a} : a \in A) = f_t(m'_t) \). For each \( t \), this function is drawn independently from some fixed (but otherwise arbitrary) distribution.

\(^{14}\)To simplify proofs, ties between the reward estimates are broken according to some fixed, deterministic ordering over the arms. This is to rule out adversarial manipulation of the tie breaking, and to ensure that all agents with the same data choose the same arm.
The fact that $\hat{\mu}_{t,a}$ falls below $1/3$ after a long sequence of low rewards appears inconsistent with the restriction that $\mu_a \in [1/3, 2/3]$. However, one could argue that agents are unaware of this restriction because they have incomplete information and/or are unsophisticated. Alternatively, all reward estimates can be projected into the $[1/3, 2/3]$ interval,\(^{15}\) assuming random tie-breaking when multiple arms achieve the highest reward estimate. This variant works with minimal changes.

Our model allows for significant flexibility in agent behavior. An optimistic (resp., pessimistic) agent may choose a reward estimate as a value towards the top (resp., bottom) of its confidence interval. Moreover, an agent can randomize its choices, by randomizing its reward estimates within their confidence intervals. This flexibility can be arm-specific as well. First, the way $\hat{\mu}_{t,a}$ depends on the data for arm $a$ can vary from one arm to another. For instance, an agent interested in restaurants can be optimistic about Chinese restaurants and pessimistic about Italian ones. Second, the reward estimates can depend on the data for other arms: e.g., an agent is more optimistic about Chinese restaurants if the Italian ones are good. Third, the estimates can be arbitrarily correlated across arms: e.g., it is sunny today, and an agent feels optimistic about all restaurants.

While our model is flexible enough to allow substantial deviations from rationality, it also encompasses Bayesian agents with appropriate beliefs. Specifically, suppose agents believe that each $\mu_a$ is drawn independently from some Beta-Bernoulli distribution $P_a$. Then the reward estimate $\hat{\mu}_{t,a}$ is the posterior mean reward given the subhistory $m_t$. This is consistent with Assumption 3.1 for a large enough prior-dependent constant $N_{\text{est}}$.\(^{16}\) Beta-Bernoulli beliefs are well-specified in that their support necessarily contains the true model. While such beliefs are inconsistent with the restriction that $\mu_a \in [1/3, 2/3]$, one could argue that Bayesian agents might be unaware of this restriction. Also, our disclosure policies guarantee that the posterior beliefs are “asymptotically consistent” with $\mu_a \in [1/3, 2/3]$, in the sense that they get arbitrarily close to $[1/3, 2/3]$ over time.\(^{17}\)

\(^{15}\)That is, truncate the reward estimate at $1/3$ if it is too low, and at $2/3$ if it is too high.

\(^{16}\)Essentially, this is because for Beta-Bernoulli priors the absolute difference between the posterior mean and the empirical mean scales as $1/\#\text{samples}$.

\(^{17}\)The formal statement is as follows: for some $\alpha, \beta \in (0, 1)$, each agent $t > T^a$ forms a posterior belief such that $\Pr[\mu_a \in [1/3, 2/3]] < O(T^{-\beta})$ for each arm $a$. This is because our disclosure policies guarantee that such agents $t$ observe sufficiently many samples of each arm.
4 A simple two-level policy

We first design a simple policy that exhibits asymptotic learning (i.e., sublinear regret). While not achieving an optimal regret rate, this policy illuminates a key feature: initial agents are partitioned into focus groups. Each agent sees the history for all previous agents in the same focus group (and nothing else). The information generated by these focus groups is then presented to later agents. We think of this policy as having two levels: the exploration level containing the focus groups, followed by the exploitation level. All agents in the latter observe full history.

We first describe the structure of a single focus group. Consider a disclosure policy that reveals the full history in each round \( t \), i.e., \( S_t = [t - 1] \); we call it the full-disclosure policy. The info-graph for this policy is a simple path. Intuitively, all agents in the path of this full-disclosure policy are in a single focus group.

**Definition 4.1.** A subset of rounds \( S \subset [T] \) is called a full-disclosure path in the info-graph \( G \) if the induced subgraph \( G_S \) is a simple path, and it connects to the rest of the graph only through the terminal node \( \max(S) \), if at all.

Full-disclosure paths are useful primitives for exploration as they guarantee that each arm is sampled with a positive-constant probability. This happens due to stochastic variation in outcomes; some agents in a focus group will get uncharacteristically bad rewards from an arm, inducing others to pull a different arm. We prove that path length at least \( L_{fd}^{K} \) suffices to guarantee this, for some parameter \( L_{fd}^{K} \) that depends only on \( K \), the number of arms (see Lemma B.1). We build on this fact throughout the paper.

Our two-level policy policy builds upon this primitive, allowing future agents to exploit the information that early agents explore, thereby closely following the well-studied “explore-then-exploit” paradigm from multi-armed bandits. For a parameter \( T_1 \) fixed later, the first \( N = T_1 \cdot L_{fd}^{K} \) agents comprise the “exploration level.” These agents are partitioned into \( T_1 \) full-disclosure paths of length \( L_{fd}^{K} \) each, where \( L_{fd}^{K} \) depends only on the number of arms \( K \). In the “exploitation level,” each agent \( t > N \) receives the full history, i.e., \( S_t = [t - 1] \).\(^{18}\) The info-graph for this disclosure policy is shown in Figure 2.

\(^{18}\)For the regret bounds, it suffices if each agent in the exploitation level only observes the history from the exploration level, or any superset thereof.
We show that this policy incentivizes the agents to perform non-adaptive exploration, and achieves a regret rate of $O_K(T^{2/3})$. The key idea is that since one full-disclosure path collects one sample of a given arm with (at least) a positive-constant probability, using many full-disclosure paths “in parallel” ensures that sufficiently many samples of this arm are collected with very high probability. The proof of the following theorem can be found in Appendix B.

**Theorem 4.2.** The two-level policy with parameter $T_1 = T^{2/3} (\log T)^{1/3}$ achieves regret

$$\text{Reg}(T) \leq O_K \left( T^{2/3} (\log T)^{1/3} \right).$$

**Remark 4.3.** All agents $t > T^{2/3} (\log T)^{1/3}$ (i.e., all but the vanishingly small fraction of agents who are in the exploration level) observe full history, and pull an arm with instantaneous regret at most $\tilde{O}(T^{-1/3})$.

**Remark 4.4.** Each full-disclosure path can be made arbitrarily longer, and more full-disclosure paths can be added (of arbitrarily length), as long as the total number of Level-1 agents increases by at most a constant factor. Same regret bounds are attained with minimal changes in the analysis.

**Remark 4.5.** For a constant $K$, the number of arms, we match the optimal regret rate for non-adaptive multi-armed bandit algorithms. If the gap parameter $\Delta$ is known to the principal, then (for an appropriate tuning of parameter $T_1$) we can achieve regret $\text{Reg}(T) \leq O_K(\log(T) \cdot \Delta^{-2})$.

One important quantity is the expected number of samples of a given arm $a$ collected by a full-disclosure path $S$ of length $L_{K}^{\text{fd}}$, i.e., the number of times this arm appears in the subhistory $H_S$. Indeed, this number, denoted $N_{K,a}^{\text{fd}}$, is the
same for all such paths. We use this quantity, here and in the subsequent sections, through a concentration inequality which aggregates the effect of having multiple full-disclosure paths (see Lemma B.2).

4.1 Counterexamples

Although simple, the two-level policy does exhibit some subtleties. First, it is important that the focus groups are independent. For example, a few initial agents observable by everyone may induce herding on a suboptimal arm. This might happen if, for example, the initial agents are celebrities, and their experiences leak to future agents outside the platform. Essentially, these “celebrities” herd on a suboptimal arm with constant probability (because they observe each other), and this herding persists afterwards (because everyone else observes the “celebrities”). We flesh out this point in the following example, analyzed in Appendix C.

Example 4.6. Posit \( K = 2 \) arms such that \( \frac{3}{4} \geq \mu_1 > \mu_2 > \frac{1}{4} \). Suppose Assumption 3.1 holds with \( N_{\text{est}} = 2 \) so that each arm is chosen in the first two rounds, and subsequently the mean reward of each arm \( a \) is estimated by the sample average (i.e., \( \hat{\mu}_a^t := \bar{\mu}_a^t \) for all rounds \( t > 2 \)). If each of the first \( R \) rounds are observable by all subsequent agents, for a large enough \( R = \Omega(\sqrt{\log(T)}) \), then with (at least) a positive-constant probability it holds that all agents \( t > R \) choose arm 2.

Second, it is important that each focus group has a linear information flow. For example, the first few agents acting in isolation may force high-probability herding within the focus group, preventing the natural exploration that we rely on. This may happen, for example, if their reviews are submitted and/or processed with a substantial delay. Suppose these initial agents are pessimistic about arm 1, so that each one in isolation pulls arm 2. This builds certainty about the mean reward of arm 2 which, for an appropriate setting of parameters, may exceed the initial reward estimate for arm 1. Then later agents viewing all this information will, with high probability, fail to pull arm 1.

Example 4.7. Suppose there are only two arms, all agents initially prefer arm 1, and have the same initial reward estimate \( \hat{\mu}_2 \) for arm 2. Consider a full-disclosure path \( P \) starting at round \( t_0 \). Suppose agent \( t_0 \) observes \( R \) “leaf agents” (each of which does not observe anybody else). Then, for any absolute constant \( \mu_1 > \hat{\mu}_2 \)
and a sufficiently large \( R = \Omega(\sqrt{\log(T)}) \), each agent in \( P \) will not try arm 2 with probability, say, at least \( 1 - O(T^{-2}) \).

Third, it is important that there are enough focus groups and agents therein, but not too many. Indeed, we need enough agents in each focus group to overpower the initial biases (as expressed by reward estimators with \(< N_{\text{est}} \) samples). Having enough focus groups ensures that the natural exploration succeeds. However, agents in the focus groups would have limited information and may make suboptimal choices, so having too many of them would induce high regret.

5 Adaptive exploration with a three-level policy

The two-level policy from the previous section implements the explore-then-exploit paradigm using a basic design with focus groups. The next challenge is to implement adaptive exploration, and go below the \( T^{2/3} \) barrier. Standard multi-armed bandit algorithms achieve this by pulling sub-optimal arm on occasion, when and if the available information requires it. However, we can not adaptively add focus groups since we must fix our policy ahead of time.

Instead, we accomplish adaptive exploration using a construction that adds a middle level to the info-graph. Agents in this middle level are partitioned into subgroups, each responsible for aggregating information from a subset of focus groups; we call these agents group aggregators. For simplicity, we assume \( K = 2 \) arms. When one arm is much better than the other, group aggregators have enough information to discern it and exploit. However, when the two arms are close, group aggregators will be induced to pull different arms (depending on the outcomes in their particular focus groups), which induces additional exploration. This construction also provides intuition for the main result, the multi-level construction presented in the next section.

**Construction 5.1.** The three-level policy is an order-based disclosure policy defined as follows. The info-graph consists of three levels: the first two correspond to exploration, and the third implements exploitation. Like in the two-level policy, the first level consists of multiple full-disclosure paths of length \( L_{\text{fd}}^K \) each, and each agent \( t \) in the exploitation level sees full history (see Figure 3). ¹⁹

¹⁹It suffices for the regret bounds if each agent in the exploitation level only observes the history
The middle level consists of $\sigma$ disjoint subsets of $T_2$ agents each, called second-level groups. All nodes in a given second-level group $G$ are connected to the same nodes outside of $G$, but not to one another.

The full-disclosure paths in the first level are also split into $\sigma$ disjoint subsets, called first-level groups. Each first-level group consists of $T_1$ full-disclosure paths, for the total of $T_1 \cdot \sigma \cdot L^\text{fd}_K$ rounds in the first layer. There is a 1-1 correspondence between first-level groups $G$ and second-level groups $G'$, whereby each agent in $G'$ observes the full history from the corresponding group $G$. More formally, agent in $G'$ is connected to the last node of each full-disclosure path in $G$. In other words, this agent receives message $\mathcal{H}_S$, where $S$ is the set of all rounds in $G$.

In more detail, the key idea is as follows. Consider the gap parameter $\Delta = |\mu_1 - \mu_2|$. If it is is large, then each first-level group produces enough data to determine the best arm with high confidence, and so each agent in the upper levels chooses the best arm. If $\Delta$ is small, then due to anti-concentration each arm gets “lucky” within at least once first-level group, in the sense that it appears much better than the other arm based on the data collected in this group. Then this arm gets explored by the corresponding second-level group. To summarize, the middle level exploits if the gap parameter is large, and provides some more exploration if it is small.

Figure 3: Info-graph for the three-level policy. Each red box in level 1 corresponds to $T_1$ full-disclosure paths of length $L^\text{fd}_K$ each.

from exploration (i.e., from all agents in the first two levels), or any superset thereof.
**Theorem 5.2.** For two arms, the three-level policy achieves regret

\[ \text{Reg}(T) \leq O\left( T^{4/7} \log T \right). \]

This holds for parameters \( T_1 = T^{4/7} \log^{-1/7}(T) \), \( \sigma = 2^{10} \log(T) \), and \( T_2 = T^{6/7} \log^{-5/7}(T) \).

**Remark 5.3.** All agents \( t > \widetilde{O}(T^{6/7}) \) (i.e., all but a vanishingly small fraction of agents who are in the first two levels) observe full history, and pull an arm with instantaneous regret \( \widetilde{O}\left( T^{-3/7} \right) \).

