ABSTRACT

We study the impact of tax and transfer polices in an overlapping generations model where agents have warm-glow bequest motives. Some dynasties of agents are risk averters, and others are dynasties of risk lovers. Making simplifying assumptions, we construct a sequence of simple models in which we are able to calculate invariant distributions for wealth holdings. In the first model, the risky assets has the same expected return as the safe asset; in the second model, only risk lovers can invest in the risky asset; and in the third model, agents can only invest in one type of asset at a time. We interpret the decision to invest in the risky asset as the career choice to become an entrepreneur. This simplifying assumption prevents risk averters from holding diversified portfolios of the risky asset and the safe asset. In the third model, moderate levels of taxes and redistribution increase growth and social by inducing some risk averters to invest in the risky asset. In the second and third models, the only productive input is capital, but we show that our analysis extends to an endogenous growth model with both labor and capital and endogenous wages and rental rates.

Keywords: Endogenous growth, Inequality, Redistribution, Overlapping generations, Invariant distribution, Social welfare function.

JEL Codes: C62, D51, H21, O4

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1. Introduction

The role of inequality and taxation in growth has been extensively studied in a variety of models. Kuznets (1955) was one of the first economists who analyzed economic growth and inequality. In his model, inequality tends to increase once the economic growth increases due to industrial revolution since low productivity labor with low inequality among workers is substituted with high productivity labor with high inequality among workers. Once the country achieves a middle-high level of industrialization, however, inequality decreases due to the development of a welfare state.

More recently, several authors analyzed theoretical models in which growth and inequality could be analyzed by general equilibrium models. Persson and Tabellini (1994) studied the impact of growth and inequality in democratic countries. In their model, redistributive taxes may act the accumulation of capital implying a reduction in the long-term growth. Alesina and Rodrik (1994) developed a model with capitalists and workers in which the optimal taxes with the highest growth rates benefitted capitalists more.

For most of the classical literature in growth and inequality, taxes and redistribution have a negative impact on growth due to the loss of capital in the economy. More recently, some authors including Persson and Tabellini (1994) and Perotti (1996) argued that inequality has a negative impact on the amount of capital invested in the rms due to a possible increment in redistributive taxes caused by a high level of inequality in the economy. Most of the authors who argued that inequality and growth are inversely correlated, as the ones mentioned before, based their studies on empirical evidence using cross-country growth regressions. However, as Temple (1999) argued, most of these studies have been criticized due to the fragility of several of their results and their ad hoc specialization.

We consider three models, one without growth and two with endogenous growth rate and heterogeneous production technologies - one with segmentation and one with risk lovers - to analyze the dynamics of taxation in OLG economies with risk lovers. In the latter models, taxes and redistribution have a negative impact on growth if the more productive technologies involve larger amount of idiosyncratic risk. However, taxation on bequest and income have different effects on growth and inequality. Moreover, in absence of taxes, the most productive technologies will dominate the economy in the long run, and inequality in the long run will depend main-
ly on the risk that the most productive one involves. In the presence of taxes and different expected technology returns, bequest or income taxes ensure the existence of an invariant distribution of wealth among the agents and an invariant growth rate of the economy. We also show that the invariant distribution with a single positive type of tax with constant marginal tax rate is also ergodic as in Piketty (1997), but only among the agents of the same type. Therefore, there is no poverty trap among the agents with the most productive and most risky technologies. Among the agents that do not have access to the most productive technologies, their wealth might not reach the top in any future date.

The study of the models are closely related. The distribution and convergence to the invariant distribution of the former model is strongly related to the latter ones. Additionally, the second one can be seen as a restricted case of the latter if the risky technology is more productive than the safe one as we assume in most part of the article, allowing us to have a better understanding of the latter model.

We study optimal taxation by introducing a central planner who chooses taxes to maximize the social welfare of the economy. We show that the social welfare function of the social planner can be written as the sum of three independent functions. The first one depends only on growth, the second one depends only on inequality, and third one depends only on the difference of the discount factors of the agent and the social planner. The first one is directly correlated with the growth rate of the economy, implying that the presence of low taxes might be optimal in some cases. The second function is inversely correlated with the inequality of the invariant distribution which implies that high taxes might be optimal in some cases. We prove that bequest taxes are always worse than income taxes for discount rate of the agents and the social planner.

We also find that, for a fixed discount rate of the consumers\(^1\), the optimal taxation is strictly decreasing on how the social planner discounts the future. Moreover, the optimal tax is such that the invariant wealth distribution tends to a completely equal one if the social planner strongly discounts the future. On the other hand, the optimal tax is zero if the social planner does not discount the future at all. The intuition behind these results is that, if a social planner discounts the future strongly, the weight of distant dates and the growth rate become almost irrele-

\(^1\) We consider the discount rate of the agent as the bequest rate since, as we will show latter on, the bequest is a form of how the agent is concerned about his/her descendants' future consumptions.
vant. Therefore, the social welfare function is dominated by the inequality effect. However, if a social planner discounts the future very weakly, the weight of future consumption will dominate the inequality effect even when both effects are considerably large. Note that, in the presence of a social planner that almost does not discount the future, inequality affects strongly the social welfare function due to the presence of a very unequal distribution of wealth that causes a high impact in the social welfare function.

We developed an overlapping generations model with bequest as in Galor and Zeira (1993). To analyze the impact of different production technologies on the accumulation of wealth, we use a model with bequest and idiosyncratic uncertainty on the technologies as in Piketty (1997). However, we consider frictions on the use of the technologies separating the agents in two groups: skilled and unskilled, and risk lovers and risk averters. As it was mentioned before, we also consider redistributive taxes as in Alesina and Rodriguez (1994). The impact of different technologies with different levels of idiosyncratic uncertainty can also be related to models with different attitudes toward risk as in Araujo, Gama, and Kehoe (2017) and Araujo, Chateauerneuf, Gama, and Novinski (2018).

The idea that high taxes should be imposed due to the existing spread between asset returns and real returns has been explored by Piketty and Saez (2014). They argued that changes in growth rate in western economies, mainly in the US, has a strong impact on inequality due to the gap among the interest rate and the real GDP growth rate. Other authors as Lindert and Williamson (2016) studied long term data to analyze the inequality in the US economy since colonial times. Bargain et al. (2015) made a deeper analysis of the tax policy and inequality from 1979-2007 in which changes in the tax policy increased income inequality causing more accumulation among the top one percent. Weide and Milanovic (2014) showed with a study on micro data of the US economy from 1960 to 2010 that high levels of inequality reduce the economic growth of the poor, but it might enhance the economic growth of the rich. Benhabib, Bisin, and Zhu (2011, 2015, 2016), Jones (2015), and Acemoglu and Robinson (2015) analyze income or wealth distribution with taxation. Gabaix, Lasry, Lions, and Moll (2016) (analytically), and Aoki and Nirei (2017) (numerically) analyze the dynamics of income distribution with taxes. And Garcia-Peñalosa and Wen (2008) analyze the effect of taxation on growth with risk averse agents. Most of works mentioned above support the idea analyzes Pareto distribution of wealth or income. Moreover, Benhabib, Bisin, and Zhu (2011, 2015, 2016) found conditions to generate fat tails for
transformation processes induced by investment risk. On the other hand, Beare and Toda (2018) show that tails of wealth distribution decay exponentially in a heterogeneous-agent dynamic general equilibrium model with idiosyncratic endowment risk. Our model has larger similarities with the former than with the latter. However, we analyze the effect of different marginal taxation rate on growth, showing that there is a trade off between growth and taxation for middle and high marginal taxation rates from which there is no empirical evidence against it.

Our results support some of the ideas mentioned above since low taxes imply higher growth rates and high levels of inequality. At the same time, a reduction of income taxes in our model will change the invariant distribution to a more unequal one, then it will cause a gradual increment of the inequality supporting several empirical studies mentioned above. Moreover, our model suggest that changes in the tax policy may be based on changes on how the social planner discounts the future compared with the other agents do so.

There is an important exception to our result that taxes reduce growth: We identify parameter values for the model where agents are restricted to invest in only one type of assets in which high enough taxes and transfers insure risk averters and induce poor risk averters to invest in the risky asset. Those risk averters who are lucky and accumulate a large enough level of wealth choose to switch to investing in the safe asset. In this case, increasing taxes and increases growth and the welfare of risk averters although it decreases the welfare of risk lovers.

The paper is organized as follows. In section 2, we define the basic model including the notion of equilibrium. In subsection 2.2, we define the basic properties of the model including the relationship between growth and inequality without taxes, and, in subsection 2.3, we analyze the basic properties with taxes. In section 2.4, we show the existence of an invariant growth rate, an invariant distribution of wealth among the agents and their basic properties. In section 2.5, we analyze the existence of optimal taxes by a social welfare function, and we also prove the basic properties of this function and of the optimal taxes. In subsection 2.6, we give some numerical examples that help us to the analysis made in section 2.5. In section 3, we analyze the case with effort cost and the extension to a model with capital and labor. Finally, in section 4, we give some concluding remarks.

2. Model with production and segmentation and taxation

Let us consider an overlapping generation economy with a continuum of two-period agents (young and old). There are two different types of agents unskilled or type a, and unskilled
or type \( l \). The former can use only one of the available technologies in the economy. On the other hand, the latter can invest in the two types of technologies available. These technologies are linear and represented by \( R_S: \{1,2\} \rightarrow \mathbb{R}_+ \) and \( R_R: \{1,2\} \rightarrow \mathbb{R}_+ \) where \( R_S \) is the safe one, the technology available for both types of agents, and \( R_R \) is the risky one, available for the skilled agents only. Then, the returns of the safe technology are represented by a constant value \( R_S > 0 \), and the returns of the risky technology are represented by the vector \((\bar{R}_R, \underline{R}_R)\) where \( \bar{R}_R > 0 \) if event 1 occurs, and \( \underline{R}_R \geq 0 \) if event 2 occurs. Note that the probability of each event is equal to \( 1/2 \) and independent among the agents. Due to the risk involved in the technologies used by the agents, each agent is exposed to idiosyncratic risk caused by the uncertainty of the technologies used. Then, the uncertainty in our model is independent among the agents, which implies that, in the aggregate economy, there is no aggregate uncertainty. Note also that the investment in one of the technologies by an agent is a one-period investment.

For any date \( t \geq 1 \), there is a single consumption good at every state \( s \) with date \( t \geq 1 \) that the young agents will use it to invest in the technologies, and the old agents will use it to consume, \( c_s^l \), and to give a bequest, \( b_s^l \), to his successor. All the agents give a bequest that is a proportion of the agent’s total wealth. In \( t = 0 \), there is no consumption since there is no old generation, and every young agent has an initial endowment \( w_0^l \), to be invested in the technologies.

### 2.1. Taxes

At each state \( s \) of length \( t \geq 1 \), there is an income tax \( \tau^{I^+_s}(\cdot) \) and bequest tax \( \tau^{B^+_s}(\cdot) \) imposed for any agent if her level of income, consumption and bequest is above some threshold \( \bar{W}_s^l \) and \( \underline{B}_s \), respectively. Additionally, there is also an income subsidy \( \tau^{I^+_s}(\cdot) \) and bequest subsidy \( \tau^{B^+_s}(\cdot) \) given to any agent with an income, consumption, and bequest below some threshold \( \underline{W}_s^l \) and \( \underline{B}_s \), respectively. For simplicity, each type of tax will be summarized by \( \tau^l_s(\cdot) = \tau^{I^+_s}(\cdot) + \tau^{I^+_s}(\cdot) \) and \( \tau^B_s(\cdot) = \tau^{B^+_s}(\cdot) + \tau^{B^+_s}(\cdot) \). From now on, each type of tax is defined by a constant marginal tax rate above the upper threshold and a constant marginal subsidy rate below the lower threshold. \( \tau^{I^+_s}, \tau^{I^+_s} \in [0,1] \) are the marginal rates related to the income policy, the ones related with consumption, and \( \tau^{B^+_s}, \tau^{B^+_s} \in [0,1] \) the ones related with bequests. Therefore, the income tax mentioned above can be written as \( \tau^l_s(x) = \tau^{I^+_s}(x - \bar{W}_s)^+ + \tau^{I^+_s}(\underline{W}_s - x)^+ \) and the be-
quest tax can be written as \( \tau^B_s(b) = \frac{\tau^{B+}_s}{1-\tau^{B+}_s}(b - \overline{B}_s)^{+} \). Note that the marginal tax rates \( \tau^{I+}_s \) and \( \tau^{B+}_s \) as well as the thresholds are exogenously defined by the central planner. On the other hand, the marginal subsidy rates \( \tau^{I-}_s \) and \( \tau^{B-}_s \) are endogenously determined in equilibrium to ensure a balanced government budget.