Let us sketch the proof; the full proof can be found in Appendix D.

The “good events”. We establish four “good events” each of which occurs with high probability.

(event 1) *Exploration in Level 1:* Every first-level group collects at least \( \Omega(T_1) \) samples of each arm.

(event 2) *Concentration in Level 1:* Within each first-level group, empirical mean rewards of each arm \( a \) concentrate around \( \mu_a \).

(event 3) *Anti-concentration in Level 1:* For each arm, some first-level subgroup collects data which makes this arm look much better than its actual mean and the other arm look much worse than its actual mean.

(event 4) *Concentration in prefix:* The empirical mean reward of each arm \( a \) concentrates around \( \mu_a \) in any prefix of its pulls. (This ensures accurate reward estimates in exploitation.)

The analysis of these events applies Chernoff Bounds to a suitable version of “reward tape” (see the definition of “reward tape” in Appendix A). For example, event 2 considers a reward tape restricted to a given first-level group.

**Case analysis.** We now proceed to bound the regret conditioned on the four “good events”. W.l.o.g., assume \( \mu_1 \geq \mu_2 \). We break down the regret analysis into four cases, based on the magnitude the gap parameter \( \Delta = \mu_1 - \mu_2 \). As a shorthand, denote \( \text{conf}(n) = \sqrt{\log(T)/n} \). In words, this is a confidence term, up to constant factors, for \( n \) independent random samples.

The simplest case is very small gap, trivially yielding an upper bound on regret.
Claim 5.4 (Negligible gap). If $\Delta \leq 3\sqrt{2} \cdot \text{conf}(T_2)$ then $\text{Reg}(T) \leq O(T^{4/7} \log^{6/7}(T))$.

Another simple case is when $\Delta$ is sufficiently large, so that the data collected in any first-level group suffices to determine the best arm. The proof follows from event$_1$ and event$_2$.

Lemma 5.5 (Large gap). If $\Delta \geq 4 \sum_{a \in A} \text{conf} \left( N_{K,a}^{fd} \cdot T_1 \right)$ then all agents in the second and the third levels pull arm 1.

In the medium gap case, the data collected in a given first-level group is no longer guaranteed to determine the best arm. However, agents in the third level see the history of all first-level groups, which enables them to correctly identify the best arm.

Lemma 5.6 (Medium gap). All agents pull arm 1 in the third level, when $\Delta$ satisfies

$$\Delta \in \left[ 4 \sum_{a \in A} \text{conf} \left( \sigma \cdot N_{K,a}^{fd} \cdot T_1 \right), \ 4 \sum_{a \in A} \text{conf} \left( N_{K,a}^{fd} \cdot T_1 \right) \right].$$

Finally, the small gap case, when $\Delta$ is between $\tilde{\Omega}(\sqrt{1/T_2})$ and $\tilde{O}(\sqrt{1/(\sigma T_1)})$ is more challenging since even aggregating the data from all $\sigma$ first-level groups is not sufficient for identifying the best arm. We need to ensure that both arms continue to be explored in the second level. To achieve this, we leverage event$_3$, which implies that each arm $a$ has a first-level group $s_a$ where it gets “lucky”, in the sense that its empirical mean reward is slightly higher than $\mu_a$, while the empirical mean reward of the other arm is slightly lower than its true mean. Since the deviations are in the order of $\Omega((\sqrt{1/T_1})$, and Assumption 3.1 guarantees the agents’ reward estimates are also within $\Omega((\sqrt{1/T_1})$ of the empirical means, the sub-history from this group $s_a$ ensures that all agents in the respective second-level group prefer arm $a$. Therefore, both arms are pulled at least $T_2$ times in the second level, which in turn gives the following guarantee:

Lemma 5.7 (Small gap). All agents pull arm 1 in the third level, when

$$\Delta \in \left( 3\sqrt{2} \cdot \text{conf}(T_2), \ 4 \sum_{a \in A} \text{conf} \left( \sigma \cdot N_{K,a}^{fd} \cdot T_1 \right) \right).$$

Wrapping up: proof of Theorem 5.2. In negligible gap case, the stated regret bound holds regardless of what the policy does. In the large gap case, the regret
only comes from the first level, so it is upper-bounded by the total number of agents in this level, which is \( \sigma \cdot L_{K}^{fd} \cdot T_1 = O(T^{4/7} \log T) \). In both intermediate cases, it suffices to bound the regret from the first and second levels, so

\[
\text{Reg}(T) \leq (\sigma T_1 \cdot L_{K}^{fd} + \sigma T_2) \cdot 4 \sum_{a \in A} \text{conf} \left( N_{K,a}^{fd} \cdot T_1 \right) = O(T^{4/7} \log^{6/7}(T)).
\]

Therefore, we obtain the stated regret bound in all cases.

6 Optimal regret with a multi-level policy

We extend our three-level policy to a more adaptive multi-level policy in order to achieve the optimal regret rate of \( \widetilde{O}(\sqrt{T}) \). This requires us to distinguish finer and finer gaps between the best and second-best arm. A naive approach would be to recursively apply the 2-level structure, creating a tree of group aggregators, each level responsible for successively larger information sets. This mimics the hierarchical information structure in many organizations, but it suffers large regret because the number of agents in focus groups grows exponentially. Furthermore, each of these agents is forced to make decisions with access to a vanishingly-small amount of history, which is undesirable in-and-of itself. In this section, we describe a method of interlacing information to reuse it without suffering from introduced correlations. This careful reuse of information is the third and final step in our journey towards policies with optimal learning rates.

On a very high level, our multi-level policy implements the limited-adaptivity framework for multi-armed bandits (Perchet et al., 2016), defined as follows. Suppose a bandit algorithm outputs a distribution \( p_t \) over arms in each round \( t \), and the arm \( a_t \) is then drawn independently from \( p_t \). This distribution can change only in a small number of rounds, called adaptivity rounds, that need to be chosen by the algorithm in advance. Optimal regret rate requires at least \( O(\log \log T) \) adaptivity rounds, where each “level” \( \ell \geq 2 \) in our construction implements one adaptivity round. The limited-adaptivity bandit algorithm from Perchet et al. (2016) is much simpler compared to our construction below, as it can ensure the desired amount of exploration directly by choosing the appropriate alternatives.

We provide two results (for two different parameterizations of the same policy). The first result analyzes the \( L \)-level policy for an arbitrary \( L \leq O(\log \log T) \), and
achieves the root-$T$ regret rate with $O(\log \log T)$ levels.

**Theorem 6.1.** There exists $L_{\max} = \Theta(\log \log T)$ such that for each $L \in \{3, 4, \ldots, L_{\max}\}$ there exists an order-based disclosure policy with $L$ levels and regret

$$\text{Reg}(T) \leq O_K(T^\gamma \cdot \text{polylog}(T)), \text{ where } \gamma = \frac{2^{L-1}}{2^L - 1}.$$ 

In particular, we obtain regret $O_K(T^{1/2} \text{polylog}(T))$ with $L = O(\log \log(T))$.

Our second policy achieves a gap-dependent regret guarantee, as per (2). This policy has the same info-graph structure as the first one in Theorem 6.1, but requires a higher number of levels $L = O(\log(T/\log \log(T)))$ and different group sizes. We will bound its regret as a function of the gap parameter $\Delta$ even though the construction of the policy does not depend on $\Delta$. In particular, this regret bound outperforms the one in Theorem 6.1 when $\Delta$ is much bigger than $T^{-1/2}$. It also has the desirable property that the policy does not withhold too much information from agents—any agent $t$ observes a good fraction of history in previous rounds.

**Theorem 6.2.** There exists an order-based disclosure policy with $L = O(\log(T)/\log \log(T))$ levels such that for every bandit instance with gap parameter $\Delta$, the policy has regret

$$\text{Reg}(T) \leq O_K \left( \min \left( \frac{1}{\Delta}, T^{1/2} \right) \cdot \text{polylog}(T) \right).$$ 

Under this policy, each agent $t$ observes a subhistory of size at least $\Omega(t/\text{polylog}(T))$.

Note for constant number of arms, this result matches the optimal regret rate (given in Equation (2)) for stochastic bandits, up to logarithmic factors.

**Remark 6.3.** The multi-level policy can be applied to the first $T/\eta(T)$ agents only, for any fixed $\eta(T) = \text{polylog}(T)$ (i.e., with reduced time horizon $T/\eta(T)$). Then the subsequent agents – which comprise all but $1/\eta(T)$-fraction of the agents – can observe the full history and enjoy instantaneous regret $O(T^{-1/2})$. The regret bounds from both theorems carry over. This extension requires only minimal modifications to the analysis, which are omitted.

Let us present the main techniques in our solution, focusing on the case of $K = 2$ arms; the full proofs are deferred to Section E.
A natural idea to extend the three-level policy is to insert more levels as multiple “check points”, so the policy can incentivize the agents to perform more adaptive exploration. In particular, each level will be responsible for some range of the gap parameter, collecting enough samples to rule out the bad arm if the gap parameter falls in this range. However, we need to introduce two main modifications in the info-graph to accommodate some new challenges.

**Interlacing connections between levels.** A tempting approach, described intuitively at the beginning of this section, generalizes the three-level policy to build an $L$-level info-graph with the structure of a $\sigma$-ary tree: for every $\ell \in \{2, \ldots, L\}$, each $\ell$-level group observes the sub-history from a disjoint set of $\sigma$ groups in level $(\ell - 1)$. The disjoint sub-histories observed by all the groups in level $\ell$ are independent, and under the small gap regime (similar to Lemma 5.7) it ensures that each arm $a$ has a “lucky” $\ell$-level group of agents that only pull $a$. This “lucky” property is crucial for ensuring that both arms will be explored in level $\ell$.

However, in this construction, the first level will have $\sigma^{L-1}$ groups, which introduces a multiplicative factor of $\sigma^{\Omega(L)}$ in the regret rate. The exponential dependence in $L$ will heavily limit the adaptivity of the policy, and prevents having the number of levels for obtaining the result in Theorem 6.2. To overcome this, we will design an info-graph structure such that the number of groups at each level stays as $\sigma^2 = \Theta(\log^2(T))$.

We will leverage the following key observation: in order to maintain the “lucky” property, it suffices to have $\Theta(\log T)$ $\ell$-th level groups that observe disjoint sub-histories that take place in level $(\ell - 1)$. Moreover, as long as the group size in levels lower than $(\ell - 1)$ are substantially smaller than group size of level $\ell - 1$, the “lucky” property does not break even if different groups in level $\ell$ observe overlapping sub-history from levels $\{1, \ldots, \ell - 2\}$.

This motivates the following interlacing connection structure between levels. For each level in the info-graph, there are $\sigma^2$ groups for some $\sigma = \Theta(\log(T))$. The groups in the $\ell$-th level are labeled as $G_{\ell, u, v}$ for $u, v \in [\sigma]$. For any $\ell \in \{2, \ldots, L\}$ and $u, v, w \in [\sigma]$, agents in group $G_{\ell, u, v}$ see the history of agents in group $G_{\ell-1, v, w}$ (and by transitivity all agents in levels below $\ell - 1$). See Figure 4 for a visualization of simple case with $\sigma = 2$). Two observations are in order:

(i) Consider level $(\ell - 1)$ and fix the last group index to be $v$, and consider the set
of groups $\mathcal{G}_{\ell-1,v} = \{G_{\ell-1,i,v} \mid i \in [\sigma]\}$ (e.g. $G_{\ell-1,1,1}$ and $G_{\ell-1,2,1}$ circled in red in the Figure 4). The agents in any group of $\mathcal{G}_{\ell-1,v}$ observe the same sub-history. As a result, if the empirical mean of arm $a$ is sufficiently high in their shared sub-history, then all groups in $\mathcal{G}_{\ell-1,v}$ will become “lucky” for $a$.

(ii) Every agent in level $\ell$ observes the sub-history from $\sigma$ $(\ell-1)$-th level groups, each of which belonging to a different set $\mathcal{G}_{\ell-1,v}$. Thus, for each arm $a$, we just need one set of groups $\mathcal{G}_{\ell-1,v}$ in level $\ell-1$ to be “lucky” for $a$ and then all agents in level $\ell$ will see sufficient arm $a$ pulls.

**Figure 4:** Connections between levels for the $L$-level policy, for $\sigma = 2$.

**Amplifying groups for boundary cases.** Recall in the three-level policy, the medium gap case (Lemma 5.6) corresponds to the case where the gap $\Delta$ is between $\Omega\left(\frac{1}{\sqrt{T_1}}\right)$ and $O\left(\frac{\log(T)}{T_1}\right)$. This is a boundary case since $\Delta$ is neither large enough to conclude that with high probability agents in both the second level and the third level all pull the best arm, nor small enough to conclude that both arms are explored enough times in the second level (due to anti-concentration). In this case, we need to ensure that agents in the third level can eliminate the inferior arm. This issue is easily resolved in the three-level policy since the agents in the third level observe the entire first-level history, which consists of $\Omega(T_1 \log(T))$ pulls of each arm and provides sufficiently accurate reward estimates to distinguish the two arms.