Therefore, the government can define the tax policy by choosing the marginal tax rates \( \tau^{I+}_s \) and \( \tau^{B+}_s \) and the thresholds \( W_s, \overline{W}_s, B_s \) and \( \overline{B}_s \). For simplicity, \( W_s = \overline{W}_s = \overline{w}_s \) where \( \overline{w}_s \) is the average income of the economy in state \( s \), and \( B_s = \overline{B}_s = b_s \) where \( b_s \) is the average bequest of the economy in state \( s \). Therefore, \( \tau^{I+}_s = \tau^{I-}_s = \tau^{I}_s \), and \( \tau^{B+}_s = \tau^{B-}_s = \tau^{B}_s \).

Additionally, we will suppose that

\[
\overline{R}_R > \overline{R}_S \geq R_S > R_R \geq 0 \quad (3.1)
\]

which implies that the \( R \)-technology involves higher levels of risk compared to the \( S \)-technology such that

\[
E[R_R] \geq \frac{1}{\delta} \geq E[R_S] \quad (3.2)
\]

where \( \delta \in (0, 1) \) is the natural bequest rate that will be explained properly latter on.

As it was mentioned above, the initial endowment is given by an initial amount of the available good, \( \{w^i_0\}_i \), and then, the problem of the agent \( i \) in the first date is defined by

\[
\max_{(c,b,\theta)} \quad \frac{1}{2} u^i(c_1, b_1) + \frac{1}{2} u^i(c_2, b_2)
\]
\[
\text{s.t.} \quad \theta_R + \theta_S \leq w^i_0, \quad 0 \leq c_1 + b_1 + \frac{\tau^B_s}{1-\tau^B_s}(b_1) \leq \overline{R}_R \theta_R + R_S \theta_S - \tau^I_s(R_R \theta_R + R_S \theta_S),
\]
\[
0 \leq c_2 + b_2 + \frac{\tau^B_s}{1-\tau^B_s}(b_2) \leq \overline{R}_R \theta_R + R_S \theta_S - \tau^I_s(R_R \theta_R + R_S \theta_S).
\]

where \( u^i(c, b) \) is the utility index of the agent \( i \). For \( t \geq 1 \), \( w^i_0 \) is substituted by \( b^i_s \) the bequest that the agent \( i \) receives from her predecessor at state \( S \). Each agent has a utility index given by \( u^i(c, b) = c^{1-\delta} b^\delta \). Using the form of the tax policy, the budget constraint when the agent \( i \) is old can be written as
0 \leq c_1 + b_1 + \frac{\tau^B_s}{1 - \tau^B_s} (b_1 - \bar{b}_s) \leq R_R \theta_R + R_S \theta_S + \tau^I_s (\bar{w}_s - R_R \theta_R - R_S \theta_S),
0 \leq c_2 + b_2 + \frac{\tau^B_s}{1 - \tau^B_s} (b_2 - \bar{b}_s) \leq R_R \theta_R + R_S \theta_S + \tau^I_s (\bar{w}_s - R_R \theta_R - R_S \theta_S).

Then, each agent receives three types of transfers that depend on the average income, consumption, and bequest. Additionally, income tax reduces the agent income by a proportion of $1 - \tau^I_s$, and consumption and bequest taxes make more expensive to consume and to give part of her income as bequest, respectively. Therefore, this constraint can be written as

$$0 \leq c_1 + \left(1 + \frac{\tau^B_s}{1 - \tau^B_s}\right) b_1 \leq (1 - \tau^I_s) (R_R \theta_R + R_S \theta_S) + \frac{\tau^B_s}{1 - \tau^B_s} \bar{b}_s,$$
$$0 \leq c_2 + \left(1 + \frac{\tau^B_s}{1 - \tau^B_s}\right) b_2 \leq (1 - \tau^I_s) (R_R \theta_R + R_S \theta_S) + \frac{\tau^B_s}{1 - \tau^B_s} \bar{b}_s.$$

Note that, in our model, the existence of income taxes can also be seen as wealth taxes since the capital is completely transformed in each state $s$. Therefore, we will focus mainly on interpret $\tau^I_s$ as income taxes. However, we understand the clear difference between both concepts, that coincide due to the capital properties of our model.

2.2. Equilibrium

Now, let us define the equilibrium for the economy as $\left((c^i, b^i, \theta^i), (\tau^I_s, \tau^B_s)\right)$ such that $(c^i, b^i, \theta^i)$ maximizes the consumer problem mentioned above for any $t \geq 0$, and for each state $s$ of length $t \geq 1$, we have

$$\int_i \tau^B_s (b^i_s) = 0,$$
$$\int_i \tau^I_s (R_{Rs} \theta^R_s + R_S \theta^I_s) = 0.$$

Due to the FOC, we know that

$$b^i_s = \frac{\delta (1 - \tau^B_s)}{(1 - \delta)} c^i_s$$

for all agent $i$, then in absence of income or bequest taxes, each agent will bequest a proportion $\delta$ of her income and consume the other part. However, in the presence of consumption or be-
quest taxes, the agent will deviate from this proportion since the cost of consuming or requesting increasing due to taxation.

Based on the effects of bequest taxes on consumer’s problem, the average consumption and bequest can be written as

\[
\bar{c}_s = \frac{(1 - \delta)\bar{w}_s}{1 - \delta \tau_s^B},
\]
\[
\bar{b}_s = \frac{\delta(1 - \tau_s^B)\bar{w}_s}{1 - \delta \tau_s^B}.
\]

Due to the form of the utility index and Equation (3.2) a skilled agent \(l_i\) will never invest in the safe technology, that is, \(\theta^l_i = 0\) for all \(i \in [0,1]\). Therefore, all agents invest only in one technology at the same time. The unskilled ones invest in the safe one, the less productive, and the skilled ones invest in the risky, the most productive one.

Since bequest taxes affect the incentives that each agent has for consumption and bequest, the average consumption and the average bequest also depends on these marginal taxation rates. Moreover, higher bequest taxes imply a larger proportion of consumption by all agents and a lower proportion of bequest, which decreases the descendant income. More specifically, we have that if \(\bar{w}_s^a\) is the after taxes mean income of the unskilled at the node \(s\), and \(\bar{w}_s^l\) is the after taxes mean income of the skilled at the node \(s\), the average income at a node \(s'\) an immediate successor of \(s\) is given by

\[
\bar{w}_{s'} = \frac{E[R_R]\delta(1-\tau_S^B)}{1 - \delta \tau_s^B} \bar{w}_s + \frac{R_S\delta(1-\tau_S^B)}{1-\delta \tau_s^B} \bar{w}_{s'},
\]

which implies the following result.

**Proposition 2.** For any fiscal policy plan with marginal tax rates given by \(\left(\bar{\tau}_s^l, \bar{\tau}_s^B\right)_s\), such that \(\bar{\tau}_s^l, \bar{\tau}_s^B < 1\), we have that any increment on the income tax rate at state \(s\) (from \(\bar{\tau}_s^l\) to \(\bar{\tau}_s^l + \epsilon\)) induces a higher growth rate than an increment on the bequest tax rate (from \(\bar{\tau}_s^B\) to \(\bar{\tau}_s^B + \epsilon\)) at state \(s'\) the immediate successor of \(s\).
2.3. Dynamic properties of the equilibrium

From now on, we will analyze the dynamic properties of the equilibrium and the existence of an invariant distribution of income. From now on, let us suppose that the fiscal policy plans satisfy that the marginal tax rates are constant over time, that is, \(\bar{\tau}_s^I = \bar{\tau}_s^I\), and \(\bar{\tau}_s^B = \bar{\tau}_s^B\) for all \(s, s'\). Then, we will simply denote the fiscal plans as \((\bar{\tau}^I, \bar{\tau}^B) \in [0,1]^2\).

Note that there is no aggregate uncertainty. It is a consequence of the continuum numbers of agent and the idiosyncratic risk that each agent has once they invest in the available technologies for them. Therefore, from now on, we will denote each state \(s\) at date \(t\) simply as \(t\), and for the successors of \(s\) at date \(t\) we will denote as \(t + 1\) for aggregate variables. However, for individual variables like optimal consumption, bequest, and income of an agent \(i\), we will denote a successor of a node \(s\) as \(s, k\) with \(k = 1, 2\).

2.4. Invariant distribution

From now on, let us assume that \(R_R = 0\) for simplicity, and, in case that this condition does not hold, we will inform you. Let us now analyze the existence of an invariant distribution of relative wealth, that is, the distribution of wealth of the agents divided by the aggregate wealth in each state. From now on, we will focus our attention in positive fiscal policies, that is, \((\bar{\tau}^I, \bar{\tau}^B)\) since in the absence of taxes there is no invariant distribution if \(R_R = 0\). Moreover, if \(R_R > 0\), the wealth of the skilled agents will have all the wealth in the economy in the long run in even when the wealth is not completely in hands of the agents who get lucky all the time.

**Proposition 6.** There is no invariant concentration of wealth if \(\bar{\tau}^I = \bar{\tau}^B = 0\) unless \(R_R = \bar{R}_R = R_S\). In this case, any initial endowment distribution such that \(x_i^1 = 0\) for all \(i \in [0,1]\) is an invariant distribution.
Proof. The case in which $R_R = 0$ is a direct consequence of the fact that, with probability one, all skilled agents will have zero consumption wealth in the long run, and, at the same time, the aggregate endowment of the economy is always positive.

To prove the case in which $R_R > 0$, notice that since the production of the skilled agents is at least as good as the unskilled agents, 1) the initial income of the invariant distribution of the skilled agents can only be zero, or 2) equal to the total aggregate initial wealth if $E[R_R] > R_S$, or 3) it can be any possible value between these two extremes if $E[R_R] = R_S$. If we analyze the case in which $x^l$ is not equal to zero almost everywhere (case 2 or 3), the aggregate production over the aggregate production of the skilled ones converges to a positive constant when $t$ goes to infinity, and, since
\[
\frac{R_R}{E[R_R]} < 1, \quad \left(\frac{R_R}{E[R_R]}\right)^n \text{ converges to zero when } n \text{ goes to infinity, and}
\]
\[
\frac{R_R}{E[R_R]}^{(k+1)n} \text{ also converge to zero when } n \text{ goes to infinity for every } k \in \mathbb{N},
\]
part of the income will be concentrated in hands of a zero-measure set of skilled agents in the long run. Therefore, the only possible case is that $x^l \equiv 0$ almost everywhere.

We have that any initial distribution converges to an invariant distribution. The following Theorem shows that any tax policy with constant and nonnegative marginal taxation rates implies that the distribution of income converges to an invariant distribution in the long run.

**Theorem 1.** Given a fixed marginal tax rate $\left(\tau^l_S, \tau^B_S\right)_S \in [0,1]^2$ for any initial distribution of endowment $\{w_0^l\}_i$, there is an invariant distribution of the proportion of wealth among the agents.

2.5. Social welfare function and optimal tax rate
We know that any increment of the marginal tax rate causes a change in the invariant distribution and the long-run growth rate of the economy. The invariant distribution will tend to be more equal among the agents, and the long-run growth rate might also decrease if a social planner taxes more income.

Cases like $\tau^I, \tau^B = 0$, or $\tau^I = 1$, or $\tau^B = 1$ are too extreme in this framework. The first one will imply the survival of a small amount of agents with arbitrarily large amount of wealth only. The second one will imply a considerably lower growth rate of the economy and in some cases even negative growth rates. The third one, consumption taxes equal to 1, will imply that any agent cannot consume any amount of good. And the fourth one, bequest taxes equal to 1, imply that any agent cannot bequest to the next generation. These two latter conditions imply that the agents will not survive in one way or another. Then, we will focus on less extreme type of taxation plans.

Therefore, the analysis of an optimal tax rate should consider inequality and growth rate. Since all agents have linear utility index, the whole dynasty will be only worried about the expected return of their investments in the long run than to have levels of wealth that are bounded from below in a positive measure set of paths. Therefore, a more suitable welfare function for a taxation plan $\hat{\tau} = (\tau^I, \tau^B) \in [0,1]^2 \setminus \{(0,0)\} = T$ could be defined by

$$W\left(\left(U^i\right)_t, \left(c^i_{\tau,t}\right)_t, \left(b^i_{\tau,t}\right)_t\right) := \sum_{t=1}^{\infty} d^t \int\log U^i \left(c^i_{\tau,t}, b^i_{\tau,t}\right) di \quad (3.8)$$

where $d \in (0,1)$ is the discounted factor used by the social planner. In this case, the social welfare function does not have problems related with the convergence of the series when the economy has positive growth rate in the long run since $\log U^i \left(c^i_{\tau,t}, b^i_{\tau,t}\right)$ is at most linear.
Note that for any fixed marginal tax rate plan \( \tau \in [0,1]^2 \), the consumption and bequest in equilibrium will depend strongly on the marginal taxation rates \( \tau \). Therefore, to avoid any confusion, we will denote the consumption and bequest in equilibrium will be denoted using the tax rate used by the social planner, that is, \( \left( c^i_{\tau,t} \right)_i, \left( b^i_{\tau,t} \right)_i \).