In the $L$-level policy, such boundary cases occur for each intermediate level $\ell \in \{2, \ldots, l-1\}$, but the issue mentioned above does not get naturally resolved since...
the ratios between the upper and lower bounds of $\Delta$ increase from $\Theta\left(\sqrt{\log(T)}\right)$ to $\Theta(\log(T))$, and it would require more observations from level $(\ell - 2)$ to distinguish two arms at level $\ell$. The reason for this larger disparity is that, except the first level, our guarantee on the number of pulls of each arm is no longer tight. For example, as shown in Figure 4, when we talk about having enough arm $a$ pulls in the history observed by agents in $G_{\ell,1,1}$, it could be that only agents in group $G_{\ell-1,1,1}$ are pulling arm $a$ and it also could be that most agents in groups $G_{\ell-1,1,1}, G_{\ell-1,1,2}, \ldots, G_{\ell-1,1,\sigma}$ are pulling arm $a$. Therefore our estimate of the number of arm $a$ pulls can be off by an $\sigma = \Theta(\log(T))$ multiplicative factor. This ultimately makes the boundary cases harder to deal with.

We resolve this problem by introducing an additional type of amplifying groups, called $\Gamma$-groups. For each $\ell \in [L], u, v \in [\sigma]$, we create a $\Gamma$-group $\Gamma_{\ell,u,v}$. Agents in $\Gamma_{\ell,u,v}$ observe the same history as the one observed by agents in $G_{\ell,u,v}$ and the number of agents in $\Gamma_{\ell,u,v}$ is $\Theta(\log(T))$ times the number of agents in $G_{\ell,u,v}$. The main difference between $G$-groups and $\Gamma$-groups is that the history of $\Gamma$-groups in level $\ell$ is not sent to agents in level $\ell + 1$ but agents in higher levels. When we are in the boundary case in which we don’t have good guarantees about the $(\ell + 1)$-level agents’ pulls, the new construction makes sure that agents in levels higher than $\ell + 1$ get to see enough pulls of each arm and all pull the best arm.

**Parameters.** Aside from the global parameter $\sigma = \Theta(\log(T))$ mentioned above, the structure of each level $\ell$ is determined by a parameter $T_{\ell}$. Specifically, we have $T_1$ full-disclosure paths in each Level-1 group. For each level $\ell = 2, \ldots, L$, each $G$-group contains $T_{\ell}$ agents, and each $\Gamma$-group contains $(\sigma - 1)T_{\ell}$ agents. Parameters $T_1, \ldots, T_L$ are specified in Appendix E, differently for the two theorems.

## 7 Robustness

We provide several results to illustrate that our constructions are robust to small amounts of misspecification. All these results require only minor changes in the analysis, which are omitted. First, we observe that all parameters in all policies can be increased by a constant factor.\(^{20}\)

\(^{20}\)For the two-level policy, this is a special case of Remark 4.4. We present it here for consistency.
Proposition 7.1 (parameters). All results hold even if all parameters increase by at most a constant factor: specifically, parameters $(L_{K}^{\text{fd}}, T_1)$ for the two-level policy (Theorem 4.2), parameters $(L_{K}^{\text{fd}}, \sigma, T_1, T_2)$ for the three-level policy (Theorem 5.2), and parameters $(L_{K}^{\text{fd}}, \sigma; T_1, \ldots, T_L)$ for the $L$-level policy (Theorems 6.1 and 6.2).

Let us consider a more challenging scenario when the structure of the communication network is altered, introducing correlation between parts of the constructions that are supposed to be isolated from one another. Recall from Example 4.6 that even a small amount of such correlation can be extremely damaging if it comes early in the game. Nevertheless, we can tolerate some undesirable correlation when it is sufficiently “local” or happens in later rounds. Informally, the existence of a local side channel between consumers does not necessarily break the regret guarantees. Families and friends can share recommendations and the reviews they’ve received if their social networks are sufficiently disjoint and information doesn’t travel too far.

Formally, we define a generalization of the two-level policy in which the exploration level can be wired in an arbitrary way, as long as it contains sufficiently many paths that are sufficiently long and sufficiently isolated. Agents in these paths may observe some agents that lie outside of these paths, but not too many, and these outside agents may not be shared among the paths. We need a definition: for a given subset $S$ of rounds, the span of $S$ is the union of $S$ and all rounds $s$ that are observable in some round $t \in S$ (i.e., rounds $s \leq t$ such that $s$ and $t$ are connected in the info-graph). We use quantity $L_{K}^{\text{fd}}$ from Lemma B.1.

Proposition 7.2 (Robustness of the two-level policy). Fix some $N < T$. Consider an order-based disclosure policy such that each agent $t > N$ sees the full history: $S_t = [t-1]$. Suppose the info-graph on the first $N$ agents contains $M$ paths of length $L_{K}^{\text{fd}}$ such that their spans are mutually disjoint and contain at most $2 \cdot L_{K}^{\text{fd}}$ rounds each. Then

$$\text{Reg}(T) \leq \tilde{O}_K \left( N + T/\sqrt{M} \right).$$

In particular, we obtain $\text{Reg}(T) \leq \tilde{O}_K \left( T^{2/3} \right)$ when $M = N = O(T^{2/3})$.

It is essential to bound the span size of the paths. Recall from Example 4.7 that too many “leaf agents” observed by everyone in a given full-disclosure path would rule out the natural exploration in this path.
A similar but somewhat weaker result extends to multi-level policies.

**Proposition 7.3** (Undesirable correlations in Level 1). Consider the info-graph of either multi-layer policy (from Theorem 5.2, 6.1, or 6.2). Suppose each full-disclosure path in Level 1 is replaced with subgraph $H$ which contains at most $2 \cdot L_{K}^{fd}$ rounds total, includes a path of length $L_{K}^{fd}$, and is connected to the rest of the info-graph via $\max(H)$ only. Then the corresponding theorem still holds.

Moreover, we can handle some undesirable correlation outside of Level 1. As a proof of concept, we focus on the three-level disclosure policy, and allow each agent in Level 2 to observe some additional Level-1 agents. These agents can be chosen arbitrarily, e.g., they could be the same for all Level-2 agents.

**Proposition 7.4** (undesirable correlations in Level 2). Consider the three-level policy from Theorem 5.2. Add edges to the info-graph: connect each Level-2 agent to at most $O(\sqrt{T_{1}})$ arbitrarily chosen agents from Level-1, where $T_{1}$ is the parameter from Theorem 5.2. The resulting order-based policy satisfies the guarantee in Theorem 5.2.

## 8 Discussion

### 8.1 Trust and Rationality

We argue that our trust and rationality assumptions in our model are substantially weaker compared to those in prior work on BIC incentivized exploration.²¹

Several issues are in play:

(i) **Whether agents understand the announced policy.** We only need an agent to understand that she is given some subhistory which is unbiased and transitive. It does not matter to the agent what subset of arrivals is covered by this subhistory, and how it is related to the other agents’ subsets. This is arguably quite comprehensible, compared to a full-blown specification of a bandit algorithm in BIC incentivized exploration.

(ii) **Whether agents trust the principal to implement the stated policy.** A third party can, at least in principle, collect rewards and subhistories from multiple agents.

²¹Recall that in Kremer et al. (2014) and the subsequent work, messaging policies can w.l.o.g. be reduced to multi-armed bandit algorithms which recommend an action to each agent and satisfy Bayesian incentive-compatibility (BIC).
and audit them for consistency. Such audits can be quite powerful: e.g., one could take random pairs of agents \(t,t'\) which are supposed to satisfy \(t \in S_t'\) and \(S_t \subset S_t'\), and check if this is indeed the case, if the rewards from agents in \(S_t\) are reported consistently, and if the reward from agent \(t\) is propagated faithfully. The possibility of such audits would incentivize the principal not to manipulate the policy.

In contrast, bandit algorithms do not readily admit “external” sanity checks, and are extremely difficult to audit. One reason is that the production code is often intertwined with many other pieces of the system, some of which may change over time or be legitimately non-public.

Moreover, debugging a bandit algorithm tends to be very intricate in realistic applications (Agarwal et al., 2017), so the implementation may deviate from the stated algorithm even if the principal intends otherwise. Faithfully revealing a subhistory is arguably trivial in comparison.

(iii) **Whether agents react as specified.** Agents in our model can treat the revealed subhistory as (just) a set of data-points, can exhibit a substantial amount of optimism or pessimism, and are not subject to the informational or cognitive load of Bayesian updates. On the other hand, agents in BIC incentivized exploration either need to trust the BIC property or to verify it. The former is arguably a lot to take on faith, and the latter typically requires a sophisticated Bayesian reasoning. Moreover, agents may be irrationally averse to recommendations without any supporting information, or to the possibility of being singled out for exploration.

### 8.2 Full disclosure and herding

The full-disclosure policy in BIC incentivized exploration reduces to the “greedy” bandit algorithm which exploits in each round. Its herding effects are most lucidly summarized by focusing on the case of two arms. Then, if arm 1 is preferable according to the prior, the algorithm never tries arm 2 with probability at least \(\mu_1^0 - \mu_2^0\), where \(\mu_a^0\) is the prior mean reward of arm \(a \in \{1,2\}\). This result holds for an arbitrary priors on rewards, possibly correlated across arms. It implies very high regret (linear in \(T\), the number of agents) under additional assumptions, e.g., for independent priors with full support. Similar results hold for a frequentist ver-
sion, where each agent chooses an arm with the highest empirical mean. These results can be found in (Chapter 11.2 in Slivkins, 2019). Various weaker versions have been “folklore” for decades.

8.3 Social learning

As mentioned in Section 2, our work can be interpreted as coordinating social learning, as we design a network on which the social learning happens. A large literature on social learning studies agents that learn over time in a shared environment, with no principal to coordinate them. A prominent topic is the presence or absence of herding phenomena. Models vary across several dimensions, to wit: how an agent acquires new information; which information is transmitted to others; what is the structure / properties of the communication network; whether agents are long-lived or only act once; whether they optimize rewards (via Bayesian rationality or frequentist behavior), or merely follow a rule-of-thumb. Below we discuss three lines of work in social learning that are most relevant, and point out the major differences compared to our model.

First, “sequential social learning” posits that agents observe private signals, but only the chosen actions of neighbors are observable in the future; see Golub and Sadler (2016) for a survey. While the early work focuses on a complete communication network, further work considers the impact of the network topology. In particular, Acemoglu et al. (2011) and Lobel and Sadler (2015) show that in a perfect Bayesian equilibrium, learning happens asymptotically if neighborhoods are sufficiently expansive or independent, features echoed in our own constructions.

To contrast these models with ours, consider the social planner that has access to all agents’ knowledge and chooses their actions. In sequential social learning, such a planner only needs to choose the best action given the previous agents’ signals, i.e., only needs to exploit, whereas in our model it also needs to explore. Also, herding in sequential social learning occurs due to restricted information flow (i.e., private signals are not observable in the future), whereas in our model

\footnote{For example, consider the case of two arms with Bernoulli rewards, with means \( \mu_1 > \mu_2 \). Assume a “warm start” such that each arm is tried \( N_0 \) times, \( N_0 < (\mu_1 - \mu_2)^{-2} \). Then arm 2 is never chosen with probability at least an absolute constant times \( \mu_1 - \mu_2 \). This holds under a mild assumption on \( \mu_1, \mu_2 \), e.g., \( 1/8 + \mu_1 - \mu_2 \leq \mu_2 < \mu_1 \leq 7/8 \).

\footnote{See, for example, Banerjee (1992); Welch (1992); Bikhchandani et al. (1992), as well as a very general result in Smith and Sørensen (2000).}
there are no private signals

Another line of work, starting from DeGroot (1974), posits that agents use “naive”, mechanical rules-of-thumb, e.g., form beliefs based on naive averaging of observations. In particular, even naive agents learn asymptotically so long as the network is not too imbalanced (e.g., Golub and Jackson, 2010). Chandrasekhar et al. (2020) show experimentally that such a behavioral model is a good predictor of human behavior in some scenarios. Dasaratha and He (2019) show similar results for sequential social learning. Theoretically, Dasaratha and He (2020) use this model of naivety to study the question of how to design the social network in a sequential learning model so as to induce optimal learning rates. They observe that silo structures akin to our two-level policy improve learning rates.

Third, “strategic experimentation”, starting from Bolton and Harris (1999); Keller et al. (2005), studies long-lived learning agents that observe both actions and rewards of one another; see Hörner and Skrzypacz (2017) for a survey. This is similar to our work in that the social planner also solves a version of multi-armed bandits. The main difference is that the agents are long-lived and engage in a complex repeated game where each player deploys an exploration policy but would prefer to free-ride on exploration by others. There are also important technical differences. Agents exactly optimize their Bayesian-expected utility (using the Markov Perfect Equilibrium as a solution concept), whereas we consider a flexible frequentist model. Also, the social-planner problem is a very different bandit problem, with Bayesian prior, time-discounting, “safe” arm that is completely known, and “risky” arm that follows a stochastic process.

9 Conclusions

We reformulate the problem of incentivized exploration as that of designing a fixed communication network for social learning. The new model substantially mitigates trust and rationality assumptions inherent in prior work on BIC incentivized exploration. We achieve optimally efficient social learning, in terms of how regret rate depends on the time horizon $T$. We do not restrict ourselves by the de-

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24 This follows from the transitivity of subhistories.
25 Our frequentist agents may behave similarly, albeit with more justification (because the subhistories they observe are unbiased and transitive). The original paper of DeGroot (1974) and much subsequent work study agents that act repeatedly, updating their beliefs over rounds.
sign choices adopted in the current recommendation platforms; instead, we hope to inform and influence the designs in the future.

We start with a two-level communication network which is very intuitive and robust to misspecifications. The idea of splitting (some of) the early arrivals into many isolated “focus groups” is plausibly practical. This construction implements the explore-then-exploit paradigm from multi-armed bandits, and achieves vanishing regret. We obtain optimal regret rate via a more intricate, multi-level communication network. The conceptual challenge here is to make exploration optimally adaptive to past observation, despite the “greedy” behavior of the agents. This requires intermediate “group aggregators” and information-sharing between groups, another plausibly actionable suggestion for recommendation platforms.