Let us analyze the equilibrium with an initial endowment consistent with the invariant distribution, that is, \( \left( (1 - \delta)x^i_\tau (1 + g_\tau)^t \right)_i, \left( \delta x^i_\tau (1 + g_\tau)^t \right)_i = \left( \frac{(1 - \delta)(1 - \tau^B)}{1 - \delta \tau^B} \right) x^i_\tau (1 + g^\tau_t)^t \), the welfare function can be re written as

\[
W \left( \left( U^i \right)_i, \left( c^i_{\tau,t} \right)_i, \left( b^i_{\tau,t} \right)_i \right) : = \sum_{t=1}^{\infty} d^t \int \log U^i \left( c^i_{\tau,t}, b^i_{\tau,t} \right) di \\
= \sum_{t=1}^{\infty} d^t \int \log \left( \frac{\delta (1 - \tau^B)}{1 - \delta \tau^B} \left( x^i_\tau + x^i_{[i+1/2]} \right) (1 + g^\tau_t)^t \right) di \\
+ \sum_{t=1}^{\infty} d^t \int (1 - \delta) \log \left( \frac{(1 - \delta)(1 - \tau^B)}{1 - \delta \tau^B} \left( x^i_\tau + x^i_{[i+1/2]} \right) (1 + g^\tau_t)^t \right) di \\
= \sum_{t=1}^{\infty} d^t \int \log \left( \left( x^i_\tau + x^i_{[i+1/2]} \right) (1 + g^\tau_t)^t \right) di \\
+ \sum_{t=1}^{\infty} d^t \int \log \left( \frac{\delta (1 - \delta)(1 - \tau^B)}{1 - \delta \tau^B} \right) di \ (3.9)
\]

where \([\cdot] : \mathbb{R} \rightarrow [0,1] \) is the function that considers the non-integral part of a number, that is, \([10.45] = 0.45 \).

Using the properties of the logarithm, we have that we can separate the social welfare function \( W \) in three different parts, one that only depends on the invariant distribution implying that it is strictly increasing with \( \tau \), another component that only depends on the growth rate of the
economy which means that is strictly decreasing with $\tau$, and a component that depends on the bequest rate of the agents. This separation is given by

$$W\left((U^i)_{i,\tau,t}, (c^i_{\tau,t}), (b^i_{\tau,t})\right) = \frac{d}{\tau} \log \left(\frac{b^i_{\tau,t}}{1 + g_{\tau,t}}\right) + \sum_{t=1}^{\infty} d^i \int_0^1 \log \left(\frac{b^i_{\tau,t}}{1 + g_{\tau,t}}\right) \, dt +$$

$$\sum_{t=1}^{\infty} d^i \int_0^1 \log \left(\frac{\delta_{\tau,t}}{1 - \delta_{\tau,t}}\right) \, dt$$

(3.10)

Therefore, we have the following result.

**Proposition 2.** Under the hypotheses mentioned above, if the initial endowment distribution is consistent with the invariant concentration of wealth for the marginal tax rate $\tau \in \mathcal{T}$ chosen by the social planner, the social welfare function, $W$, in the equilibrium allocation can be written as

$$W\left((U^i)_{i,\tau,t}, (c^i_{\tau,t}), (b^i_{\tau,t})\right) = X(d, \tau) + G(d, \tau) + D(d),$$

where $X: [0,1] \times \mathcal{T} \to \mathbb{R}$, $G: [0,1] \times \mathcal{T} \to \mathbb{R}$, and $D: [0,1] \to \mathbb{R}$ are differentiable functions in $(0,1) \times (0,1)^3$, strictly increasing in the first component, and, in the second component, $G$ is strictly decreasing.

As it was mentioned above, there are two different things that are affecting the social welfare function, the invariant concentration of wealth ($X$) and the growth of the economy ($G$). Therefore, for a fixed discount rate for the social planner $d \in (0,1)$ and the bequest rate $\delta \in (0,1)$ there is a trade off between growth and inequality since growth rate tends to increase and inequality tends to decrease when taxes are diminished. However, each type of taxation has different implications on growth and inequality. In general, income taxes are the ones that reduce more inequality, and bequest taxes are the ones that generate higher consumption in the first dates. Therefore, it is not completely natural to determine each combination of taxes are better.

The characterization of the social welfare function it is extremely useful to understand the phenomena behind the marginal taxation rate, the inequality, the growth rate of the economy and
the relationship between the bequest rate and how the social planner discounts the future. As it can observed, inequality and growth are in opposite direction in the social welfare function. When you increase a marginal tax rate, growth and inequality will always reduce. However, their impact in the social welfare function is not comonotonic since the growth term of the social welfare function is comonotic with the growth of the economy due to the monotonicity of the logarithmic function. On the other hand, the social welfare function is anticomonotonic with respect to inequality since the logarithmic function is a strictly concave function that always decreases if you consider a more diverse type of distribution or variable. Consequently, the social planner must find a balance between low taxes to have large economic growth and high taxes to reduce inequality.

Since G and X are logarithm functions related to the growth and inequality, we conjecture that both functions are strictly concave functions.

**Conjecture 3.** G(d,·) and X(d,·) are strictly concave functions for every $d \in (0,1)$.

Therefore, we have that the social welfare function is a strictly concave function respect to $\tau$.

**Conjecture 4.** The function $W\left((U^t), (c_{t,t}^d), (b_{t,t}^d)\right)$ is strictly concave respect to $\tau$ for every $d \in (0,1)$, and, then, there is only one marginal tax rate, $\tau_d$, that maximizes the social welfare function.

---

This is the case of martingales in probability, that is, a process that has the same expected conditional value based on previous information, but it diversifies it values over time. In this case, if $(M_t,F_t)$ with $F_t \subset F_s$ for all $t \leq s$ is a martingale, that is, $M_t$ is a $F_t$-measurable function such that $E[M_{t+s}|F_t] = M_t$ almost certainly, then, $E[g(M_{t+s})|F_t] \leq g(M_t)$ almost certainly if $g$ is a concave function. Note that in this case, the invariant concentration of wealth is not a martingale. It is more as a analogy of what happens.
Intuitively, if a social planner is more worried about distant consumptions, it will give more attention to growth than inequality since the latter is maintained over time since we analyze the inequality of the invariant distribution, and the former is strongly related with distant consumption since by definition, an increment in the discount factor of the social planner, increases the weight of the future events and the only thing that changes over time in this equilibrium allocation is the growth rate of the economy. Therefore, it is natural to think that a social planner who decides to be more concerned about the future will choose a lower marginal tax rate than a social planner who do not.

**Conjecture 5.** In absence of bequest taxes, let be $\tau_d$ the optimal tax for a social planner given by $d \in (0,1)$. If $d_1 < d_2$ then $\tau_{d_1} > \tau_{d_2}$.

Mathematically, if the social planner is concerned more about distant consumptions, that is, he moves from $d_1$ to $d_2$ with $d_1 < d_2$, the value of $X$ and $X'$ increase only by small fraction (of the order of $\frac{1}{1-d}$). If the marginal tax rate is maintained. However, the value of $G$ increases a lot and, more importantly, its derivative due to the existence of a linear factor in the sum (of the order of at least of $\sum t^2d^{t-1}$). Therefore, we have that the optimal tax seems to be sensible for changes in the discount rate of the social planner.

The sensibility of the optimal taxes with the discount factor of the social planner does not imply that the optimal marginal tax rate of the economy must be such that the economy must have a positive growth rate at every possible discount factor $d \in (0,1)$. In the following subsection, we compute some numerical examples in which the optimal tax has a negative growth rate for a large class of discount factors. Nevertheless, it does not imply that, under our conditions,
the optimal tax for every discount factor \( d \in (0,1) \) is such that the growth rate of the economy is negative. Moreover, we have the following result.

**Proposition 6.** Under the conditions mentioned above, there exist \( d_1 \) and \( d_2 \) in \((0,1)\) such that \( d_1 < d_2 \), and

1. for every \( d < d_2 \), \( g_{\tau_d} < 0 \), and
2. for every \( d > d_2 \), \( g_{\tau_d} > 0 \).

Moreover, \( \lim_{d \to 0^+} g_{\tau_d} = \delta E[R_R] - 1 > 0 \), \( \lim_{d \to 0^+} x_{\tau_d} = 0 \) almost everywhere, and \( \lim_{d \to 1^-} g_{\tau_d} = \frac{\delta}{2} (E[R_R] + R_S) - 1 < 0 \), \( \lim_{d \to 1^-} x_{\tau_d} = 1 \) almost everywhere.

All the previous conjectures mentioned above have their economic and mathematical intuition. However, due to specific form of the invariant distribution for every possible marginal tax rate, it is not possible to analyze these properties analytically as it was made before. Moreover, we also found numerical evidence that support the conjectures and these results mentioned above. These numerical examples will be explain in the following subsection in which we will analyze more deeply the properties of the optimal marginal tax and the robustness of the model to analyze the trade off between inequality and growth.

Since consumption taxes increases growth rate but keeps inequality high, it is not clear if it is better to tax on consumption or income. However, bequest taxes seem to be worst since they induce the lowest growth rate with high inequality levels.
Proposition 7. For any marginal taxation plan $\hat{\tau} \in \mathcal{T}$ with positive bequest taxes, any taxation plan $\hat{\tau}' \in \mathcal{T}$ such that $\frac{\hat{\tau}' - \tau}{1-\hat{\tau}} > \frac{\delta}{1-\hat{\tau}}$ induces a strictly higher welfare compared with the welfare obtained by the taxation plan $\hat{\tau}$.

Finally, note that, the invariant distribution of income, the invariant growth rate is quite sensible to changes in the technology returns. If there is a global change and the technologies suffer modifications, the invariant distribution will change increasing or decreasing the amount of inequality if the expected value of the two technologies are more distant or of the technologies have larger idiosyncratic uncertainty. This phenomenon is consistent with several empirical works in which inequality or growth changes due to an industrial revolution or stagnation of the productivity.

2.6. Numerical Example

If we analyze the social welfare function in Example 1 with an initial distribution of wealth such that the aggregate wealth of each group is maintained over time, we found that the optimal tax rate for a discounted factor for the social planner $d = \delta$ is 80.3% that is a little bit more than critical tax in which the economy has a stationary aggregate wealth. Therefore, if the social planner discounts the future similarly as the agents do, the economy will collapse in the long run. The explanation for this phenomenon is that the social planner is not concerned about very distant low consumptions making that low rates of contraction of the economy could be optimal because reduces the unequally in the first dates. A mathematical explanation of this is that the discount factor of the social planner converges to zero faster than the collapse of the economy. Nevertheless, if the social planner discounts less the future, that is $d > \delta$, the optimal tax rate is lower than the critical rate. More precisely, if $d = \frac{1+\delta}{2}$, the optimal marginal tax rate
will be around 52.4%. Moreover, if \( d = \frac{2+\delta}{3} \), the optimal marginal tax rate will be approximately 36%. Figure 1 shows how the welfare function changes for all the possible tax rates for the three different discount factors mentioned above.

![Welfare vs taxes](image)

**Figure 1:** Welfare vs income taxes.

From the numerical examples showed above, we know that an increment in the discount factor of the social planner implies an increment in growth rate of the economy and a more unequal invariant distribution of wealth. However, in any case, the inequality does not necessarily increases over time since it will depend strongly on the initial distribution of wealth. Therefore if we start from a more unequal distribution than the invariant distribution for the chosen marginal tax rate, the inequality will decrease over time.

As we mentioned before, the social welfare function has a strictly concave behavior in the discount factor of the social planner. Moreover, the set of discount factors in which the economy will have a negative growth rate is considerably larger than the one in which the economy
has a positive growth rate. In fact, for every \( d \in (0, \alpha) \), the growth rate of the economy, \( g_{\tau \delta} \), is negative, and for every \( d \in (\alpha, 1) \), the growth rate of the economy, \( g_{\tau \delta} \), is positive with \( \alpha \sim 0.6 \).

\[
\begin{align*}
\text{Figure 2: Income inequality for different income taxation rates.}
\end{align*}
\]

As can be observed in Figure 1 and Figure 2, the level of inequality implemented by a social planner with a discount factor equal to the agent is quite low (2.85% of the total income for the top 1%, 20.7% for the top 10%, and 34.15% for the top 20%) if it is compared to very equal countries as Japan where the top 1% earns around 10% of the national income. For a social planner that with a discount factor equal to \( d = \frac{1+\delta}{2} \), the inequality is clearly larger with 9.69% for the top 1%, 34.85% for the top 10%, and 47.6% for the top 20% which seems to be similar to Japan. Finally, for a social planner with a discount factor equal to \( d = \frac{2+\delta}{3} \), the inequality is clearly larger than the other two cases with 21.44% for the top 1%, 48.37% for the top 10%, and 59.09% for the top 20% which seems to be similar to the US where the top 1% earns around 20% of the national income. All these results support the idea that the social planner should be more patience than the agents and that most of the governments are indeed more patience than their
population since most of them are generally quite worried about increasing the growth path of the economy than to almost eliminate any inequality.