One could consider several well-motivated extensions of our model. However, they are not well-understood even in the BIC version, and arguably should first be resolved therein. To wit, one could (i) allow reward-dependent biases in agents’ reporting of the rewards. (ii) allow long-lived agents that strive to optimize their long-term utility, (iii) consider heterogenous agents, with public or private idiosyncratic signals, and (iv) posit some unavoidable information leakage among the agents, e.g., according to a pre-specified social network. The first two extensions have not been studied even in the BIC version; the other two have been studied for BIC incentivized exploration, but are not yet well-understood.

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Appendix A Preliminaries

We use the standard concentration and anti-concentration inequalities: respectively, Chernoff Bounds and Berry-Esseen Theorem. The former states that \( \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \), the average of \( n \) independent random variables \( X_1, \ldots, X_n \), converges to its expectation quickly. The latter states that the CDF of an appropriately scaled average \( \bar{X} \) converges to the CDF of the standard normal distribution pointwise. In particular, the average strays far enough from its expectation with some guaranteed probability. The theorem statements are as follows:

**Theorem A.1.** Fix \( n \). Let \( X_1, \ldots, X_n \) be independent random variables, and let \( \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \).

Then:

(a) (Chernoff Bounds) Assume \( X_i \in [0,1] \) for all \( i \). Then

\[
\Pr[|\bar{X} - \mathbb{E}[\bar{X}]| > \epsilon] \leq 2 \exp(-2n \epsilon^2).
\]

(b) (Berry-Esseen Theorem) Assume \( X_1, \ldots, X_n \) are identically distributed, with

\[
\sigma^2 := \mathbb{E}[(X_1 - \mathbb{E}[X_1])^2] \quad \text{and} \quad \rho := \mathbb{E}[(X_1 - \mathbb{E}[X_1])^3] < \infty.
\]

Let \( F_n \) be the cumulative distribution function of \( \frac{(\bar{X} - \mathbb{E}[\bar{X}])\sqrt{n}}{\sigma} \) and \( \Phi \) be the cumulative distribution function of the standard normal distribution.

\[
|F_n(x) - \Phi(x)| \leq \frac{\rho}{2\sigma^3 \sqrt{n}} \quad \forall x \in \mathbb{R}.
\]

We use the notion of reward tape to simplify the application of (anti-)concentration inequalities. This is a \( K \times T \) random matrix with rows and columns corresponding to arms and rounds, respectively. For each arm \( a \) and round \( t \), the value in cell \((a,t)\) is drawn independently from Bernoulli distribution \( D_a \). W.l.o.g., rewards in our model are defined by the rewards tape: namely, the reward for the \( j \)-th pull of arm \( a \) is taken from the \((a,j)\)-th entry of the reward matrix.

**Appendix B The two-level policy (Theorem 4.2)**

We state two lemmas (which are also used to analyze the three- and multi-level policies). First, full-disclosure paths sample each arm with constant probability.

**Lemma B.1.** There exist numbers \( L_k^{fd} > 0 \) and \( p_k^{fd} > 0 \) that depend only on \( K \), the number of arms, with the following property. Consider an arbitrary disclosure policy, and let \( S \subset [T] \) be a full-disclosure path in its info-graph, of length \( |S| \geq L_k^{fd} \). Under Assumption 3.1, with probability at least \( p_k^{fd} \), it holds that subhistory \( \mathcal{H}_S \) contains at least one sample of each arm \( a \).
Proof. Fix any arm $a$. Let $L_{K}^{\text{fd}} = (K-1) \cdot N_{\text{est}} + 1$ and $p_{K}^{\text{fd}} = (1/3)^{K_{est}}$. We will condition on the event that all the realized rewards in $L_{K}^{\text{fd}}$ rounds are 0, which occurs with probability at least $p_{K}^{\text{fd}}$ under Assumption 3.1. In this case, we want to show that arm $a$ is pulled at least once. We prove this by contradiction. Suppose arm $a$ is not pulled. By the pigeonhole principle, we know that there is some other arm $a'$ that is pulled at least $N_{\text{est}} + 1$ rounds. Let $t$ be the round in which arm $a'$ is pulled exactly $N_{\text{est}} + 1$ times. By Assumption 3.1, we know that

$$\hat{\mu}_{a'}^{t} \leq 0 + C_{\text{est}}/\sqrt{N_{\text{est}}} \leq C_{\text{est}} < 1/3.$$ 

On the other hand, we have $\hat{\mu}_{a}^{t} \geq 1/3 > \hat{\mu}_{a'}^{t}$. This contradicts with the fact that in round $t$, arm $a'$ is pulled, instead of arm $a$. \hfill $\square$

The second lemma concerns $N_{K,a}^{\text{fd}}$, the expected number of samples of a given arm $a$ collected by a full-disclosure path $S$ of length $L_{K}^{\text{fd}}$. It is a simple corollary of Chernoff Bounds (Theorem A.1(a)).

**Lemma B.2.** Suppose the info-graph contains $T_{1}$ full-disclosure paths of $L_{K}^{\text{fd}}$ rounds each. Let $N_{a}$ be the number of samples of arm $a$ collected by all paths. Then with probability at least $1 - \delta$,

$$|N_{a} - N_{K,a}^{\text{fd}}| \leq L_{K}^{\text{fd}} \cdot \sqrt{T_{1} \log(2K/\delta)/2} \quad \text{for all } a \in A.$$ 

We are now ready to prove Theorem 4.2.

We will set $T_{1}$ later in the proof, depending on whether the gap parameter $\Delta$ is known. For now, we just need to know we will make $T_{1} \geq \frac{4L_{K}^{\text{fd}}}{(p_{K}^{\text{est}})^{2}} \log(T)$. Since this policy is agnostic to the indices of the arms, we assume w.l.o.g. that arm 1 has the highest mean.

The first $T_{1} \cdot L_{K}^{\text{fd}}$ rounds will get total regret at most $T_{1} \cdot L_{K}^{\text{fd}}$. We focus on bounding the regret from the second level of $T - T_{1} \cdot L_{K}^{\text{fd}}$ rounds. We consider the following two events. We will first bound the probability that both of them happen and then we will show that they together imply upper bounds on $|\hat{\mu}_{a}^{t} - \mu_{a}|$'s for any agent $t$ in the second level. Recall $\hat{\mu}_{a}^{t}$ is the estimated mean of arm $a$ by agent $t$ and agent $t$ picks the arm with the highest $\hat{\mu}_{a}^{t}$.

Define $W_{1}^{a}$ to be the event that the number of arm $a$ pulls in the first level is at least $N_{K,a}^{\text{fd}} T_{1} - L_{K}^{\text{fd}} \sqrt{T_{1} \log(T)}$. As long as we set $T_{1} \geq \frac{4L_{K}^{\text{fd}}}{(p_{K}^{\text{est}})^{2}} \log(T)$, this implies that the number of arm $a$ pulls is then at least $N_{K,a}^{\text{fd}} T_{1}/2$. Define $W_{1}$ to be the intersection of all these events (i.e. $W_{1} = \bigcap_{a} W_{1}^{a}$). By Lemma B.2, we have $\Pr[W_{1}] \geq 1 - \frac{\Delta}{T_{1}} \geq 1 - \frac{1}{T}$.

Next, we show that the empirical mean of each arm $a$ is close to the true mean. To facilitate our reasoning, let us imagine there is a tape of length $T$ for each arm $a$, with each cell containing an independent draw of the realized reward from the distribution $D_{a}$. Then for each arm $a$ and any $\tau \in [T]$, we can think of the sequence of the first $\tau$ realized rewards of $a$ coming from the prefix of $\tau$ cells in its reward tape. Define $W_{2}^{a,\tau}$ to be the event that the empirical mean of the first $\tau$ realized rewards in the tape of arm $a$ is at most $\sqrt{\frac{2\log(T)}{\tau}}$ away from $\mu_{a}$. Define $W_{2}$ to be the intersection of these events (i.e. $\bigcap_{a,\tau \in [T]} W_{2}^{a,\tau}$). By Chernoff bound,

$$\Pr[W_{2}^{a,\tau}] \geq 1 - 2 \exp(-4\log(T)) \geq 1 - 2/T^{4}.$$ 

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By union bound, \( \Pr[W_2] \geq 1 - KT \cdot \frac{2}{T} \geq 1 - \frac{2}{T} \).

By union bound, we know \( \Pr[W_1 \cap W_2] \geq 1 - 3/T \). For the remainder of the analysis, we will condition on the event \( W_1 \cap W_2 \).

For any arm \( a \) and agent \( t \) in the second level, by \( W_1 \) and \( W_2 \), we have

\[
|\bar{\mu}_t^a - \mu_a| \leq \frac{2 \log(T)}{N_{K,a}^{f^d} T_1/2}.
\]

By \( W_1 \) and Assumption 3.1, we have

\[
|\bar{\mu}_t^a - \hat{\mu}_t^a| \leq \frac{C_{est}}{N_{K,a}^{f^d} T_1/2}.
\]

Therefore,

\[
|\mu_t^a - \mu_a| \leq \sqrt{\frac{2 \log(T)}{N_{K,a}^{f^d} T_1/2}} + \frac{C_{est}}{N_{K,a}^{f^d} T_1/2} \leq 3 \sqrt{\frac{\log(T)}{p^d K T_1}}.
\]

So the second-level agents will pick an arm \( a \) which has \( \mu_a \) at most \( \sqrt{\log(T)} \cdot p^d K T_1 \) away from \( \mu_1 \).

To sum up, the total regret is at most

\[
T_1 \cdot L_{K}^{f^d} + T \cdot (1 - \Pr[W_1 \cap W_2]) + T \cdot 6 \sqrt{\log(T) \cdot p^d K T_1}.
\]

By setting \( T_1 = T^{2/3} \log(T)^{1/3} \), we get regret \( O(T^{2/3} \log(T)^{1/3}) \).

**Appendix C  Analysis of Example 4.6**

We consider three events, denoted \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \). Event \( \mathcal{E}_1 \) is that after the first \( N_1 = 2 \) rounds, arm 1 has empirical mean at most \( \mu < \mu_2 \) and arm 2 empirical mean at least \( \mu_2 \). (The proof can work for other constant \( N_1 \), too.) We pick \( \mu' \) such that \( \mu_2 - \mu' = \Omega(1) \). Event \( \mathcal{E}_2 \) focuses on the next \( N - N_1 \) rounds. It asserts that arm 2 is the only one chosen in these rounds, and the empirical mean in any prefix of these rounds is at least \( \mu_2 \). Event \( \mathcal{E}_3 \) is that the last \( T - N \) agents all choose arm 2.

We lower-bound \( \Pr[\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3] \) by a positive constant by considering \( \Pr[\mathcal{E}_1], \Pr[\mathcal{E}_2 | \mathcal{E}_1] \) and \( \Pr[\mathcal{E}_3 | \mathcal{E}_1, \mathcal{E}_2] \). First, \( \mathcal{E}_1 \) happens with a constant probability as arm 1 getting 0 in its first pull and arm 2 getting 1 in its first pull is a sub case of \( \mathcal{E}_1 \).

Now we condition on \( \mathcal{E}_1 \) happening. We show that \( \mathcal{E}_2 \) happens with a positive-constant probability. We focus on the case when the first \( N_2 \) pulls of arm 2 in rounds \( \{N_1 + 1, \ldots, N\} \) are all 1’s for some large enough constant \( N_2 \) and then use Chernoff bound and union bound on the rest \( N - N_1 - N_2 \) pulls.

Now we condition on \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \). We consider a “reward tape” generating rewards of arm 2, where the \( t \)-th “cell” in the tape corresponds to the reward of arm 2 in round \( t \) if
this arm is chosen in this round. For each \( t > N \), let \( C_t \) be the subset of cells in the tape that correspond to rounds \( S_t \cap (N,T] \), where \( S_t \) is the set of rounds observable by agent \( t \). We can show that with very high probability, the empirical mean over \( C_t \) is larger than \( \mu' \) for all \( t \). Let us focus on this event, call it \( \mathcal{E}_{\text{tape}} \). We show that under \( \mathcal{E}_{\text{tape}} \), each agents \( t > N \) chooses arm 2, using induction on \( t \). This is because \( C_t \), together with the history of the first \( N \) rounds, is exactly the subhistory seen by agent \( t \), if all agents in round \( \{N+1,\ldots,t-1\} \) pull arm 2.

### Appendix D The three-level policy (Theorem 5.2)

#### High-probability events

The following lemmas can be derived from combining Lemma B.2 and union bound.

**Lemma D.1** (Concentration of first-level number of pulls.). Let \( W_1 \) be the event that for all groups \( s \in [\sigma] \) and arms \( a \in [1, 2] \), the number of arm \( a \) pulls in the \( s \)-th first-level group is in the range of

\[
\left[N_{K,a}^{\text{fd}} T_1 - L_K \sqrt{T_1 \log(T)}, N_{K,a}^{\text{fd}} T_1 + L_K \sqrt{T_1 \log(T)}\right],
\]

where \( N_{K,a}^{\text{fd}} \) is the expected number of arm \( a \) pulls in a full-disclosure path run of length \( L_K^{\text{fd}} \).

Then \( \Pr[W_1] \geq 1 - \frac{4\sigma}{T^2} \).

**Proof of Lemma D.1.** For the \( s \)-th first-level group, define \( W_1^{a,s} \) to be the event that the number of arm \( a \) pulls in the \( s \)-th first-level group is between \( N_{K,a}^{\text{fd}} T_1 - L_K^{\text{fd}} \sqrt{T_1 \log(T)} \) and \( N_{K,a}^{\text{fd}} T_1 + L_K^{\text{fd}} \sqrt{T_1 \log(T)} \). By Lemma B.2

\[
\Pr[W_1^{a,s}] \geq 1 - 2 \exp(-2\log(T)) \geq 1 - 2/T^2.
\]

By union bound, the intersection of all these events, \( \bigcap_{a,s} W_1^{a,s} \), has probability at least \( 1 - \frac{4\sigma}{T^2} \). \( \square \)

To state the events, it will be useful to think of a hypothetical reward tape \( T_{s,a}^{1} \) of length \( T \) for each group \( s \) and arm \( a \), with each cell independently sampled from \( \mathcal{D}_a \). The tape encodes rewards as follows: the \( j \)-th time arm \( a \) is chosen by the group \( s \) in the first level, its reward is taken from the \( j \)-th cell in this arm’s tape. The following result characterizes the concentration of the mean rewards among all consecutive pulls among all such tapes, which follows from Chernoff bound and union bound.

**Lemma D.2** (Concentration of empirical means in the first level). For any \( \tau_1, \tau_2 \in [T] \) such that \( \tau_1 < \tau_2 \), \( s \in [\sigma] \), and \( a \in [1, 2] \), let \( W_2^{s,a,\tau_1,\tau_2} \) be the event that the mean among the cells indexed by \( \tau_1, (\tau_1 + 1), \ldots, \tau_2 \) in the tape \( T_{s,a}^{1} \) is at most \( \sqrt{\frac{2\log(T)}{\tau_2 - \tau_1 + 1}} \) away from \( \mu_a \). Let \( W_2 \) be the intersection of all these events (i.e. \( W_2 = \bigcap_{a,s,\tau_1,\tau_2} W_2^{s,a,\tau_1,\tau_2} \)). Then \( \Pr[W_2] \geq 1 - \frac{4\sigma}{T^2} \).

**Proof of Lemma D.2.** By Chernoff bound,

\[
\Pr[W_2^{s,a,\tau_1,\tau_2}] \geq 1 - 2 \exp(-4\log(T)) \geq 1 - 2/T^4.
\]
By union bound, we have $\Pr[W_2] \geq 1 - 4\sigma/T^2$. 

Our policy also relies on the anti-concentration of the empirical means in the first round. We show that for each arm $a \in [1, 2]$, there exists a group $s_a$ such that the empirical mean of $a$ is slightly above $\mu_a$, while the other arm $(3-a)$ has empirical mean slightly below $\mu_{(3-a)}$. This event is crucial for inducing agents in the second level to explore both arms when their mean rewards are indistinguishable after the first level.

**Lemma D.3** (Co-occurrence of high and low deviations in this first level). For any group $s \in [\sigma]$, any arm $a$, let $\bar{\mu}_{a,s}$ be the empirical mean of the first $N_{K,a}^{T_1}$ cells in tape $T_{a,s}^1$. Let $W_{3}^{s,a,\text{high}}$ be the event $\bar{\mu}_{a,s} \geq \mu_a + 1/\sqrt{N_{K,a}^{T_1}}$ and let $W_{3}^{s,a,\text{low}}$ be the event that $\bar{\mu}_{a,s} \leq \mu_a - 1/\sqrt{N_{K,a}^{T_1}}$. Let $W_3$ be the event that for every $a \in \{1, 2\}$, there exists a group $s_a \in [\sigma]$ in the first level such that both $W_{3}^{s_a,a,\text{high}}$ and $W_{3}^{s_a,3-a,\text{low}}$ occur. Then $\Pr[W_3] \geq 1 - 2/T$.

**Proof of Lemma D.3.** By Berry-Esseen Theorem and $\mu_a \in [1/3, 2/3]$, we have for any $a$, 

$$\Pr[W_{3}^{s,a,\text{high}}] \geq (1 - \Phi(1/2)) - \frac{5}{\sqrt{N_{K,a}^{T_1}}} > 1/4.$$ 

The last inequality follows when $T$ is larger than some constant. Similarly we also have 

$$\Pr[W_{3}^{s,a,\text{low}}] > 1/4.$$ 

Since $W_{3}^{s,a,\text{high}}$ is independent with $W_{3}^{s,3-a,\text{low}}$, we have 

$$\Pr[W_{3}^{s,a,\text{high}} \cap W_{3}^{s,3-a,\text{low}}] = \Pr[W_{3}^{s,a,\text{high}}] \cdot \Pr[W_{3}^{s,3-a,\text{low}}] > (1/4)^2 = 1/16.$$ 

Notice that $(W_{3}^{s,a,\text{high}} \cap W_{3}^{s,3-a,\text{low}})$ are independent across different $s$’s. By union bound, we have 

$$\Pr[W_3] \geq 1 - 2(1 - 1/16)^\sigma \geq 1 - 2/T.$$ 

Lastly, we will condition on the event that the empirical means of both arms are concentrated around their true means in any prefix of their pulls. This guarantees that the policy obtains an accurate estimate of rewards for both arms after aggregating all the data in the first two levels.

**Lemma D.4** (Concentration of empirical means in the first two levels). With probability at least $1 - \frac{4}{T^2}$, the following event $W_4$ holds: for all $a \in \{1, 2\}$ and $\tau \in [N_{T,a}]$, the empirical means of the first $\tau$ arm $a$ pulls is at most $\sqrt{\frac{2\log(T)}{\tau}}$ away from $\mu_a$, where $N_{T,a}$ is the total number of arm $a$ pulls by the end of $T$ rounds.

**Proof of Lemma D.4.** For any arm $a$, let’s imagine a hypothetical tape of length $T$, with each cell independently sampled from $D_a$. The tape encodes rewards of the first two levels as follows: the $j$-th time arm $a$ is chosen in the first two levels, its reward is taken from the
$j$-th cell in the tape. Define $W_{4}^{a,\tau}$ to be the event that the mean of the first $t$ pulls in the tape is at most $\sqrt{\frac{2\log(T)}{t}}$ away from $\mu_a$. By Chernoff bound,

$$\Pr[W_{4}^{a,\tau}] \geq 1 - 2\exp(-4\log(T)) \geq 1 - 2/T^4.$$ 

By union bound, the intersection of all these events has probability at least:

$$\Pr[W_4] \geq 1 - \frac{4}{T^3}.$$ 

Let $W = \bigcap_{i=1}^{4} W_i$ be the intersection of all 4 events. By union bound, $W$ occurs with probability $1 - \mathcal{O}(1/T)$. Note that the regret conditioned on $W$ not occurring is at most $\mathcal{O}(1/T) \cdot T = \mathcal{O}(1)$, so it suffices to bound the regret conditioned on $W$.

**Case Analysis**

Now we assume the intersection $W$ of events $W_1, \cdots, W_4$ happens. We will first provide some helper lemmas for our case analysis.

**Lemma D.5.** For the $s$-th first-level group and arm $a$, define $\bar{\mu}_{a,s}^1$ to be the empirical mean of arm $a$ pulls in this group. If $W$ holds, then

$$|\bar{\mu}_{a,s}^1 - \mu_a| \leq \sqrt{\frac{4\log(T)}{N_{K,a}^{id}T_1}}.$$ 

**Proof.** The events $W_1$ and $W_{a,s,1,\tau}^{a,s,1,\tau}$ for $\tau = N_{K,a}^{id}T_1 - L_{K}^{id}\sqrt{T_1\log(T)}, \ldots, N_{K,a}^{id}T_1 + L_{K}^{id}\sqrt{T_1\log(T)}$ together imply that

$$|\bar{\mu}_{a,s}^1 - \mu_a| \leq \sqrt{\frac{2\log(T)}{N_{K,a}^{id}T_1 - L_{K}^{id}\sqrt{T_1\log(T)}}} \leq \sqrt{\frac{4\log(T)}{N_{K,a}^{id}T_1}}.$$ 

The last inequality holds when $T$ is larger than some constant. 

**Lemma D.6.** For each arm $a$, define $\bar{\mu}_a$ to be the empirical mean of arm $a$ pulls in the first two levels. If $W$ holds, then

$$|\bar{\mu}_a - \mu_a| \leq \sqrt{\frac{4\log(T)}{\sigma N_{K,a}^{id}T_1}}.$$ 

Furthermore, if there are at least $T_2$ pulls of arm $a$ in the first two levels,

$$|\bar{\mu}_a - \mu_a| \leq \sqrt{\frac{2\log(T)}{T_2}}.$$ 

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Proof. The events $W_1$ and $W_4^{a,\tau}$ for $\tau \geq (N_{K,a}^{fd}T_1 - L_{K}^{fd} \sqrt{T_1 \log(T)})\sigma$ together imply that

$$|\bar{\mu}_a - \mu_a| \leq \frac{2\log(T)}{\sigma(N_{K,a}^{fd}T_1 - L_{K}^{fd} \sqrt{T_1 \log(T)})} \leq \frac{4\log(T)}{\sigma N_{K,a}^{fd}T_1}.$$

The last inequality holds when $T$ is larger than some constant. \hfill \square

**Lemma D.7.** For the $s$-th first-level group and arm $a$, define $\bar{\mu}_a^{1,s}$ to be the empirical mean of arm $a$ pulls in this group. For each $a \in \{1, 2\}$, there exists a group $s_a$ such that

$$\bar{\mu}_a^{1,s_a} > \mu_a + \frac{1}{4N_{K,a}^{fd}T_1} \quad \text{and} \quad \bar{\mu}_a^{1,s_a} < \mu_{3-a} - \frac{1}{4N_{K,3-a}^{fd}T_1}.$$

Proof. For each $a \in \{1, 2\}$, $W_3$ implies that there exists $s_a$ such that both $W_3^{s_a,a,high}$ and $W_3^{s_a,3-a,low}$ happen. The events $W_3^{s_a,a,high}$, $W_1$, $W_2^{s_a,a,tau,N_{K,a}^{fd}T_1}$ for $\tau = N_{K,a}^{fd}T_1 - L_{K}^{fd} \sqrt{T_1 \log(T)} + 1, \ldots, N_{K,a}^{fd}T_1 - 1$ and $W_2^{s_a,a,N_{K,a}^{fd}T_1,\tau}$ for $\tau = N_{K,a}^{fd}T_1, \ldots, N_{K,a}^{fd}T_1 + L_{K}^{fd} \sqrt{T_1 \log(T)}$ together imply that

$$\bar{\mu}_a^{1,s_a} \geq \mu_a + \left(\frac{N_{K,a}^{fd}T_1}{N_{K,a}^{fd}T_1} - L_{K}^{fd} \sqrt{T_1 \log(T)}\right) \cdot \frac{1}{\sqrt{N_{K,a}^{fd}T_1}} \cdot \sqrt{2\log(T)} \cdot \frac{1}{\sqrt{L_{K}^{fd} \sqrt{T_1 \log(T)}}}$$

$$\geq \mu_a + \frac{1}{4N_{K,a}^{fd}T_1}.$$

The second to the last inequality holds when $T$ is larger than some constant. Similarly, we also have

$$\bar{\mu}_a^{1,s_a} < \mu_{3-a} - \frac{1}{4N_{K,3-a}^{fd}T_1}.$$

Now we proceed to the case analysis.

**Proof of Lemma 5.5 (Large gap case).** Observe that for any group $s$ in the first level, the empirical means satisfy

$$\bar{\mu}^{1,s}_1 - \bar{\mu}^{1,s}_2 \geq \mu_1 - \mu_2 - \sqrt{\frac{4\log(T)}{N_{K,1}^{fd}T_1}} \cdot \sqrt{\frac{4\log(T)}{N_{K,2}^{fd}T_1}} \geq \sqrt{\frac{4\log(T)}{N_{K,1}^{fd}T_1}} + \sqrt{\frac{4\log(T)}{N_{K,2}^{fd}T_1}}.$$
For any agent $t$ in the $s$-th second-level group, by Assumption 3.1, we have

$$\hat{\mu}_1^t - \hat{\mu}_2^t > \bar{\mu}_1^{1,s} - \bar{\mu}_2^{1,s} - \frac{C_{\text{est}}}{\sqrt{N_{K,1}^d T_1/2}} - \frac{C_{\text{est}}}{\sqrt{N_{K,2}^d T_1/2}}.$$

$$\geq \sqrt{\frac{4 \log(T)}{N_{K,1}^d T_1}} + \sqrt{\frac{4 \log(T)}{N_{K,2}^d T_1}} - \frac{C_{\text{est}}}{\sqrt{N_{K,1}^d T_1/2}} - \frac{C_{\text{est}}}{\sqrt{N_{K,2}^d T_1/2}} > 0.$$

Therefore, we know agents in the $s$-th second-level group will all pull arm 1.

Now consider the agents in the third level group. Recall $\bar{\mu}_a$ is the empirical mean of arm $a$ in the history they see. We have

$$\bar{\mu}_1 - \bar{\mu}_2 \geq \mu_1 - \mu_2 - \sqrt{\frac{4 \log(T)}{\sigma N_{K,1}^d T_1}} - \sqrt{\frac{4 \log(T)}{\sigma N_{K,2}^d T_1}} \geq \sqrt{\frac{4 \log(T)}{\sigma N_{K,1}^d T_1}} + \sqrt{\frac{4 \log(T)}{\sigma N_{K,2}^d T_1}}.$$

Similarly as above, by Assumption 3.1, we know $\hat{\mu}_1^t - \hat{\mu}_2^t > 0$ for any agent $t$ in the third level. Therefore, the agents in the third-level group will all pull arm 1.