**Figure 3:** Welfare vs changes on productivity of $R_R$.

In Figure 3, we can notice that changes in productivity of the risky technology, that in our case implies changes in the spread of the risky one, cause a change in the optimal taxation. In this case, a more productive risky technology leads to lower optimal taxation rate. An explanation to this is that a more productive economy due to its risky technology needs lower taxation rates compared to economies less productive to achieve its maximum welfare. Therefore, in this case, for the social planner is optimal to increase the inequality due to the increment of productivity. This is also observed in a large variety of economies around the world, one of these examples are the US and the largest economies in Europe, such as Germany, France and England. The former has a larger productivity than the latter and it has also a considerably larger income inequality.
2.7. Case with risk lovers instead of skilled agents

In the case in which $E[R_R] > R_S$, if the risk averters cannot use the risky technology, all the results are also valid.

2.7.1. Welfare analysis and optimal taxation from numerical examples

If we analyze the social welfare function in Example 1 with an initial distribution of wealth such that the aggregate wealth of each group is maintained over time, we found that the optimal tax rate for a discounted factor for the social planner $d = \delta$ is 76.7% that is a little bit more than critical tax in which the economy has a stationary aggregate wealth. Therefore, if the social planner discounts the future similarly as the agents do, the economy will collapse in the long-run. The explanation for this phenomena is that the social planner is not concerned about very distant low consumptions making that low rates of contraction of the economy could be optimal because reduces the unequally in the first dates. A mathematical explanation of this is that the discount factor of the social planner converges to zero faster than the collapse of the economy. Nevertheless, if the social planner discounts less the future, that is $d > \delta$, the optimal tax rate is lower than the critical rate. More precisely, if $d = \frac{1+\delta}{2}$, the optimal marginal tax rate will be around 49.3%. Moreover, if $d = \frac{2+\delta}{3}$, the optimal marginal tax rate will be approximately 34.8%. Figure 4 shows how the welfare function changes for all the possible tax rates for the three different discount factors mentioned above.
Figure 4: Social welfare function vs income taxes for different values of \( d = \delta, \frac{\delta+1}{2}, \frac{\delta+2}{3}. \)

As we mentioned before, the social welfare function has a strictly concave behavior in the discount factor of the social planner as in the previous case. For every \( d \in (0, \alpha) \), the growth rate of the economy, \( g_{\tau_8} \) is negative, and for every \( d \in (\alpha, 1) \), the growth rate of the economy, \( g_{\tau_8} \) is positive with \( \alpha \sim 0.55 \). Therefore, in this case, to have a similar optimal taxation rate as in the previous case, the social planner must be more impatient.
Figure 5: Changes in welfare vs different income taxes for different productivities of $R_R$.

In Figure 5, we can notice that changes in productivity of the risky technology, that in our case implies changes in the spread of the risky one, cause a change in the optimal taxation like in the case with segmentation. The only difference is that the optimal taxation rate is always lower in this case, implying more unequal optimal distribution for the solution of the social planner. An explanation for this phenomenon is that the large desire of the risk lovers for extreme consumption reduce makes that the social planner consider better for the society a distribution of wealth more unequal. However, the behavior of the risk lovers does not affect much since must of the taxes only varies between three and four points (from 52.4% to 49.3%), which is a consequence of the fact that the social planner knows that the are risk lovers in the economy and considers their utility function distorted by a logarithmic function to avoid non convergence of the welfare function. Therefore, the social planner is only partially influenced by the desire of the risk lovers to specialize.
To compare taxation on income, consumption, and bequest for different discount factors of the social planner, \( d = \delta, \delta, \delta, \delta, \frac{\delta+1}{2}, \delta, \frac{\delta+2}{3}, \frac{\delta+3}{4}, \frac{\delta+4}{5} \), we parametrized the marginal tax rate as follows \( \bar{\tau}_c = \bar{\tau}_b = \frac{\tau}{1-\tau} \) where \( \tau \in [0,1] \) is the marginal tax rate for income taxes, where \( \bar{\tau}_c \) is the marginal tax rate for consumption taxes, and where \( \bar{\tau}_b \) is the marginal tax rate for bequest taxes.

\[
R_S = 1.2, R_R = 4.2
\]

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<th>( \delta = 0.25, d = \delta )</th>
<th>( \delta = 0.25, d = 2\delta )</th>
<th>( \delta = 0.25, d = 1.9\delta )</th>
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<tr>
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<td>0.97</td>
<td>0.334</td>
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<td>10.288</td>
<td>11.7561</td>
</tr>
<tr>
<td>Growth</td>
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<td>1.5009</td>
<td>1.4725</td>
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<th>( \delta = 0.5, d )</th>
<th>( \delta = 0.5, d )</th>
<th>( \delta = 0.5, d = 1.9\delta )</th>
</tr>
</thead>
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<td>0.999</td>
<td>0.954</td>
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<tr>
<td>Optimal beq. tax</td>
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<td>0.99</td>
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<td>9.2533</td>
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<tr>
<td>Growth</td>
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<td>Growth</td>
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**Table 1**: Optimal taxation plans for different bequest rates, \( \delta \), and discount factor of the social planner, \( d \).
We observe that for most of the discount factor of the social planner the best type of taxation is the consumption taxes. More specifically, for a discount factor given by \( d = \frac{\delta}{2}, \delta, \frac{\delta + 1}{2}, \frac{\delta + 2}{3}, \frac{\delta + 3}{4}, \frac{\delta + 4}{5} \). Therefore, it seems that very impatient social planner prefers, \( d = \frac{\delta}{6}, \frac{\delta}{4} \), income taxes instead of consumption ones. However, for social planners that are from moderately impatient to patient, consumption taxes are always preferred to obtain a welfare optimum. Additionally, the optimal tax rate, in the latter case is higher than 95% for all the marginal taxation rate analyzed.

Note that the optimal taxation rate cannot be equal to 1 since this marginal tax rate has no incentives for consumption even for a government being patient. Note also that income taxes increase the value of the welfare function for low and extremely high levels of taxation, and consumption taxes increases the value of the welfare function for middle and relatively high levels of taxation for \( d = \frac{\delta}{2}, \delta, \frac{\delta + 2}{3} \).

In all cases, taxes on bequest seems to be the worst type of taxation in terms of welfare. There are several things that justify this phenomenon. First, the growth rate is considerably lower compared to the other cases. Additionally, even when consumption increases in the short run, the increment is not considerably high compared to the losses obtained by the lower growth rate since the difference among the growth rates are quite expressive. Note also that the increment of the consumption does not necessarily increase the utility of the agents in the first periods since they also care about the bequest left to the next generation. Finally, bequest and consumption taxes seem to be bad tools to reduce inequality in the economy, see Table 1.
Additionally, for low levels of taxation, income or consumption taxes might be preferred depending on \( d \). In most of the cases, taxing consumption seems to be preferred, but, for \( d = \frac{\delta + 1}{2} \), taxing income improves the welfare function for low levels of taxation.

3. Model with risk loving, production, and effort cost

Let us define a model based on Section 3 only with income taxes, and instead of skilled and unskilled type of agents, we will have risk averters and risk lovers as in Section 2 in which both agents have access to the technology, but each agent that decides to invest in the risky technology will have an effort cost, \( L \geq 0 \), a fixed effort cost that the agent must take to have a positive probability of winning the highest return. In absence of this cost, the agent will have a null return in the next period. Therefore, if an agent decides to invest in the risky one, it is always optimal to pay the effort cost.

Note that most of the properties established in Section 3 such as the invariant distribution are still true if \( E[R_R] < R_S \) if the growth rate in the long run is positive. However, if the growth rate in the long run is negative and \( \tau > 0.5 \); all risk lovers will invest in the safe one unless the effort cost is zero, \( L = 0 \).

**Proposition 8.** Under \( E[R_R] < R_S < \frac{1}{\delta} \) and \( \frac{R_R^2}{\delta} > \frac{2}{\delta} \), we have:

1. If \( \tau > 0.5 \), any initial distribution with large enough aggregate endowment converges to an constant invariant distribution in which all the agent invest in the safe technology.

2. If \( \tau \leq 0.5 \) and \( R_S < \frac{1}{\delta} \), there is a constant invariant distribution that can only be attained if the initial endowment of the risk lovers and the aggregate endowment are quite low.
Let us suppose that each agent has access to both technologies. However, they cannot have access to both at the same time. That is, if an agent decides to invest in one of the technologies, she cannot invest in the other technology. This can be justified by the fact that each agent has a limited amount of different things that the agent can do at the same time.

For the risk lovers, no matter how much the agent is being taxed, if the risky return is at least as productive as the safe one, she will invest in the risky one for any marginal taxation rate if \( L = 0 \). If \( L > 0 \), the risk lover invests in the safe one if his initial wealth is quite low (if \( L = 0.04879 \), the indifference level is around 0.08, if \( L = 0.01 \), it is around 0.034). However, if there is a positive growth in the economy, in the long run, all risk lovers will invest in the risky one since the poorest risk lovers it is the mean wealth multiplied by the marginal tax rate, an amount that is certainly larger than the indifference threshold if the economy increases in the long run.

From now on, we will assume that \( E[R_R] > \frac{1}{\delta} > \frac{E[R_R]+R_S}{2} > R_S \).

For the risk averters, the marginal taxation rate and the level of wealth of the agent will affect her optimal solution. More precisely, we have that:

**Proposition 9.** Given a marginal taxation rate \( \tau \in (0,1) \), there is a constant

\[
\alpha_{\tau}^* = \frac{\tau (R_R - 2R_S \exp(L)) \pm \sqrt{(R_R - 2R_S \exp(L))^2 - 4(\exp(L)-1)R_S^2}}{R_S^2(1-\tau)}
\]  

(4.1)
such that:

1. if \( w_{t}^{a_i} > \alpha_{\tau}^* \bar{w}_{t+1} \) or \( w_{t}^{a_i} < \alpha_{\tau}^* \bar{w}_{t+1} \), the agent \( a_i \) invests in the safe technology at date \( t + 1 \),

2. if \( w_{t}^{a_i} \in (\alpha_{\tau}^* \bar{w}_{t+1}, \alpha_{\tau}^* \bar{w}_{t+1}) \), the agent \( a_i \) invests in the risky technology at date \( t + 1 \), and

28
3. if \( w^a_t = \alpha^a_{\tau+1} \bar{w}_{t+1} \), the agent \( a_i \) is indifferent between both type of investments at date \( t + 1 \).

For the risk lovers, we have a similar proposition.

**Proposition 10.** Given a marginal taxation rate \( \tau \in (0,1) \) and the aggregate wealth in \( t + 1 \), \( \bar{w}_{t+1} \), there is a constant

\[
\beta^*_{t+1, \tau} = \frac{-(R_R - 2R_S)\tau + \sqrt{(R_R - 2R_S)^2 + \frac{L}{\bar{w}_{t+1}}(R_R^2 - 2R^2_S)}}{(R^2_R - 2R^2_S)(1-\tau)} \tag{4.2}
\]

such that:

1. if \( w^l_t < \beta^*_{t+1, \tau} \bar{w}_{t+1} \), the agent \( l_i \) invests in the safe technology at date \( t + 1 \),
2. if \( w^l_t > \beta^*_{t+1, \tau} \bar{w}_{t+1} \), the agent \( l_i \) invests in the risky technology at date \( t + 1 \), and
3. if \( w^l_t = \beta^*_{t+1, \tau} \bar{w}_{t+1} \), the agent \( l_i \) is indifferent between both type of investments at date \( t + 1 \).

Therefore, taxation has a positive impact on growth since it makes that more agents are been taxed. And since, with a very low probability, any risk averter could have a wealth high enough to avoid the risky technology, we have the following result:

**Proposition 11.**

1. If \( \tau \geq \frac{R_R}{R_S} + 1 - \frac{R^2_R}{2R^2_S} \), any increment of the marginal tax rate increases the growth rate of the economy. The proportion of risk averters that invest in the safe technology is a decreasing positive function of the marginal tax that converges to 0 when \( \tau \) converges to one.
2. If \( \tau < \frac{R_R}{R_S} + 1 - \frac{R_R^2}{2R_S^2} \), all the risk averters invest in the safe technology. Therefore, the economy converges to an invariant distribution as in Section 3.