Proof of Lemma 5.6 (Medium gap case). Recall $\bar{\mu}_a$ is the empirical mean of arm $a$ in the first two levels. We have

$$\bar{\mu}_1 - \bar{\mu}_2 \geq \mu_1 - \mu_2 - \sqrt{\frac{4 \log(T)}{\sigma N_{K,1}^d T_1}} - \sqrt{\frac{4 \log(T)}{\sigma N_{K,2}^d T_1}} \geq \sqrt{\frac{4 \log(T)}{\sigma N_{K,1}^d T_1}} + \sqrt{\frac{4 \log(T)}{\sigma N_{K,2}^d T_1}}.$$

For any agent $t$ in the third level, by Assumption 3.1, we have

$$\hat{\mu}_1^t - \hat{\mu}_2^t > \bar{\mu}_1 - \bar{\mu}_2 - \frac{C_{\text{est}}}{\sqrt{\sigma N_{K,1}^d T_1/2}} - \frac{C_{\text{est}}}{\sqrt{\sigma N_{K,2}^d T_1/2}}.$$

$$\geq \sqrt{\frac{4 \log(T)}{\sigma N_{K,1}^d T_1}} + \sqrt{\frac{4 \log(T)}{\sigma N_{K,2}^d T_1}} - \frac{C_{\text{est}}}{\sqrt{\sigma N_{K,1}^d T_1/2}} - \frac{C_{\text{est}}}{\sqrt{\sigma N_{K,2}^d T_1/2}} > 0.$$

So we know agents in the third-level group will all pull arm 1.

Proof of Lemma 5.7 (Small gap case). In this case, we need both arms to be pulled at least $T_2$ rounds in the second level. For every arm $a$, consider the $s_a$-th second-level group, with
given by Lemma D.7. We have

$$\hat{\mu}_a^{1,s_a} - \hat{\mu}_{3-a}^{1,s_a} > \mu_a + \frac{1}{4N_{kd}^{1}} - \frac{1}{4N_{kd}^{3-a}}$$

$$\geq \frac{1}{8N_{kd}^{1}} + \frac{1}{8N_{kd}^{2}} > 0$$

For any agent $t$ in the $s_a$-th second-level group, by Assumption 3.1, we have

$$\hat{\mu}_t^{1,s_a} - \hat{\mu}_{3-a}^{1,s_a} > \mu_a - \mu_{3-a} - C_{est} \frac{1}{\sqrt{N_{kd}^{1}T_1/2}} - \frac{C_{est}}{\sqrt{N_{kd}^{2}T_1/2}}$$

$$\geq \frac{1}{8N_{kd}^{1}} + \frac{1}{8N_{kd}^{2}} > 0.$$

So we know agents in the $s_a$-th second-level group will all pull arm $a$. Therefore in the first two levels, both arms are pulled at least $T_2$ times. Now consider the third-level. We have

$$\hat{\mu}_1 - \hat{\mu}_2 \geq \mu_1 - \mu_2 - 2\sqrt{\frac{2\log(T)}{T_2}} \geq \sqrt{\frac{2\log(T)}{T_2}}.$$

Similarly as above, by Assumption 3.1, we know $\hat{\mu}_t^1 - \hat{\mu}_t^2 > 0$ for any agent $t$ in the third level. So we know agents in the third-level group will all pull arm 1.

**Appendix E  The multi-level policy**

In this subsection, we analyze our $L$-level policy for $L > 3$, proving Theorems 6.1 and 6.2. We first analyze it for the case of $K = 2$ arms. The bulk of the analysis, joint for both theorems, is presented in Appendix E.1. We provide two different endings where the details differ: Appendix E.2 and Appendix E.3, respectively. We extend the analysis to $K > 2$ arms in Appendix E.4.

The parameters are set as follows. In Theorem 6.1, recall from the theorem statement that we restrict $L$ to be at most $L_{max} = \Theta(\log \log T)$. Specifically, we define

$$L_{max} = \log \left( \frac{\ln T}{\frac{\log T}{\log \sigma^4}} \right).$$

The group number parameter $\sigma$ is set at $\sigma = 2^{10}\log(T)$ for both theorems. Parameters $T_1, \ldots, T_L$ are specified differently for the two theorems, see (8), (9) and (10).
Let us recap the construction of the \(L\)-level policy. There are two types of groups: \(G\)-groups and \(\Gamma\)-groups. Each level has \(\sigma^2\) \(G\)-groups. Label the \(G\)-groups in the \(\ell\)-th level as \(G_{\ell,u,v}\) for \(u,v \in [\sigma]\). Level 2 to level \(L\) also have \(\sigma^2\) \(\Gamma\)-groups. Label the \(\Gamma\)-groups in the \(\ell\)-th level as \(\Gamma_{\ell,u,v}\) for \(u,v \in [\sigma]\). Each first-level group (\(G_{1,u,v}\) for \(u,v \in [\sigma]\)) has \(T_1\) full-disclosure path of \(L_{K,K}^{\text{fd}}\) rounds in parallel. For \(\ell \geq 2\), there are \(T_{\ell}\) agents in group \(G_{\ell,u,v}\) and there are \(T_{\ell}(\sigma - 1)\) agents in group \(\Gamma_{\ell,u,v}\).

The info-graph is defined as follows. Agents in the first level only observe the history defined in the full-disclosure path run. For agents in group \(G_{\ell,u,v}\) with \(\ell \geq 2\), they observe all the history in the first \(\ell - 2\) levels (both \(G\)-groups and \(\Gamma\)-groups) and history in group \(G_{\ell-1,v,w}\) for all \(w \in [\sigma]\). Agents in group \(\Gamma_{\ell,u,v}\) observe the same history as agents in group \(G_{\ell,u,v}\).

### E.1 Joint analysis for \(K = 2\) arms

The bulk of the analysis is joint for Theorems 6.1 and 6.2. While the parameters \(T_{\ell}\) are set differently for the two theorems, we will only assume

\[
T_1 \leq \sigma^4 \leq \frac{T_{\ell}}{T_{\ell-1}} \quad \text{for} \quad \ell \in \{2, \ldots, L - 1\},
\]

which will hold for both parameter settings. Wlog we assume \(\mu_1 \geq \mu_2\) as the recommendation policy is symmetric to both arms.

Similarly as the proof of Theorem 5.2, we start with some “clean events”.

- **Concentration of the number of arm \(a\) pulls in the first level:**
  For \(a \in \{1, 2\}\), define \(N_{K,a}^{\text{fd}}\) to be the expected number of arm \(a\) pulls in one run of full-disclosure path used in the first level. By Lemma B.1, we know \(P_{K,a}^{\text{fd}} \leq N_{K,a}^{\text{fd}} \leq L_{K,a}^{\text{fd}}\). For group \(G_{1,u,v}\), define \(W_1^{a,u,v}\) to be the event that the number of arm \(a\) pulls in this group is between \(N_{K,a}^{\text{fd}} - L_{K,a}^{\text{fd}} \sqrt{T_1 \log(T)}\) and \(N_{K,a}^{\text{fd}} + L_{K,a}^{\text{fd}} \sqrt{T_1 \log(T)}\). By Chernoff bound,

\[
\Pr[W_1^{a,u,v}] \geq 1 - 2\exp(-2\log(T)) \geq 1 - 2/T^2.
\]

Define \(W_1\) to be the intersection of all these events (i.e. \(W_1 = \bigcap_{a,u,v} W_1^{a,u,v}\)). By union bound, we have

\[
\Pr[W_1] \geq 1 - \frac{4\sigma^2}{T^2}.
\]

- **Concentration of the empirical mean for arm \(a\) in the history observed by agent \(t\):**
  For each agent \(t\) and arm \(a\), imagine there is a tape of enough arm \(a\) pulls sampled before the recommendation policy starts and these samples are revealed one by one whenever agents in agent \(t\)’s observed history pull arm \(a\). Define \(W_2^{a,t_1,t_2}\) to be the event that the mean of \(\tau_1\)-th to \(\tau_2\)-th pulls in the tape is at most \(\sqrt{3\log(T)\tau_2^{-\tau_1+1}}\) away from \(\mu_a\). By Chernoff bound,

\[
\Pr[W_2^{a,t_1,t_2}] \geq 1 - 2\exp(-6\log(T)) \geq 1 - 2/T^6.
\]
Define $W_2$ to be the intersection of all these events (i.e. $W_2 = \bigcap_{t,a,\tau_1,\tau_2} W_{t,a,\tau_1,\tau_2}$). By union bound, we have

$$\Pr[W_2] \geq 1 - \frac{4}{T^5}.$$ 

- **Anti-concentration of the empirical mean of arm $a$ pulls in the $\ell$-th level for $\ell \geq 2$:**

For $2 \leq \ell \leq L - 1$, $u \in [\sigma]$ and each arm $a$, define $n_{\ell,u,a}$ to be the number of arm $a$ pulls in groups $G_{\ell,u,1},...,G_{\ell,u,\sigma}$. Define $W_{3}^{\ell,u,a,\text{high}}$ as the event that $n_{\ell,u,a} \geq T_{\ell}$ implies the empirical mean of arm $a$ pulls in group $G_{\ell,u,1},...,G_{\ell,u,\sigma}$ is at least $\mu_{a} + 1/\sqrt{n_{\ell,u,a}}$. Define $W_{3}^{\ell,u,a,\text{low}}$ as the event that $n_{\ell,u,a} \geq T_{\ell}$ implies the empirical mean of arm $a$ pulls in group $G_{\ell,u,1},...,G_{\ell,u,\sigma}$ is at most $\mu_{a} - 1/\sqrt{n_{\ell,u,a}}$.

Define $H_{\ell}$ to be random variable the history of all agents in the first $\ell - 1$ levels and which agents are chosen in the $\ell$-th level. Let $h_{\ell}$ be some realization of $H_{\ell}$. Notice that once we fix $H_{\ell}$, $n_{\ell,u,a}$ is also fixed.

Now consider $h_{\ell}$ to be any possible realized value of $H_{\ell}$. If fixing $H_{\ell} = h_{\ell}$ makes $n_{\ell,u,a} < T_{\ell}$, then $\Pr[W_{3}^{\ell,u,a,\text{high}}|H_{\ell} = h_{\ell}] = 1$ If fixing $H_{\ell} = h_{\ell}$ makes $n_{\ell,u,a} \geq T_{\ell}$, by Berry-Eseen Theorem and $\mu_{a} \in [1/3, 2/3]$, we have

$$\Pr[W_{3}^{\ell,u,a,\text{high}}|H_{\ell} = h_{\ell}] \geq (1 - \Phi(1/2)) - \frac{5}{\sqrt{T_{\ell}}} > 1/4.$$ 

Similarly we also have

$$\Pr[W_{3}^{\ell,u,a,\text{low}}|H_{\ell} = h_{\ell}] > 1/4$$

Since $W_{3}^{\ell,u,a,\text{high}}$ is independent with $W_{3}^{\ell,u,3-a,\text{low}}$ when fixing $H_{\ell}$, we have

$$\Pr[W_{3}^{\ell,u,a,\text{high}} \cap W_{3}^{\ell,u,3-a,\text{low}}|H_{\ell} = h_{\ell}] > (1/4)^2 = 1/16.$$ 

Now define $W_{3}^{\ell,a} = \bigcup_{u}(W_{3}^{\ell,u,a,\text{high}} \cap W_{3}^{\ell,u,3-a,\text{low}})$. Since $(W_{3}^{\ell,u,a,\text{high}} \cap W_{3}^{\ell,u,3-a,\text{low}})$ are independent across different $u$’s when fixing $H_{\ell} = h_{\ell}$, we have

$$\Pr[W_{3}^{\ell,a}|H_{\ell} = h_{\ell}] \geq 1 - (1 - 1/16)^{\sigma} \geq 1 - 1/T^2.$$ 

Since this holds for all $h_{\ell}$’s, we have $\Pr[W_{3}^{\ell,a}] \geq 1 - 1/T^2$. Finally define $W_{3} = \bigcap_{\ell,a} W_{3}^{\ell,a}$. By union bound, we have

$$W_{3} \geq 1 - 2L/T^2.$$ 

- **Anti-concentration of the empirical mean of arm $a$ pulls in the first level:**

For first-level groups $G_{1,u,1},...,G_{1,u,\sigma}$ and arm $a$, imagine there is a tape of enough arm $a$ pulls sampled before the recommendation policy starts and these samples are revealed one by one whenever agents in these groups pull arm $a$. Define $W_{4}^{n,a,\text{high}}$ to be the event that first $N_{1}^{T_{1}T_{1}}$ pulls of arm $a$ in the tape has empirical mean at least
\[ \mu_a + 1 / \sqrt{N_{K,a}^{\text{fd}} T_1 \sigma} \] and define \( W_{4,a}^{u,a,\text{low}} \) to be the event that first \( N_{K,a}^{\text{fd}} T_1 \sigma \) pulls of arm \( a \) in the tape has empirical mean at most \( \mu_a - 1 / \sqrt{N_{K,a}^{\text{fd}} T_1 \sigma} \). By Berry-Esseen Theorem and \( \mu_a \in [1/3, 2/3] \), we have

\[
\Pr[W_{4,a}^{u,a,\text{high}}] \geq (1 - \Phi(1/2)) - \frac{5}{\sqrt{N_{K,a}^{\text{fd}} T_1 \sigma}} > 1/4.
\]

The last inequality follows when \( T \) is larger than some constant. Similarly we also have

\[
\Pr[W_{4,a}^{u,a,\text{low}}] > 1/4.
\]

Since \( W_{4,a}^{u,a,\text{high}} \) is independent with \( W_{4,a}^{u,3-a,\text{low}} \), we have

\[
\Pr[W_{4,a}^{u,a,\text{high}} \cap W_{4,a}^{u,3-a,\text{low}}] = \Pr[W_{4,a}^{u,a,\text{high}}] \cdot \Pr[W_{4,a}^{u,3-a,\text{low}}] > (1/4)^2 = 1/16.
\]

Now define \( W_{4,a}^4 \) as \( \bigcup_u (W_{4,a}^{u,a,\text{high}} \cap W_{4,a}^{u,3-a,\text{low}}) \). Notice that \( (W_{4,a}^{u,a,\text{high}} \cap W_{4,a}^{u,3-a,\text{low}}) \) are independent across different \( u \)'s. So we have

\[
\Pr[W_{4,a}^4] \geq 1 - (1 - 1/16)^a \geq 1 - 1/T^2.
\]

Finally we define \( W_4 \) as \( \bigcap_a W_{4,a}^4 \). By union bound,

\[
\Pr[W_4] \geq 1 - 2/T^2.
\]

By union bound, the intersection of these clean events (i.e. \( \bigcap_{i=1}^4 W_i \)) happens with probability \( 1 - O(1/T) \). When this intersection does not happen, since the probability is \( O(1/T) \), it contributes \( O(1/T) \cdot T = O(1) \) to the regret.

Now we assume the intersection of clean events happens and prove upper bound on the regret.

By event \( W_1 \), we know that in each first-level group, there are at least \( N_{K,a}^{\text{fd}} T_1 - L_{\Gamma}^{\text{fd}} \sqrt{T_1 \log(T)} \) pulls of arm \( a \). We prove in the next claim that there are enough pulls of both arms in higher levels if \( \mu_1 - \mu_2 \) is small enough. For notation convenience, we set \( \epsilon_0 = 1, \epsilon_1 = \frac{1}{4 \sqrt{N_{K,a}^{\text{fd}} T_1 \sigma}}, \) and \( \epsilon_\ell = 1/(4 \sqrt{T_{\ell} \sigma}) \) for \( \ell \geq 2 \).

**Claim E.1.** For any arm \( a \) and \( 2 \leq \ell \leq L \), if \( \mu_1 - \mu_2 \leq \epsilon_{\ell-1} \), then for any \( u \in [\sigma] \), there are at least \( T_{\ell} \) pulls of arm \( a \) in groups \( G_{\ell,u,1}, G_{\ell,u,2}, \ldots, G_{\ell,u,\sigma} \) and there are at least \( T_{\ell} \sigma (\sigma - 1) \) pulls of arm \( a \) in the \( \ell \)-th level \( \Gamma \)-groups.

**Proof.** We are going to show that for each \( \ell \) and arm \( a \) there exists \( u_a \) such that agents in groups \( G_{\ell,1,u_a}, \ldots, G_{\ell,\sigma,u_a} \) and \( \Gamma_{\ell,1,u_a}, \ldots, \Gamma_{\ell,\sigma,u_a} \) all pull arm \( a \). This suffices to prove the claim.

We prove the above via induction on \( \ell \). We start by the base case when \( \ell = 2 \). For each arm \( a \), \( W_4 \) implies there exists \( u_a \) such that \( W_{4,a}^{u,a,\text{high}} \) and \( W_{4,a}^{u,3-a,\text{low}} \) happen. For an agent \( t \) in groups \( G_{2,1,u_a}, \ldots, G_{2,\sigma,u_a} \) and \( \Gamma_{2,1,u_a}, \ldots, \Gamma_{2,\sigma,u_a} \). \( W_{4,a}^{u,a,\text{high}}, W_{4,a}^{u,3-a,\text{low}} \) and \( W_2 \) together imply
that
\[
\hat{\mu}_a^n \geq \mu_a + \left( \frac{N_{fd}^{K,a} T_1 \sigma}{\sqrt{N_{fd}^{K,a} T_1 \sigma}} - L_K \sqrt{T_1 \log(T) \sigma} \cdot \sqrt{\frac{3 \log(T)}{L_K \sqrt{T_1 \log(T) \sigma}}} \right)
\cdot \frac{1}{(N_{fd}^{K,a} T_1 + L_K \sqrt{T_1 \log(T) \sigma})}
\geq \mu_a + \frac{1}{4 \sqrt{N_{fd}^{K,a} T_1 \sigma}}.
\]

The second last inequality holds when \( T \) is larger than some constant. Similarly, we also have
\[
\hat{\mu}_{3-a}^n < \mu_{3-a} - \frac{1}{4 \sqrt{N_{fd}^{K,3-a} T_1 \sigma}}.
\]

Then we have
\[
\hat{\mu}_a^n - \hat{\mu}_{3-a}^n > \mu_a - \mu_{3-a} + \frac{1}{4 \sqrt{N_{fd}^{K,a} T_1 \sigma}} - \frac{1}{4 \sqrt{N_{fd}^{K,3-a} T_1 \sigma}}
\geq -\epsilon_1 + \frac{1}{4 \sqrt{N_{fd}^{K,a} T_1 \sigma}} + \frac{1}{4 \sqrt{N_{fd}^{K,3-a} T_1 \sigma}}
\geq \frac{1}{8 \sqrt{N_{fd}^{K,a} T_1 \sigma}} + \frac{1}{8 \sqrt{N_{fd}^{K,3-a} T_1 \sigma}}.
\]

By Assumption 3.1, we have
\[
\hat{\mu}_a^n - \hat{\mu}_{3-a}^n > \hat{\mu}_a^n - \hat{\mu}_{3-a}^n - \frac{C_{est}}{\sqrt{N_{fd}^{K,a} T_1 \sigma/2}} - \frac{C_{est}}{\sqrt{N_{fd}^{K,3-a} T_1 \sigma/2}}
\geq \frac{1}{8 \sqrt{N_{fd}^{K,a} T_1 \sigma}} + \frac{1}{8 \sqrt{N_{fd}^{K,3-a} T_1 \sigma}} - \frac{C_{est}}{\sqrt{N_{fd}^{K,a} T_1 \sigma/2}} - \frac{C_{est}}{\sqrt{N_{fd}^{K,3-a} T_1 \sigma/2}}
> 0.
\]

The last inequality holds since \( C_{est} \) is a small enough constant defined in Assumption 3.1. Therefore we know agents in groups \( G_{2,1,ua}, \ldots, G_{2,o,a,ua} \) and \( \Gamma_{2,1,ua}, \ldots, \Gamma_{2,o,a,ua} \) all pull arm \( a \).

Now we consider the case when \( \ell > 2 \) and assume the claim is true for smaller \( \ell \)'s. For each arm \( a \), \( W_{3} \) implies that there exists \( u_a \) such that \( W_{3}^{(\ell-1),u_a,high} \) and \( W_{3}^{(\ell-1),u_a,low} \) happen. Recall \( n^{(\ell-1),u_a,a} \) is the number of arm \( a \) pulls in groups \( G_{\ell-1,1,u_a,\ldots}, G_{\ell-1,o,a,ua} \). The induction hypothesis implies that \( n^{(\ell-1),u_a,a} \geq T_{\ell-1} \). \( W_{3}^{(\ell-1),u_a,high} \) together with \( n^{(\ell-1),u_a,a} \geq T_{\ell-1} \) implies that the empirical mean of arm \( a \) pulls in group \( G_{\ell-1,1,u_a,\ldots}, G_{\ell-1,o,a,ua} \) is at least \( \mu_a + 1/\sqrt{n^{(\ell-1),u_a,a}} \). For any agent \( t \) in groups \( G_{\ell,1,u_a,\ldots}, G_{\ell,o,a,ua} \) and \( \Gamma_{\ell,1,u_a,\ldots}, \Gamma_{\ell,o,a,ua} \), it observes
history of groups $G_{\ell,1,\omega_1},\ldots,G_{\ell,1,\omega_\sigma}$ and all groups in levels below level $\ell - 1$. Notice that
the groups in the first $\ell - 2$ levels have at most $(T_1 L^d_K + T_2 + \cdots + T_{\ell-2}) \sigma^3 \leq T_{\ell-1}/(12 \log(T)) \leq n^{\ell-1,u_a,a}/(12 \log(T))$ agents. By $W_2$, we have

$$
\tilde{\mu}_a \geq \mu_a + \left( \frac{1}{\sqrt{n^{\ell-1,u_a,a}}} - (T_1 L^d_K + T_2 + \cdots + T_{\ell-2}) \sigma^3 \right) \sqrt{\frac{3 \log(T)}{(T_1 L^d_K + T_2 + \cdots + T_{\ell-2}) \sigma^3}} \cdot \frac{1}{n^{\ell-1,u_a,a} + (T_1 L^d_K + T_2 + \cdots + T_{\ell-2}) \sigma^3} > \mu_a + \frac{1}{4 \sqrt{n^{\ell-1,u_a,a}}.}
$$

The third last inequality holds when $T$ larger than some constant. Similarly, we also have

$$
\tilde{\mu}_3-a < \mu_3-a - \frac{1}{4 \sqrt{n^{\ell-1,u_a,3-a}}.}
$$

Then we have

$$
\tilde{\mu}_a - \tilde{\mu}_3-a > \mu_a - \mu_3-a + \frac{1}{4 \sqrt{n^{\ell-1,u_a,a}}} + \frac{1}{4 \sqrt{n^{\ell-1,u_a,3-a}}} \geq -\epsilon_{\ell-1} + \frac{1}{8 \sqrt{n^{\ell-1,u_a,a}}} + \frac{1}{8 \sqrt{n^{\ell-1,u_a,3-a}}.}
$$

The last inequality holds because $n^{\ell-1,u_a,a}$ and $n^{\ell-1,u_a,3-a}$ are at most $T_{\ell-1} \sigma$. By Assumption 3.1, we have

$$
\tilde{\mu}_a - \tilde{\mu}_3-a > \mu_a - \mu_3-a - \frac{C_{\text{est}}}{4 \sqrt{n^{\ell-1,u_a,a}}} - \frac{C_{\text{est}}}{4 \sqrt{n^{\ell-1,u_a,3-a}}} > \frac{1}{8 \sqrt{n^{\ell-1,u_a,a}}} + \frac{1}{8 \sqrt{n^{\ell-1,u_a,3-a}} - \frac{C_{\text{est}}}{4 \sqrt{n^{\ell-1,u_a,a}}} - \frac{C_{\text{est}}}{4 \sqrt{n^{\ell-1,u_a,3-a}}} > 0.
$$

The last inequality holds since $C_{\text{est}}$ is a small enough constant defined in Assumption 3.1. Therefore agents in groups $G_{\ell,1,\omega_1},\ldots,G_{\ell,\omega,\omega_a}$ and $\Gamma_{\ell,1,\omega_1},\ldots,\Gamma_{\ell,\omega,\omega_a}$ all pull arm $a$.

\begin{claim}
For any $2 \leq \ell \leq L$, if $\epsilon_{\ell-1} \sigma \leq \mu_1 - \mu_2 < \epsilon_{\ell-2} \sigma$, there are no pulls of arm 2 in groups with level $\ell,\ldots,L$.
\end{claim}

\begin{proof}
We argue in 2 cases $\epsilon_{\ell-1} \sqrt{\sigma} \leq \mu_1 - \mu_2 \leq \epsilon_{\ell-2}$ for $\ell \geq 2$ and $\epsilon_{\ell-2} \leq \mu_1 - \mu_2 \leq \epsilon_{\ell-2} \sqrt{\sigma}$ for $\ell > 2$. Since our recommendation policy’s first level is slightly different from other levels, we need to argue case $\epsilon_{\ell-1} \sqrt{\sigma} \leq \mu_1 - \mu_2 \leq \epsilon_{\ell-2}$ for $\ell = 2$ and case $\epsilon_{\ell-2} \leq \mu_1 - \mu_2 \leq \epsilon_{\ell-2} \sqrt{\sigma}$ for $\ell = 3$ separately.

- $\epsilon_{\ell-1} \sigma \leq \mu_1 - \mu_2 \leq \epsilon_{\ell-2}$ for $\ell = 2$ (i.e. $\epsilon_1 \sigma \leq \mu_1 - \mu_2 \leq \epsilon_0$): We know agents in level at least 2 will observe at least $N^d_K t_1/2$ pulls of arm $a$ for $a \in \{1,2\}$. By $W_2$, for any agent
in level at least 2, we have
\[ |\hat{\mu}_a^t - \mu_a| \leq \sqrt{\frac{3 \log(T)}{\sigma N_{K,a} T_1}}. \]

By Assumption 3.1, we have
\[
\hat{\mu}_1^t - \hat{\mu}_2^t \geq \frac{C_{\text{est}}}{\sqrt{\sigma N_{K,1} T_1}} - \frac{C_{\text{est}}}{\sqrt{\sigma N_{K,2} T_1}} - \frac{2 C_{\text{est}}}{\sqrt{\sigma N_{K,2} T_1}} - \frac{C_{\text{est}}}{\sqrt{\sigma N_{K,1} T_1}} - \frac{C_{\text{est}}}{\sqrt{\sigma N_{K,2} T_1}}
\]
\[ \geq \frac{\sqrt{\sigma}}{4\sqrt{N_{K,1} T_1}} + \frac{\sqrt{\sigma}}{4\sqrt{N_{K,2} T_1}} - \frac{3 \log(T)}{\sigma N_{K,2} T_1} - \frac{C_{\text{est}}}{\sqrt{\sigma N_{K,1} T_1}} - \frac{C_{\text{est}}}{\sqrt{\sigma N_{K,2} T_1}}
\]
\[ \geq 0. \]

Therefore agents in level at least 2 will all pull arm 1.