Note that if \( E[R_R] > R_S \), \( \frac{R_R}{R_S} + 1 - \frac{R_R^2}{2R_S^2} < 1 \) implying that for any economy, there is a positive marginal tax rate \( \tau < 1 \) such that, at any give date \( t \), a proportion of the risk averters invest in the risky technology. However, if

\[
R_R \geq (1 + \sqrt{3})R_S
\]  \hspace{1cm} (4.3)

\( \frac{R_R}{R_S} + 1 - \frac{R_R^2}{2R_S^2} \leq 0 \) implying that there are risk averters investing in the risky technology at any period \( t \).

However, if the economy collapses in the long run, all risk averters invest in the safe investment eventually because \( \beta_{t,\tau}^* \) goes to infinity. On the other hand, the risk averters might not invest in the same form since the satisfy Inada condition.

3.1. Invariant Distribution

Due to Proposition 9 and Proposition 11, we can ensure that a process similar to the one made in the other models can be done in this case since the wealth of each risk averter (more precisely, almost every risk averter) can be computed by a recursive process if we suppose that there is an invariant concentration of wealth.

Numerically, it requires only to compute a preliminary proportion of agents that invest in each technology in each period \( t \) by using an increasing function \( \alpha^* \) defined above. Then, we
compute \( \alpha_t \frac{w_{t+1}}{w_t} \) and the distribution of wealth step by step. Now, we restart the process with the proportion of investment induced by the distribution that we have just found.

By doing this, we can find the invariant distribution of wealth invested in each technology, and, as a consequence, the invariant growth rate of the economy, which are the only things that we need to know the invariant concentration of wealth among the agents.

**Lemma 13.** Under a fixed and positive marginal tax rates and a wealth dynamic process such that the growth rate \( g_{\tau,t} \) satisfies that \( g_{\tau,t} \to g_{\tau} \) when \( t \to \infty \), the wealth concentration of wealth converges to an invariant distribution.

This result implies the existence of the invariant concentration of wealth and it also suggests that if the concentration is considerably close to the invariant one, it converges in the long run to the invariant one which is what we found in the numerical examples that will be afterwards.

**Proposition 14.** Under a fixed and positive marginal tax rates, there is an invariant concentration of wealth.

3.2. **Numerical examples**

Based on the numerical examples defined before, we consider \( R_R = 4.5, R_S = 1.6, \delta = 0.5, L = 0.04879 \). We start with a distribution constant distribution of wealth among the agents.

In this case, increments on taxation generate different effects on growth depending on the marginal tax rate that we start on. For very low levels of marginal taxation rate, increment in
taxation rates might decrease the growth rate due to the transfers from the risk lovers to risk averters. The former are investing completely in the risky and more productive type of investment, and the latter are investing part in the risky and part in the safe investment. Therefore, the increment in the marginal taxation rate implies less investments in the risky and more investments in the safe one.

For marginal tax rates between 0.3 and 0.5, the growth rate increases when the marginal tax rate increases. In this case, the risk averters are investing considerably more in the risky than in the safe one since they need a larger number of successful periods investing in the risky investment to reach the indifference threshold, \( \alpha^*_r, \tau \). Intuitively, the threshold being attainable for the risk averters means that the insurance effect caused by taxes is observed, that is, some risk averters decide to invest in a risky and more productive type of investment because the government ensures that he/she will receive minimum level of wealth if his/her investment does not give any return. In this case, given an increment of the marginal tax rate, the proportion of agents that decide to invest in the risky one compensates the transfers of wealth from the risk lovers to the risk averters who decide to invest in the safe one.

For marginal taxation rate slightly above 0.5, there is no fat tails in the economy. Moreover, there is always an upper bound depending on the aggregate wealth that is unattainable for any agent, including the ones who always invest in the risky investment and succeed. Therefore, there is \( \hat{\tau} \in (0.5, 0.6) \) the lowest positive marginal taxation rate such that the growth rate in the long run is maximum, that is, \( g_{\tau,t} \to \frac{R\delta}{2} - 1 = 0.125 = 12.5 \) when \( t \to \infty \).

When the marginal taxation rate is larger than 0.75, the poorest risk averter satisfies \( w_{t}^{\alpha} = \tau \overline{w}_t \sim \alpha^*_{r,\tau} \overline{w}_{t+1} \) therefore, an increment on the marginal taxation rate will imply that these agents decide to invest in the safe and less productive investment implying reductions in
the growth rate. This phenomenon continues until the growth rate is positive. Once the marginal tax rate is such that the growth rate in the long run is slightly negative, we are under the conditions in which the poorest risk lovers start to invest in the safe one reducing even more the growth rate which makes that more agents (risk averters and lovers) witch to the safe one. Then, in the long run, almost all agents invest in the safe one instead of the risky one implying that the economy has the lowest growth rate possible, that is, $g_{\tau,t} \to R_S \delta - 1 = -0.2 = -20\%$ when $t \to \infty$.

**Figure 6:** Growth rate over time for different income tax rates.

Note that, based on Figure 6, the economy converges to a constant growth rate in the long run for all the marginal tax rates analyzed. Moreover, we observed that for almost all the marginal tax rates analyzed, the convergence of the growth rate holds. However, it might be a problem for marginal tax rates around 0.83 since the induced growth rate oscillates between positive and negatives growth rates. We note that this problems might be convergence problems or numerical problems induced by computational errors.
From Figure 6, we noticed that for low marginal tax rate, all risk averters invest in the safe one implying a constant distribution. When \( \tau = 0.3, 0.4, \) and 0.5 the risk averters invest in the risky one when they are poor and in the safe one when they are above the threshold \( \alpha^*_t, \tau. \) The biggest difference between these two distributions is that the threshold is considerably higher when \( \tau = 0.4 \) (moreover when \( \tau = 0.5 \)) implying that a larger proportion of wealth is being invested in the risky in this case which causes the phenomenon mentioned before, an increment on the growth rate. This happens because this effect overcome the transfers made from the risk lovers to the risk averters that invest in the safe one- a proportion of agents that is small in this case and decrease every time that you increase the marginal tax rate.

When \( \tau = 0.6 \) and 0.7, all risk averters invest in the risky one all the time since there is no over-accumulation of wealth (fat tails) in this case. Therefore, a risk averter that is infinitely successful by investing in the risky one has a wealth in period \( t \) bounded by \( 2.6\overline{w}_{t+1} \) for \( \tau = 0.6, \) and \( 2.6\overline{w}_{t+1} \) for \( \tau = 0.7. \)

When \( \tau = 0.8, \) the poorest risk averters, the ones that their parents only received the transfers in the previous period, the invest in the safe one. However, once they invest in the safe one, the transfers made by taxes increases the wealth of their successors in such a way that they decide to invest in the risky one generating the invariant distribution observed. This phenomenon continues making that a larger proportion of risk averters decide to invest in the safe one instead of the risky one. However, once the growth rate is negative for a long number of periods, the risk lovers will switch too implying that the invariant distribution is even lower. In this case (\( \tau = 0.9, \)) all risk lovers and all risk averters will eventually invest only in the safe one implying that the invariant distribution is constant.
3.3. Extension to a model with capital, labor, and innovation

Using the model exposed before, we can extend it to a capital, labor, and model as follows. $K_t^i = (R_R \theta_R^i + R_S \theta_S^i) \delta b_t^i$ for $t \geq 1$ and $K_0^i = 1$ is the amount of capital of the agent $i$’s firm that depreciates completely, $\theta_t^i = \theta_{t-1}^i \left( \frac{K_t^i}{K_{t-1}^i} \right)^{1-\alpha}$ with $\alpha \in (0,1)$ for $t \geq 1$ and $\theta_0^i = K_0^i$ is the innovation factor, $L_t^i \in [0,1]$ without any utility for leisure which implies that $L_t^i = 1$ for all $t$, $r_t$ is the price of the capital at date $t$, and $w_t$ is the salary. The technology of the firm $i$ at $t$ is given by

$$y_t^i = \theta_t^i \left( K_t^i \right)^\alpha \left( L_t^i \right)^{1-\alpha}$$

The consumer constraint is given by

$$c_{t+1} + b_{t+1} \leq w_t^i L_t^i + r_t^i K_t^i$$
In equilibrium, since the firm has constant returns to scale, \( w_t^i = (1 - \alpha)\theta_t^i (K_t^i)^{\alpha} (L_t^i)^{-\alpha} \), 
\( w_t^i = \alpha \theta_t^i (K_t^i)^{\alpha - 1} (L_t^i)^{1-\alpha} \). Therefore, in equilibrium, the consumer problem of each agent is defined as before, and all the results related to the dynamics of the wealth are still valid.

4. Conclusions

We developed an overlapping generation model with endogenous growth rate and heterogeneous technology productions. In this model, taxes and redistribution has a negative impact on growth if the more productive technologies involve larger amount of idiosyncratic risk. Moreover, in absence of taxes, the most productive technologies will dominate the economy in the long run and the long run inequality will depend mainly in the risk that it involves. In the presence of taxes, taxes ensure the existence of an invariant distribution of wealth among the agents and also an invariant growth rate of the economy. We also showed that there is no poverty trap among the agents with the most productive. Among the agent that do not have access to the most productive technologies, their wealth may not reach the top in any future date.

Redistribute taxes has a negative effect in growth rate and inequality. To establish an optimal taxation, we introduced a central planner that considers the consumption of the agents at equilibrium. We showed that the social welfare function can be written as the sum of three independent functions, one depending on growth, one depending on inequality, and one depending on the difference of the discount factors of the agent and the social planner. The first function is comonotonic with the growth rate of the economy, implying that in presence of low taxes might be optimal. The second one is anticomonotonic with the inequality of the invariant distribution which implies that high taxes might be optimal in some cases.
We also found that, for a fixed discount rate for every agent in the economy, the optimal taxation is strictly decreasing on how the social planner discounts the future. Moreover, the optimal tax will be such that the invariant wealth distribution tends to an equal one if the social planner strongly discounts the future, and, on the other hand, the optimal tax is zero when the social planner does not discount the future at all. The intuition behind these results is that, if a social planner discounts the future strongly, the weight of distant dates becomes almost irrelevant, and analogously with the growth rate of the economy. Therefore, the social welfare function will be dominated by the inequality effect. However, if a social planner almost does not discount the future, the weight of future consumption will dominate the inequality effect even when both effects are increasing. Additionally, our model suggests that changes in the tax policy may be based on changes on the form of the social planner discounts the future compared to how the other agents do so.

5. Bibliography


Appendix A: Case with risk loving and without production

Let us define an overlapping generation model with bequest and a finite number of successor for each state based on Kehoe and Levine (1984). Let us consider an economy with two events in each state and a continuum of agents characterized by two different types of behaviors (risk lovers and risk averse) each of them with measure 1. Additionally, let us consider two assets in positive net supply with returns \((\overline{R}, \overline{R})\) and \((\underline{R}, \overline{R})\) with \(\overline{R} \geq \underline{R} > 0\) in each state with \(t \geq 1\). There is no consumption in \(t = 0\) and every agent has an initial amount of the assets \((\hat{\theta}_{0,1}, \hat{\theta}_{0,2}) \geq 0\), and there is a consumption good at every state \(s = (\eta_1, \eta_2, \ldots, \eta_t)\) with date \(t \geq 1\) where \(\eta_k \in \{1, 2\}\) for all \(k \in \{1, 2, \ldots, t\}\). In the first period of life, an agent receives a bequest from his predecessor \(b_i \geq 0\) and an endowment \(\omega_i \geq 0\), and he decides on purchases of assets for any state \(s\) with \(t \geq 1\). And in second period of life, he decides on consumption and the bequest that he leaves to his descendant.

At each state \(s\), there is a wealth tax \(\tau_s^+ (\cdot)\) that will be imposed on any agent above some threshold \(\overline{W}_s\), and there is also a wealth subsidy \(\tau_s^- (\cdot)\) that will be given for any agent below some threshold \(\underline{W}_s\). Therefore, the tax and the subsidy will be summarized by \(\tau_s (\cdot) = \tau_s^+(\cdot) - \tau_s^- (\cdot)\). From now on, \(\tau_s^+\) is a tax with an exogenous constant marginal rate \((\tau_s^+ \in [0, 1])\) over the wealth above the threshold \(\overline{W}_s\), \(\tau_s^-\) is a subsidy with a constant marginal rate \((\tau_s^-)\) over the wealth below the benchmark \(\underline{W}_s\). Therefore, \(\tau_s (x) = \tau_s^+(x - \overline{W}_s)^+ - \tau_s^- (\underline{W}_s - x)^+\). \(\tau_s^-\) is endogenously determined in equilibrium to ensure a balanced government budget.

For a risk averter, \(\alpha_i \in [0, 1]\), and a state \(s\) for some \(t \geq 1\) his maximization problem is
\[
\text{max } U^a(c, b) = \frac{1}{2}((1-\delta) \log c_1 + \delta \log b_1) + \frac{1}{2}((1-\delta) \log c_2 + \delta \log b_2)
\]
\[
\text{s.t. } q_{s,1}^i \theta_1 + q_{s,2}^i \theta_2 \leq \omega_s^i + b_s^i
\]
\[
c_1 + b_1 \leq \sum_{j \in \{1,2\}} (R_{(s,1),j} + q_{(s,1),j}) \theta_j - \tau \left( \sum_{j \in \{1,2\}} (R_{1,j} + q_{(0,1),j}) \theta_j \right)
\]
\[
c_2 + b_2 \leq \sum_{j \in \{1,2\}} (R_{(s,2),j} + q_{(s,2),j}) \theta_j - \tau \left( \sum_{j \in \{1,2\}} (R_{2,j} + q_{(0,2),j}) \theta_j \right)
\]
\[
0 \leq c_1, b_1, c_2, b_2
\]

where \( q >> 0 \), \( R_{(s,j),1} = \bar{R} = R_{(s,j),2} = R = R_{(s,1),2} \), \( \omega_s^i \) is the initial endowment in the state \( s \) and \( \delta \in [0,1] \).

For a risk lover, \( l \in [0,1] \), agent one will maximize the same utility function with a different constraints:
\[
q_{0,1}^i \theta_1 + q_{0,2}^i \theta_2 \leq q_{0,1}^i \hat{\theta}_{0,1}^i + q_{0,2}^i \hat{\theta}_{0,2}^i
\]
\[
c_1 + b_1 \leq \sum_{j \in \{1,2\}} (R_{(0,0),j} + q_{(0,0),j}) \theta_j - \tau \left( \sum_{j \in \{1,2\}} (R_{1,j} + q_{(0,1),j}) \theta_j \right)
\]
\[
c_2 + b_2 \leq \sum_{j \in \{1,2\}} (R_{(0,2),j} + q_{(0,2),j}) \theta_j - \tau \left( \sum_{j \in \{1,2\}} (R_{2,j} + q_{(0,2),j}) \theta_j \right)
\]
\[
0 \leq c_1, b_1, c_2, b_2
\]

where \( \hat{\theta}_0 \) satisfies
\[
\int_{i \in [0,1]} \hat{\theta}_{0,j}^i di + \int_{i \in [0,1]} \hat{\theta}_{0,j}^i di = 1 \forall j = 1, 2
\]

Notice that the risk averter will have similar consumption among the events. In fact, they will distribute the same proportion of wealth in each event. For the risk lovers, their behavior is extremely different, they will specialize as much as possible in one event depending on the price of the two assets available in the economy. If the prices are such that it is cheaper to invest in event 1 instead of event 2, all the agents will specialize in event 1. However, if the cost to invest is the same, a positive measure of the agents can specialize in each of the events. Notice that, in the model, the agent separate part of their wealth to the his predecessor as bequest. For both type of agents, the optimal conditions ensure that they will spend the same proportion of the available wealth at any event, in this case this proportion is \( (\delta) \).
A.1. Equilibrium and optimal allocation

Now, let us define the equilibrium for the model.

**Definition A.1.** For each state $s$, we require

$$
\int \theta^i_{s,j} di = 1, 2
$$

$$
\int c^i_{s,k} di = R + \bar{R} + \bar{\omega}_{(s,k)} \forall k = 1, 2,
$$

$$
\int \tau_s \left( \sum_{j=1,2} (R_{i,j} + q_{(s,j),i} \theta^i_{s,j}) \right) = 0,
$$

where

1. $(c^i_s, b^i_s, \theta^i_s)$ is the optimal solution for the agent $i$ and
2. $\bar{\omega}_{(s,k)} = \int \omega_{(s,k)} di$ is the aggregate endowment in state $(s,k)$.

Notice that bequest could change the wealth distribution in a specific state even if the endowment distribution is uniform among the agents. However, it does not mean that, in some subtrees, there will be an over accumulation of endowment since the bequest is not a transfer of good from different generation, it is a transfer nominal transfer that depends on the consumption in the economy that does not depend on bequest in the aggregate level, see Definition 1. What bequest could cause is a change on the wealth distribution increasing of reducing the inequality.

Before we characterize the equilibrium allocation, let us notice that, since there is no short-sales constraints, there is no possibility of arbitrage in an equilibrium allocation which ensures that the budget constraint can be rewritten using Arrow security assets in each state $s$; therefore, there is a price vector $\pi^{(s,1)}, \pi^{(s,2)} \in \Delta^1$ such that the budget constraint is equivalent to

$$
\pi^{(s,1)} (c_1 + b_1) + \pi^{(s,2)} (c_2 + b_2) \leq \omega^i_s + b^i_s
$$

$$
c_1, b_1, c_2, b_2 \geq 0
$$

for a state of length $t \geq 1$, and

$$
\pi^{(0,1)} (c_1 + b_1) + \pi^{(0,2)} (c_2 + b_2) \leq q_{0,1} \theta^i_{0,1} + q_{0,2} \theta^i_{0,2}
$$

$$
c_1, b_1, c_2, b_2 \geq 0,
$$

for $t = 0$. 

42
Note that $q_{s,t} = \pi_{(s,t)}(\bar{R} + q_{(s,t),1}) + \pi_{(s,t)}(\bar{R} + q_{(s,t),2})$ and $q_{s,2} = \pi_{(s,1)}(\bar{R} + q_{(s,1),2}) + \pi_{(s,2)}(\bar{R} + q_{(s,2),2})$.

For a risk averter,
\[
\frac{\delta}{1-\delta} c_{(s,t)}^{\alpha_i} = b_{(s,t)}^{\alpha_i} \quad \text{and} \quad \frac{\delta}{1-\delta} c_{(s,2)}^{\alpha_i} = b_{(s,2)}^{\alpha_i}
\]
for each state $s$. Therefore, for $t = 0$,
\[
c_{(0,k)}^{\alpha_i} = \frac{1-\delta}{2\pi_{(0,k)}} \left( q_{0,k}^{\alpha_i} + q_{0,2}^{\alpha_i} \right) \text{for } k = 1,2,
\]
for $t \geq 1$,
\[
c_{(s,k)}^{\alpha_i} = \frac{1-\delta}{2\pi_{(s,k)}} (\omega_{s}^{\alpha_i} + b_{s}^{\alpha_i}) \text{ for } k = 1,2.
\]

Now, each risk lover specializes as much as possible in one event and distributes his consumption as in equation (2.1) since both agents distribute their wealth between consumption and bequest in the same form. Therefore, for $t = 0$, if $q_{0,1} < q_{0,2}, \pi_{(0,1)} < \pi_{(0,2)}$, all agents will consume only on the first event and therefore
\[
c_{(0,1)}^{\alpha_i} = \frac{1-\delta}{\pi_{(0,1)}} \left( q_{0,1}^{\alpha_i} + q_{0,2}^{\alpha_i} \right) \text{ and } c_{0,2}^{\alpha_i} = 0,
\]
for $t \geq 1$ under $q_{s,1} < q_{s,2}, \pi_{(s,1)} < \pi_{(s,2)}$, we have
\[
c_{(s,1)}^{\alpha_i} = \frac{1-\delta}{\pi_{(s,1)}} (\omega_{s}^{\alpha_i} + b_{s}^{\alpha_i}) \text{ and } c_{(s,2)}^{\alpha_i} = 0,
\]
and analogously to the case in which $q_{s,1} > q_{s,2}$, for any state $s$. For the case in which $q_{s,t} = q_{s,2}$, $\pi_{(s,1)} = \pi_{(s,2)}$ and each risk lover is indifferent to consume on event 1 or in event 2 therefore, there could be a set of agents with positive measure\(^3\) that consumes on the first event and also another set with positive measure that consumes on the second event.

For $t = 0$, if all risk lovers specialize in event one ($\pi_{(0,1)} \leq \pi_{(0,2)}$),

---

\(^3\) Respect to the Lebesgue measure.
\[
\bar{\omega}_{(0,2)} + 2\frac{1-\delta}{\pi_{(0,1)}} \int_0^1 \left( q_{(0,1)} \hat{\omega}^{(0,1)} + q_{(0,2)} \hat{\omega}^{(0,2)} \right) di \leq \bar{\omega}_{(0,1)},
\]
(2.2)

Or if all risk lovers specialize in event two \((\pi_{(0,1)} \geq \pi_{(0,2)})\), we have
\[
\bar{\omega}_{(0,1)} + 2\frac{1-\delta}{\pi_{(0,2)}} \int_0^1 \left( q_{(0,1)} \hat{\omega}^{(0,1)} + q_{(0,2)} \hat{\omega}^{(0,2)} \right) di \leq \bar{\omega}_{(0,2)},
\]
(2.3)

If equations (2.2) and (2.3) are not hold, in equilibrium, \(\pi_{(0,1)} = \pi_{(0,2)}\) implying that there is a set of agents with positive measure that specializes in event 1 and another set with positive measure that specializes in event 2. And for a state \(s\), if all risk lovers specialize in event one \((\pi_{(s,1)} \leq \pi_{(s,2)})\), it will imply \((\pi_{(0,1)} \leq \pi_{(0,2)})\)
\[
\bar{\omega}_{(s,2)} + 2\frac{1-\delta}{\pi_{(s,1)}} \int_0^1 \left( \bar{\omega}^l_s + \bar{b}^l_s \right) di \leq \bar{\omega}_{(s,1)},
\]
(2.4)

where \(\bar{\omega}^l_s = \int_0^1 \omega^l_s di\) is the aggregate endowment of the risk lovers in the state \(s\) and \(\bar{b}^l_s = \int_0^1 b^l_s di\) is the aggregate bequest received by the risk lovers in state \(s\). Or in event two, we have
\[
\bar{\omega}_{(s,1)} + 2\frac{1-\delta}{\pi_{(s,2)}} \int_0^1 \left( \bar{\omega}^l_s + \bar{b}^l_s \right) di \leq \bar{\omega}_{(s,2)},
\]
(2.4)

If equations (2.4) and (2.5) are not hold, in equilibrium, \(\pi_{(s,1)} = \pi_{(s,2)}\) implying that there is a set of agents with positive measure that specializes in event 1 and another set with positive measure that specializes in event 2.

2.1. Analysis of equilibria and invariant distribution of wealth

Before the analysis of the invariant distribution of wealth along the time, let us notice some particularities of the model. The first and most important is that the equilibrium is no unique in most of the cases, at each state there is an infinite number of distributions in which the risk lovers could specialize if \(\pi_{(s,1)} = \pi_{(s,2)}\).
**Proposition A.2.** If there is an equilibrium for the economy in which there is at least one state \( S \) in which we have:

1. \( \pi_{(s,1)} = \pi_{(s,2)} \) and
2. Equations (2.4) and (2.5) are not satisfied.

Then, there is an infinite number of equilibria for the economy.

The intuition behind this result is that if the conditions mentioned in Proposition 1 are satisfied, there is a mass with positive weight of risk lovers that specialize in event 1 in the successor of \( s \) and there is also a positive mass of risk lovers that specialize in event 2. And since the only property of this sets is that the aggregate consumption must satisfy the market clearing, see Definition 1, the multiplicity of equilibria is a result of the large class of sets of agents that satisfy the property.

If these conditions are not satisfied for all states, there is a uniqueness of the equilibrium. However, in this equilibrium all risk lovers specialize in one event implying that Equation 2.4 or 2.5 is satisfied\(^4\), implying the existence of unbounded aggregate wealth in the long run in all the paths of the tree with probability 1.

**Proposition A.3.** If the conditions of Proposition 1 are not satisfied in any state of the tree, the set of paths \( \eta = (\eta_1, \eta_2, \ldots) \) in which there is a subsequence \( \{s_k\} \) such that \( \bar{\omega}_{n_k} \to \infty \) when \( k \to \infty \) has probability one.

**Proof.** Since the aggregate endowment for the Risk Lovers is positive in each state, Equations 2.4 and 2.5 ensure that if the agents specialize in an event, the aggregate risk must be larger than the wealth of the risk lovers. Then, the 0-1 Kolmogorov law ensures that with probability one the risk lovers will rich enough to make that aggregate endowment be extremely large infinitely many times.

As a consequence of Proposition 2, the aggregate uncertainty is also unbounded in the long run.

\(^4\) Note that this two conditions are mutually exclusive.
Therefore, the analysis of any invariant distribution can not be done in the cases in which there is uniqueness of equilibrium.

A.2. Invariant distribution with constant distribution of endowment among time

Let us suppose that there is no aggregate risk, that is $\overline{\omega}_{(s,1)} = \overline{\omega}_{(s,2)}$, moreover, let us suppose that $\omega_i = \omega$ for all state $s$ and all agent $i$. Under these conditions, $\overline{\pi}_{(s,1)} = \overline{\pi}_{(s,2)}$ for all state $s$, moreover, half of the risk lovers will specialize in event one and half of them in event two. Therefore, the expected aggregate bequest for the Risk lovers and the expected aggregate endowment for the risk lover maintain constant over time.