• \( \epsilon_{\ell-1} \sigma \leq \mu_1 - \mu_2 \leq \epsilon_{\ell-2} \sigma \) for \( \ell > 2 \): By Claim E.1, for any agent \( t \) in level at least \( \ell \), that agent will observe at least \( T_{\ell-1} \) arm \( a \) pulls. By \( W_2 \), we have
\[ |\hat{\mu}_a^t - \mu_a| \leq \sqrt{\frac{3 \log(T)}{T_{\ell-1}}}. \]

By Assumption 3.1, we have
\[
\hat{\mu}_1^t - \hat{\mu}_2^t \geq \frac{\hat{\mu}_1^t - \hat{\mu}_2^t}{\sqrt{T_{\ell-1}}} - \frac{2 C_{\text{est}}}{\sqrt{T_{\ell-1}}}
\]
\[ \geq \frac{\mu_1 - \mu_2}{\sqrt{T_{\ell-1}}} - \frac{2 \sqrt{3 \log(T)}}{T_{\ell-1}} - \frac{2 C_{\text{est}}}{\sqrt{T_{\ell-1}}}
\]
\[ \geq \frac{\sigma}{16 T_{\ell-1}} - \frac{2 \sqrt{3 \log(T)}}{T_{\ell-1}} - \frac{2 C_{\text{est}}}{\sqrt{T_{\ell-1}}}
\]
\[ \geq 0. \]

Therefore agents in level at least \( \ell \) will all pull arm 1.

• \( \epsilon_{\ell-2} < \mu_1 - \mu_2 < \epsilon_{\ell-2} \sigma \) for \( \ell = 3 \) (i.e. \( \epsilon_1 < \mu_1 - \mu_2 < \epsilon_1 \sigma \)): By Claim E.1, for any agent \( t \) in level at least 3, that agent will observe at least \( T_1 N_{K,a} \sigma^2 / 2 \) arm \( a \) pulls (just from
the first level). By $W_2$, we have

$$|\hat{\mu}_a^t - \mu_a| \leq \sqrt{\frac{3\log(T)}{\sigma^2 N_{K,a}^{\text{fid}} T_{1/2}}}.$$ 

By Assumption 3.1, we have

$$\hat{\mu}_1^t - \hat{\mu}_2^t \geq \hat{\mu}_1^t - \mu_1^t - \frac{C_{\text{est}}}{\sqrt{\sigma^2 N_{K,1}^{\text{fid}} T_{1/2}}} - \frac{C_{\text{est}}}{\sqrt{\sigma^2 N_{K,2}^{\text{fid}} T_{1/2}}} \geq \mu_1 - \mu_2 - \frac{3\log(T)}{\sqrt{\sigma^2 N_{K,1}^{\text{fid}} T_{1/2}}} - \frac{3\log(T)}{\sqrt{\sigma^2 N_{K,2}^{\text{fid}} T_{1/2}}} - \frac{C_{\text{est}}}{\sqrt{\sigma^2 N_{K,1}^{\text{fid}} T_{1/2}}} - \frac{C_{\text{est}}}{\sqrt{\sigma^2 N_{K,2}^{\text{fid}} T_{1/2}}} \geq \frac{1}{4\sqrt{\sigma N_{K,1}^{\text{fid}} T_1}} \sqrt{\frac{3\log(T)}{\sigma^2 N_{K,2}^{\text{fid}} T_1}} - \frac{C_{\text{est}}}{\sqrt{\sigma^2 N_{K,1}^{\text{fid}} T_{1/2}}} - \frac{C_{\text{est}}}{\sqrt{\sigma^2 N_{K,2}^{\text{fid}} T_{1/2}}} > 0.$$ 

Therefore agents in level at least 3 will all pull arm 1.

• $\epsilon_{\ell-2} < \mu_1 - \mu_2 < \epsilon_{\ell-2} \sigma$ for $\ell > 3$: Since $\mu_1 - \mu_2 < \epsilon_{\ell-2} \sigma < \epsilon_{\ell-3}$, by Claim E.1, for any agent $t$ in level at least $\ell$, that agent will observe at least $T_{\ell-2} \sigma^2$ arm $a$ pulls (just from level $\ell - 2$). By $W_2$, we have

$$|\hat{\mu}_a^t - \mu_a| \leq \sqrt{\frac{3\log(T)}{\sigma^2 T_{\ell-2}}}.$$ 

By Assumption 3.1, we have

$$\hat{\mu}_1^t - \hat{\mu}_2^t \geq \hat{\mu}_1^t - \mu_1^t - \frac{2C_{\text{est}}}{\sqrt{\sigma^2 T_{\ell-2}}} \geq \mu_1 - \mu_2 - \frac{2\sqrt{3\log(T)}}{\sigma^2 T_{\ell-2}} - \frac{2C_{\text{est}}}{\sqrt{\sigma^2 T_{\ell-2}}} \geq \frac{1}{4\sqrt{T_{\ell-2}}} \sqrt{\frac{3\log(T)}{\sigma^2 T_{\ell-1}}} - \frac{2C_{\text{est}}}{\sqrt{T_{\ell-1}}} > 0.$$ 

Therefore agents in level at least $\ell$ will all pull arm 1. □
E.2 Finishing the proof of Theorem 6.1 for \( K = 2 \) arms

We set the parameters \( T_{\ell} \) for each level \( \ell \in \{1, \ldots , L-1\} \):

\[
T_{\ell} = T^{\gamma_{\ell}}/\sigma^3, \quad \text{where} \quad \gamma_{\ell} := \frac{2^{L-1} + 2^{L-2} + \ldots + 2^{L-\ell}}{2^{L-1} + 2^{L-2} + \ldots + 1} = \frac{2^L - 2^{L-\ell}}{2^{L-1}}.
\]  

(8)

Therefore, \( T_{\ell}/T_{\ell-1} \geq T^{1/2^\ell} \geq \sigma^4 \), as required by Eq. (7).

The \( L \)-th layer comprises all remaining nodes, hence

\[
T_L = (T - T_1 \cdot L_K^{fd} \cdot \sigma^2 - (T_2 + \ldots + T_{\ell-1}) \sigma^3)/\sigma^3.
\]  

(9)

Proof. By Claim E.2, the regret conditioned the intersection of clean events is at most

\[
\max \left( T_1 L_K^{fd} \sigma^2, \max_{\ell \geq 2} \epsilon_{\ell-1} \sigma (T_1 L_K^{fd} \sigma^2 + T_2 \sigma^3 + \ldots + T_{\ell} \sigma^3) \right)
\]

\[
\leq \max \left( T_1 L_K^{fd} \sigma^2, \max_{\ell \geq 2} 2 \epsilon_{\ell-1} T_{\ell} \sigma^4 \right)
\]

\[= O \left( T^{2^{L-1}/(2^{L-1})} \log^2 (T) \right). \]  

E.3 Finishing the proof of Theorem 6.2 for \( K = 2 \) arms

We set the parameters as follows:

\[
L = \log(T)/\log(\sigma^4),
T_{\ell} = \sigma^{4\ell} \text{ for } \ell \in \{1, \ldots , L-1\}.
\]  

(10)

\( T_L \) is defined via Eq. (9). Note that these settings satisfy Eq. (7), as required.

Proof. Recall from Appendix E.1 that \( \epsilon_{\ell} = \Theta(1/\sqrt{\ell} \sigma) \) for \( \ell \in [L-1] \) and \( \epsilon_0 = 1 \).

Consider two cases:

- \( \Delta < \epsilon_{L-1} \sigma \). In this case, notice that even always picking the sub-optimal arm gives expected regret at most \( T (\mu_1 - \mu_2) = T \Delta = O(T^{1/2} \text{polylog}(T)) \). On the other hand, \( T^{1/2} = O(\text{polylog}(T)/\Delta) \). So, regret is \( O(\min(1/\Delta, T^{1/2}) \text{polylog}(T)) \).

- \( \Delta \geq \epsilon_{L-1} \sigma \). In this case, we can find \( \ell \in \{2, \ldots, L\} \) such that \( \epsilon_{\ell-1} \sigma \leq \Delta < \epsilon_{\ell-2} \sigma \). By
Claim E.2, we can upper bound the regret by
\[
\Delta \cdot (T L_a^2 \sigma^2 + T_2 \sigma^3 + \cdots T_{\ell-1} \sigma^3)
= O(\Delta T_{\ell-1} \sigma^3)
= O(\Delta T \sigma^2)
= O(\Delta \cdot \sigma^6)
= O(\Delta \cdot \sigma^8)
= O(polylog(T)/\Delta).
\]

We also have \(1/\Delta \leq (\varepsilon_{\ell-1} / \sigma) = O(T^{1/2})\). So, regret is \(O(min(1/\Delta, T^{1/2}) \ polylog(T))\).

Finally we discuss the subhistory sizes. We know that agents in level \(\ell\) observes the history of all agents below level \(\ell - 2\) (including level \(\ell - 2\)). It is easy to check that the ratio between the number of agents below level \(\ell\) and the number of agents below level \(\ell - 2\) is bounded by \(O(polylog(T))\). Therefore our statement about the subhistory sizes holds.

\[\square\]

### E.4 Extending the analysis to \(K > 2\) arms.

Here we discuss how to extend Theorems 6.2 and 6.2 to \(K > 2\) arms. The analysis is very similar to the \(K = 2\) case, so we only sketch the necessary changes.

**Proof Sketch.** We still wlog assume arm 1 has the highest mean (i.e. \(\mu_1 \geq \mu_a \forall a \in A\). We first extend the clean events (i.e. \(W_1, W_2, W_3, W_4\)) in Appendix E.1 to the case when \(K\) is larger than 2. \(W_1\) and \(W_2\) extend naturally: we still set \(W_1 = \cap_{s,a} W_1^{u,a}\) and \(W_2 = \cap_{t,a,t_1,t_2} W_2^{t,a,t_1,t_2}\). The difference is that now \(a\) is taken over \(K\) arms instead of 2 arms. For \(W_3\), we change the definition \(W_3^{\ell,a} = \bigcup_u \left( W_3^{\ell,u,a,high} \cap \left( \cap_{a' \neq a} W_3^{\ell,u,a',low} \right) \right)\) and \(W_3 = \cap_{\ell,a} W_3^{\ell,a}\). We extend \(W_4\) in a similar way: define \(W_4^a = \bigcup_u \left( W_4^{u,a,high} \cap \left( \cap_{a' \neq a} W_4^{u,a',low} \right) \right)\) and \(W_4 = \bigcap_a W_4^a\). Since \(K\) is a constant, it’s easy to check that the same proof technique shows that the intersection of these clean events happen with probability \(1 - O(1/T)\). So the case when some clean event does not happen contributes \(O(1)\) to the regret.

Now we proceed to extend Claim E.1 and Claim E.2. The statement of Claim E.1 should be changed to “For any arm \(a\) and \(2 \leq \ell \leq L\), if \(\mu_1 - \mu_a \leq \varepsilon_{\ell-1}\), then there are at least \(T_{\ell}\) pulls of arm \(a\) in groups \(G_{\ell,u,1}, G_{\ell,u,2}, \ldots, G_{\ell,u,\sigma}\) and there are at least \(T_{\ell}\sigma(\sigma - 1)\) pulls of arm \(a\) in the \(\ell\)-th level \(\Gamma\)-groups”. The statement of Claim E.2 should be changed to “For any \(2 \leq \ell \leq L\), if \(\varepsilon_{\ell-1} \sigma \leq \mu_1 - \mu_a < \varepsilon_{\ell-2}\sigma\), there are no pulls of arm \(a\) in groups with level \(\ell, \ldots, L\)".

The proof of Claim E.2 can be easily changed to prove the new version by changing “arm 2” to “arm \(a\). The proof of Claim E.1 needs some additional argument. In the proof of Claim E.1, we show that \(\mu_a - \mu_{3-a} > 0\) for agent \(t\) in the chosen groups. When extending to more than 2 arms, we need to show \(\mu_a - \mu_{3-a} > 0\) for all arm \(a' \neq a\). The proof of Claim E.1 goes through if \(\mu_1 - \mu_{a'} \leq \varepsilon_{\ell-2}\) since then there will be enough arm \(a'\) pulls.
in level $\ell - 1$. We need some additional argument for the case when $\mu_1 - \mu_{a'} > \epsilon_{\ell-2}$. Since $\mu_1 - \mu_{a'} > \epsilon_{\ell-2} > \epsilon_{\ell-1} \sigma$, we can use the same proof of Claim E.2 (which rely on Claim E.1 but for smaller $\ell$'s) to show that there are no arm $a'$ pulls in level $\ell$ and therefore $\hat{\mu}_t^{a'} - \hat{\mu}_t^a > 0$.

Finally we proceed to bound the regret conditioned on the intersection of clean events happens. The analysis for $K = 2$ bounds it by consider the regret from pulling the suboptimal arm (i.e. arm 2). When extending to more than 2 arms, we can do the exactly same argument for all arms except arm 1. This will blow up the regret by a factor of $(K-1)$ which is a constant. \qed