For risk averters, the wealth distribution is given by $w^e_0 = q_{0,1}\hat{\omega}_{0,1} + q_{0,2}\hat{\omega}_{0,2}$ in $t = 0$. And therefore, $w^e_{(0,1)} = w^e_{(0,2)} = \omega + \frac{\delta}{2\pi_{(0,1)}}w^e_0$ for $t = 0$ where

$$\pi_{(0,1)} = \pi_{(0,2)} = \pi_{(s,2)} = \pi = \frac{1}{2} \left( \frac{1 + \frac{\delta}{1-\delta}}{1} \left( 2\omega + \overline{R} + \overline{R} \right) \right)$$

and

$$w^e_{(s,1)} = w^e_{(s,2)} = \omega + \frac{\delta}{2\pi_{(s,1)}}s^e_{s^-}$$

for all state $s$ of length $t \geq 1$ where $s^-$ is the predecessor of the state $s$. Then, $w^e_{(s,1)} = w^e_{(s,2)} = \frac{\delta}{2\pi}k + \frac{\delta}{2\pi}w^e_0$ if $q_{0,1} + q_{0,2} = 2\omega + \overline{R} + \overline{R}$, which implies

$$\lim_{t \to \infty} w^e_{(s,1)} = \frac{1}{1 - \frac{\delta}{2\pi}}\omega = \left( 1 + \frac{\delta}{1-\delta} \right) \left( \omega + \frac{1}{2} \overline{R} \right)$$

that $w^e_{(s,1)} = w^e_{(s,2)} = \frac{\delta}{2\pi}w^e_0$ and the wealth distribution converge to a constant value that just only depends on $\omega$.

Therefore, the invariant distribution does not change with variations of the bequest level since bequest does not change the proportion of the wealth that the risk averters have or the aggregate endowment available in the economy. As a consequence, an increment on the bequest proportion will only make more expensive to buy one unit of good in any of the possible events, $\pi$, which implies an increment of the assets' price $(q_{(s,1)}, q_{(s,2)})_s$.

For risk lovers, let us suppose symmetry on how they specialize.
If the risk lovers are indifferent between investing in either events, half of the agents will specialize in event 1 and half in event 2, and half of the agents with the same level of wealth will specialize in event 1 and half in event 2.

This condition can be implemented if each risk lover's decision is independent of the other risk lovers' decisions.

Now the wealth distribution for risk lover can be characterized as follows:

- For $t = 0$, we have that $w_0^{\ell} = q_{0,0}^{\ell} G_{0,1}^{\ell} + q_{0,2}^{\ell} G_{0,2}^{\ell}$.
- For $t = 1$, we have that $w_{(0,1)}^{\ell} = \omega$ and $w_{(0,2)}^{\ell} = \omega + \frac{\delta}{\pi} w_0^{\ell}$ with a proportion of $\frac{1}{2}$ of the risk lovers, and vice-versa with the same proportion.
- For $t = 2$, we have, for a state $s$, $w_s^{\ell}$ has the following distribution
  - $\omega$ with measure $\frac{1}{2}$
  - $\omega + \frac{\delta}{\pi} \omega$ with measure $\frac{1}{4}$
  - $\omega + \frac{\delta}{\pi} w_0^{\ell}$ with measure $\frac{1}{4}$
- For $t = 3$, we have, for a state $s$, $w_s^{\ell}$ has the following distribution
  - $\omega$ with measure $\frac{1}{2}$
  - $\omega + \frac{\delta}{\pi} \omega$ with probability (or proportion) $\frac{1}{4}$,
  - $\omega + \frac{\delta}{\pi} \left( \omega + \frac{\delta}{\pi} w_0^{\ell} \right)$ with measure $1/8$
  - $\omega + \frac{\delta}{\pi} \left( \omega + \frac{\delta}{\pi} w_0^{\ell} \right)$ with measure $1/8$
- Therefore, recursively, we have, for any state $s$ with length $t \geq 1$, that $w_s^{\ell}$ has the following distribution
Under RL, the invariant distribution is unique and that any initial portfolio distribution will converge to \((w^d_\omega)_t\) in distribution. Therefore, we have:

**Theorem A.4.** Under RL, there is an invariant distribution for the agents that, given any initial wealth distribution \(W_0\), the wealth distribution converges to the invariant distribution \(W_\infty\) when \(t\) goes to infinity.

As it can be observed in Figure 1 and 2, the shape of the invariant distribution depends on the level of bequest of the agents. For large levels of bequest, \(\delta \geq \frac{2\omega}{4\omega + (\bar{R} + \bar{R})}\), gains given by the natural specialization of the risk lovers make that the agents whose predecessors invest in the events that in fact occurred, become extremely wealthy over time. However, if the level of bequest is low, \(\delta < \frac{2\omega}{4\omega + (\bar{R} + \bar{R})}\) that is the bequest is less than the agent endowment over the average aggregate endowment available for the agent, the gains mentioned above have little impact in the predecessors' wealth and as a consequence, when agent bets correctly, his successor's wealth will increase his wealth by an amount that converges to zero in the long run and therefore, for an agent who bets correctly always his wealth will converge to

\[
- \sum_{k=0}^{m} \left( \frac{\delta}{\pi} \right)^k \omega \text{ with measure (or proportion) } 1/2^{m+1} \text{ (for } m = 0, ..., t-1),
\]

\[
- \sum_{k=0}^{t-2} \left( \frac{\delta}{\pi} \right)^k + \left( \frac{\delta}{\pi} \right)^{t-1} w^d_0 \text{ with measure } 1/2^t
\]

And as it can be seen before, the wealth distribution for risk lovers will tend (in distribution) to \((w^d_\omega)_t\) with measure (or proportion) \(1/2^{t+1}\) (for \(t \in \mathbb{N}\)).

\[\delta \geq \frac{2\omega}{4\omega + (\bar{R} + \bar{R})}\]

\[\delta < \frac{2\omega}{4\omega + (\bar{R} + \bar{R})}\]

---

5 We will denote by the Invariant Distribution of the Risk Lovers (IDRL).
Then, a low level of bequest will decrease the wealth of a risk lover with a wealth higher than even when he always invests in the events that will happen.

Comparing the two type of agents, we can observe that bequest affects more the risk lovers than the risk averters since different levels of bequest cause a real effect on the consumption plan increasing the consumption for low levels of bequest and decreasing the consumption for high levels. However, the aggregate consumption of the risk lovers is preserved by all levels of bequest since the proportion of the aggregate wealth of the risk lovers is preserved under changes of bequest. As a consequence, the increment of the assets' price due to an increment on bequest also affects the risk lovers increasing their wealth nevertheless, it does not increase the aggregate consumption of them.

Note that RL implies the convergence to the invariant distribution for any initial wealth distribution and the uniqueness of the equilibrium if the initial wealth distribution of risk lovers is discrete.

In case of RL is not satisfied, Theorem 2 proves that there is no invariant distribution, it means that RL is necessary and sufficient for the existence of an invariant distribution of the economy.

**Theorem A.5.** If RL is not satisfied, there is no invariant distribution for the wealth.

**Proof.** First, let us suppose that the initial distribution of wealth is discrete. Without loss of generality, let us suppose that RL is true for all the predecessors of a state $s$ with length $t \geq 0$ and it is not satisfied for the state $s$, the wealth distributions of the successors of the state $s$ will have different weights for the agents with the minimal level of wealth ($\omega$), which implies that the non existence of an invariant distribution.

Now, let us prove the general case. Using the Kolmogorov 0-1 law it is easy to see that any possible invariant distribution must be discrete. Therefore, if we assume that there is an invariant distribution in the case of, we can apply the first part of the theorem, which will imply a contradiction.
Note that RL does not imply uniqueness of the equilibrium however, it is enough to ensure that all the possible wealth distributions in equilibrium will converge to the invariant distribution $w_\infty$ even if RL is not satisfied in all states.

**Theorem A.6.** If for each path $\sigma = (\eta_1, \eta_2, \ldots)$ there is a $t$ such that RL is valid for all state of the path from the time $t$ onwards, the wealth distribution will tend to $w_\infty$.

**Proof.** Let us consider a path $\sigma$ then, there is a $t$ such that RL is valid for $\sigma' = (\sigma_1, \ldots, \sigma_t, \ldots, \sigma_r)$ where $r \geq t - 1$. Now, consider an equilibrium for the economy it is clear to achieve market clearing, we must have that the weight of agents with the lowest level of wealth ($\omega$) is a probability $\alpha \in (0, 1)$ with the following property $\int_{E_{B, \omega}} w_{\omega}^j \, di = \int_{E_{B'}} w_{\omega}^j \, di = 1/2 \int w_{\omega}^j = 1/2 \omega$ where $P[\beta_{\omega}^j] = \alpha_j$.

For $\sigma'^{t+1}$, the weight of agents with the minimum level of wealth satisfies the previous conditions and therefore, the aggregate wealth of the group that never loose is $1/4 \int w_{\omega}^j \, di = 1/4 \int w_{\omega}^{j+1} = 1/4 \omega$.

And due to the fact that once a risk lover losses, he will always be part of the discrete distribution of wealth, and, in the long run, the mass of agents in this part, the discrete one, will converge to 1 and the aggregate wealth of them will converge to $\left(1 + \frac{\delta}{1 - \delta}\right)(\omega + 1/2(\bar{R} + R))$.

Now, let us define $\beta_r$ as the measure of agents in the discrete part of the distribution at time $r$. To conclude the proof, we must prove that the mass of agents with the minimum level of wealth converge to $1/2$ when $r$ goes to infinity. Note that the measure of this set is bounded, from below, by half $\beta_r$ and, from above, by half of $\beta_r$ plus the weight of risk lovers who gain in $\sigma_1, \sigma_2, \ldots, \sigma_r$. And as it was mentioned before, $r'$ converges to 1 and the weight of the agents who always get “right” goes to zero implying that the weight of risk lovers who has the minimum
level of wealth goes to $1/2$ when the time goes to infinity and due to RL, the distribution will converge to $W_\infty$.

**A.3. Invariant distribution with taxes**

$$\tilde{W}_t = W_t = \left(1 + \frac{\delta}{1 - \delta}\right)\left(\omega + \frac{1}{2}(\bar{R} + \bar{R})\right),$$

Now, for taxes with $W_0$, the wealth distribution converges to the invariant distribution $W_{t,\infty} = W_\infty$ when $t$ goes to infinity.

**Conjecture A.7.** Under the condition RL, there is an invariant distribution for the agents that, with any initial wealth distribution $W_0$, the wealth distribution converges to the invariant distribution $W_{t,\infty} = W_\infty$ when $t$ goes to infinity.

The main reason for this result is that risk lovers will foresight the taxes that the government will implement when they are old so they will end up with negative returns with one event such that the subsidy will compensate completely this return. Therefore, risk lovers will not change their consumption with taxes.

Taxes on bequest will have an impact on the invariant distribution reducing inequality by reducing the difference of each agent compared to the average bequest.

**Appendix B. Proofs**

**A.1. Proof of Theorem 1**

Let us prove a preliminary result that ensures that for any initial distribution $(w_0^l) \gg 0$, the aggregate income in hands of the $l$ agents over the aggregate wealth in hands of the $a$ agents, $\frac{\tilde{w}_t^l}{\tilde{w}_t}$, converge to a positive constant even if $R_R \neq 0$ or $R_S \neq \bar{R}_S$.

**Lemma A.8.** For taxes defined by nonnegative marginal tax rates $(\tau^l, \bar{\tau}^B) > 0$ with technology returns such that satisfy Equations 3.1 and 3.2, $\lim_{t \to \infty} \frac{\tilde{w}_t^l}{\tilde{w}_t^a} = \gamma(\tau^l, \bar{\tau}^B)$ where $\gamma(\tau^l, \bar{\tau}^B) \in [1, \infty)$.
Proof. To simplify the proof, we will assume that $\bar{\tau}^l = \tau > 0$, $\bar{\tau}^R = 0$. The idea of the proof is to show that the function $f: [0, \infty) \to [0, \infty)$ defined by

$$z_{\tau}^{(e+1)} = \frac{E[R_R]}{E[R_S]} b_{\tau}^{(e)} = f(z_{\tau}^e) = \left( \frac{E[R_R]}{E[R_S]} \right) \left( \frac{1-\frac{\tau}{2} z_{\tau}^{(e)} + \frac{\tau}{2}}{z_{\tau}^{(e)} + (1-\frac{\tau}{2})} \right)$$

satisfies that $f(0) > 0$,

$$\lim_{z \to \infty} f(z) = \left( \frac{E[R_R]}{E[R_S]} \right) \frac{2-\tau}{\tau}, \quad f'(z) > 0 \forall z \in [0, \infty), \quad f'(\infty) = 0 \text{ and } f' \text{ is a decreasing function.}$$

Under these conditions, $f$ has only one fixed point $y_\tau$ defined by

$$z_{\tau} = \left( \frac{1}{2} - \frac{1}{\tau} \right) \left( \frac{E[R_R]}{E[R_S]} - 1 \right) + \sqrt{\left( \frac{1}{2} - \frac{1}{\tau} \right)^2 \left( \frac{E[R_R]}{E[R_S]} - 1 \right)^2 + \frac{E[R_R]}{E[R_S]}}$$

(3.5)

and, for each $z^0 \in (0, \infty)$, $z_{\tau}^e$ converge to $z_{\tau}$. Since $z_{\tau}^e$ is the proportion of the aggregate production of the $l$ agents and the $a$ agents, the sequence $\left\{ \frac{w_l}{w_a} \right\}$, also converge, and since $z_{\tau}^{(e+1)} = 1$,

$$\gamma_{\tau}^{(e)} = \frac{w_l}{w_a} \geq 1 \quad \gamma_{\tau} = \frac{z_{\tau}^{(e)}(1-\frac{\tau}{2}) + \frac{\tau}{2}}{z_{\tau}^{(e)} + (1-\frac{\tau}{2})} \in [1, \infty)$$

implying that $\gamma_{\tau} \in [1, \infty)$. The proof is analogous to the other two types of taxation.

From the Proof of Lemma A.8, we can interpret $z_{\tau}$ as the ratio of the aggregate production of the skilled agents and the aggregate production of the unskilled ones. Therefore, note that $\gamma$ can be seen as monotonic function of $z_{\tau}$, implying that $\gamma$ is a $C^1$ function for $\tau \in [0,1]$ that decreases when $\tau$ and $E[R_S]$ increases, and that increases when $E[R_R]$ increases.

Since the aggregate production depends on aggregate wealth of each of the groups, the convergence of the ratio of the skilled and unskilled aggregate wealth ensures the convergence of the growth path.
Corollary A.9. For any fixed tax rate \( (\tau^I, \tau^B) > 0 \), the growth rate of the economy, \( g(\tau^I, \tau^B, t) \), converges when \( t \) goes to infinity.

Due to the convergence of how each group invest in each technology, the growth rate of the economy will also converge. Then, the proportion of income of the poorest skilled agent converges, which implies that the proportion of income of a skilled agent that has received at least once the lower return \( R_R = 0 \) also converges.

Proof of Theorem 1. For simplicity, let us consider a positive marginal income tax rate \( \tau^I = \tau \) only. The convergence of the proportion of wealth of the \( l \) agents to an invariant distribution is a direct consequence of the conditions mentioned above. In fact, the invariant distribution of the proportion of the \( l \) agents is

\[
\sum_{k=0}^{n-1} \frac{R_R^k \tau}{2} \frac{(1-\tau)^k \delta^k}{(1 + g_{\tau,l})^k}
\]

for the \( n^{th} \) poorest group of \( l \) agents with weight \( \left( \frac{1}{2} \right)^{n+1} \) for \( n \in \mathbb{N} \).

To conclude the proof, we must ensure that Equation 3.7 converges for any initial \( w_0^{\alpha_i} \).

Since \( R_s < E[R_R] \),

\[
\frac{R' \delta^{t-1}}{w_i} \rightarrow 0 \quad \text{when} \quad t \rightarrow \infty, \text{ implying that } \quad \frac{\delta^{t-1} R'_s (1-\tau)^l w_0^{\alpha_i}}{w_i} \rightarrow 0 \quad \text{when} \quad t \rightarrow \infty.
\]

\[
\sum_{l=0}^{t} \left( \frac{\delta^k (1-\tau)^k \tau R_s^k}{\prod_{n=1}^{k} (1 + g_{\tau,l})} \right)
\]

converges when \( t \) goes to infinity since

\[
0 < \frac{\delta (1-\tau) R_s}{(1 + g_{\tau,l})^2} < 1 - \frac{\tau}{2}
\]

for all \( l \in \mathbb{N} \). Therefore, the proportion of the wealth of the \( \alpha \) agents in the limit is
\[
\begin{align*}
\frac{1}{1 - \delta (1 - \tau) R_s} \tau &= \frac{\tau}{2} (1 + g_\tau) \quad \text{and} \\
\frac{1}{1 + g_\tau - \delta (1 - \tau) R_s} &= \frac{1}{\gamma_\tau + 1}
\end{align*}
\]

The proof is analogous to the other two types of taxation.

Based on the proof of Theorem 1, with \( \tau^l = \tau > 0 \), and \( \tau^B = 0 \), the proportion of the income of the poorest \( l \) agents is \( \tau/2 \), and the weight of this group is 1/2. The income of the second poorest group of \( l \) agents only depends on the average income and the income of the poorest \( l \) agents in the previous period. Therefore, the proportion of the second poorest group of \( l \) agents

\[
is \frac{R_R(\frac{1}{2}) \delta}{1 + g_{\tau,t-1}} + \tau \left( \frac{1}{2} - \frac{R_R(\frac{1}{2}) \delta}{1 + g_{\tau,t-1}} \right) = \frac{\tau}{2} \left( R_R(\frac{1}{2}) \delta + 1 \right),
\]

and its weight is 1/4. If we continue this process, we obtained that proportion of the \( n^{th} \) poorest group of \( l \) agents

\[
is \sum_{k=0}^{n-1} \frac{R_R^{k+t} (1-\tau)^k \delta^k}{\prod_{j=1}^k (1 + g_{\tau,t-k})},
\]

and the weight of this group is \( \frac{1}{2^{n+1}} \) if \( t \geq n \).

For the \( a \) agents, since \( R_s = \overline{R_s} \), the proportion of the income of an \( a_l \) agent is given by

\[
is \sum_{k=0}^{t-1} \frac{\delta^{k+t} R_{\overline{S}}^k}{\prod_{\tau=1}^k (1 + g_{\tau,t})} + \delta^{t-1} R_{\overline{S}}^t (1 - \tau)^t \frac{w_a}{w_t}
\]

for each node \( s \) of length \( t \geq 0 \). Since \( \gamma^t \rightarrow \gamma_t \in [0, \infty) \); when \( t \rightarrow \infty \), \( g_{\tau,t} \rightarrow g_\tau \in [R_\delta - 1, E[R_t] \delta - 1] \) when \( t \rightarrow \infty \), which concludes the proof. The proof with positive income and consumption taxes is analogous.

A.2. Proof of Proposition 2

*Proof of Proposition 2.* Due to Equation 3.10, we can define \( X(\cdot, \cdot) \) as

\[
X(d, \tau) = \frac{d}{1 - d} \int \log \left( x_\tau^l + x_\tau^{\tau[l+1]} \right), \quad G(\cdot, \cdot) \quad \text{as} \quad G(d, \tau) = \sum_{t=1}^{\infty} d^t \int \log \left( \left( 1 + g_\tau \right)^t \right) dt \quad \text{which is clear-}
\]
ly a decreasing function in \( \tau \), and \( D(\cdot,\cdot) \) as \( D(d) = \sum_{t=1}^{\infty} d \int \log \left( \frac{\delta \delta_{1-\delta}(1-\tau^d)}{1-\delta \tau} \right) dt. \)

In this case, each function depends directly or indirectly on \( \delta \) since the bequest rate affects the distribution of the invariant distribution and the growth rate of the economy by increasing inequality and the growth rate when \( \delta \) increases.

The properties of \( X \) are consequence of the dominated convergence theorem and the properties of the invariant concentration of wealth (in the aggregate, is constant, and it is more unequal every time that you decrease the marginal tax rate). The properties of \( G \) are consequence of the invariant growth rate and the fact that the series that defines this function is absolutely convergent.

Finally, the properties of \( D \) can be easily obtained due to its functional form.

**A.3. Proof of Proposition 6**

*Proof of Proposition 6.* To prove the second part, it is enough to analyze asymptotic behavior of \( X \) and \( G \) when \( d \) goes to zero and goes to one. When \( d \) goes to zero, \( G \) becomes null and \( X \) does not, then, the \( W \) only depends on \( X \) for \( d \) small enough which implies our result. When \( d \) goes to one, \( X \) becomes null and \( G \) does not. However, in this case \( X \) is unbounded from below. Therefore, the optimal taxes are small but no zero. Nevertheless, when the discount factor of the social planner goes to zero, we can find taxes that converge to zero that keep constant \( X \) and increases the value of \( G \) implying our result.

Finally, the first part is a consequence of the second one.

**A.4. Proof of Proposition 7**
Proof of Proposition 7. Consider a taxation plan \( \hat{\tau}' \in \mathcal{I} \) such that \( 1 > \hat{\tau}' > \hat{\tau}' + \frac{\delta \tau B}{1 - \delta \tau B} \). Since we analyze the welfare function in the invariant distribution, we can assume that \( \bar{w}_0 = 1 \) in both cases.

For the poorest skilled agents, we have that their level of after tax nominal income with the taxation plan \((\hat{\tau}', \tau B)\) is given by \( \hat{\tau}' = \bar{w}_0 \hat{\tau}' + \bar{w}_0 \frac{\tau B}{1 - \tau B} \left( \delta (1 - \tau B) \right) = \bar{w}_0 \left( \hat{\tau}' + \frac{\delta \tau B}{1 - \delta \tau B} \right) < \bar{w}_0 \hat{\tau}' = \hat{\tau}' \) which is the after tax income with the new taxation plan.

Note that that since the function \( f_k(x) = \frac{x}{(1 - x)^k} \) is an increasing function for \( x \in [0,1) \) for all \( k, l \in \mathbb{N} \) and \( \hat{\tau}' < \hat{\tau}' \), we have that \( \hat{\tau}' (1 - \hat{\tau}')^k (1 - \tau B)^l < \hat{\tau}' (1 - \hat{\tau}')^k (1 - \tau B)^l \). The second poorest group of skilled agents, we have that their level of after tax nominal income with the taxation plan \((\hat{\tau}', \tau B)\) is given by \( R \hat{\tau}' \bar{w}_0 \delta (1 - \tau B) (1 - \hat{\tau}') + \bar{w}_0 \hat{\tau}' + \bar{w}_0 \frac{\tau B}{1 - \tau B} \left( \delta (1 - \tau B) \right) = \bar{w}_0 \left( R \hat{\tau}' \delta (1 - \tau B) (1 - \hat{\tau}') + \hat{\tau}' + \frac{\delta \tau B}{1 - \delta \tau B} \right) < \bar{w}_0 \left( R \hat{\tau}' \delta (1 - \tau B) (1 - \hat{\tau}') + \hat{\tau}' \right).

For the \( n \)-poorest group of skilled agents, we have

\[
\sum_{k=0}^{n-1} \left( R \hat{\tau}' \delta (1 - \tau B) \right) (1 - \hat{\tau}')^k (1 - \tau B)^l \left( \hat{\tau}' + \frac{\delta \tau B}{1 - \delta \tau B} \right) < \bar{w}_0 \left( \sum_{k=0}^{n-1} \left( R \hat{\tau}' \delta (1 - \tau B) \right) (1 - \hat{\tau}')^k (1 - \tau B)^l \right) < \bar{w}_0 \left( \sum_{k=0}^{n-1} \left( R \hat{\tau}' \delta (1 - \tau B) \right) (1 - \hat{\tau}')^k \right)
\]

Then, the income of the invariant distribution in each period is always lower with the taxation plan \((\hat{\tau}', \tau B)\) than with \((\tau', 0)\). For the unskilled ones, the result is also true because of the convergence of an analogous series as the one described above. To conclude the proof, notice that the utility of the agent \( i \) when the after taxes nominal incomes in each state are \( w_1^i \) and \( w_2^i \) is

\[
\frac{1}{2} u^i(c_1^i, b_1^i) + \frac{1}{2} u^i(c_2^i, b_2^i) = \frac{1}{2} (c_1^i)^{1-\delta} (b_1^i)^{\delta} + \frac{1}{2} (c_2^i)^{1-\delta} (b_2^i)^{\delta} = \frac{1}{2} (1 - \delta) \left( w_1^i \right)^{1-\delta} \left( \delta (1 - \tau B) w_1^i \right)^{\delta} + \frac{1}{2} (1 - \delta) \left( w_2^i \right)^{1-\delta} \left( \delta (1 - \tau B) w_2^i \right)^{\delta} = \left( 1 - \delta \right)^{1-\delta} \delta \left( 1 - \tau B \right) w_1^i + \frac{1}{2} \left( 1 - \delta \right)^{1-\delta} \delta \left( 1 - \tau B \right) w_2^i < \frac{1}{2} \left( 1 - \delta \right)^{1-\delta} \delta w_1^i + \frac{1}{2} \left( 1 - \delta \right)^{1-\delta} \delta w_2^i.
\]