Confidence Intervals for Nonparametric Empirical Bayes Analysis

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Abstract

In an empirical Bayes analysis, we use data from repeated sampling to imitate inferences made by an oracle Bayesian with extensive knowledge of the data-generating distribution. Existing results provide a comprehensive characterization of when and why empirical Bayes point estimates accurately recover oracle Bayes behavior. In this paper, we develop flexible and practical confidence intervals that provide asymptotic frequentist coverage of empirical Bayes estimands, such as the posterior mean or the local false sign rate. The coverage statements hold even when the estimands are only partially identified or when empirical Bayes point estimates converge very slowly.

Keywords: Empirical Bayes, mixture models, local false sign rate, partial identification, bias-aware inference

1 Introduction

Empirical Bayes methods enable frequentist estimation that emulates a Bayesian oracle. Suppose we observe $Z$ generated as below, and want to estimate $\theta_G(z)$,

$$
\mu \sim G, \quad Z \sim p(\cdot | \mu), \quad \theta_G(z) = \mathbb{E}_G [h(\mu) | Z = z],
$$

(1)

for some known function $h(\cdot) \in \mathbb{R}$. Given knowledge of $G$, $\theta_G(z)$ can be directly evaluated via Bayes’ rule. An empirical Bayesian does not know $G$, but seeks an approximately optimal estimator $\hat{\theta}(z) \approx \theta_G(z)$ using independent draws $Z_1, Z_2, ..., Z_n$ from the distribution (1).

The empirical Bayes approach was first introduced by Robbins [1956] and has proven to be successful in a wide variety of settings with repeated observations of similar phenomena, such as genomics [Efron et al., 2001, Love et al., 2014], education [Lord, 1969, Gilraine et al., 2020] and actuarial science [Bühlmann and Gisler, 2005]. Table 1 provides concrete applications of model (1) for these subject areas. In all examples, the posterior mean $\theta_G(z) = \mathbb{E}_G [\mu | Z = z]$ is a statistic of interest, as it describes the (mean squared error) optimal shrinkage rule for estimating $\mu$. In the genomics application, it is also of interest to determine the local false sign rate $\theta_G(z) = \mathbb{P}_G [\mu z \leq 0 | Z = z]$, i.e., the posterior probability that $\mu$ has a different sign than $Z$.

As elaborated later, there is by now a large literature proposing a suite of estimators $\hat{\theta}(z)$ for $\theta_G(z)$. Many of these estimators have theoretical guarantees under nonparametric
specification of $G$, say $G \in \mathcal{G}$, where $\mathcal{G}$ is a convex class of distributions. The goal of this paper is to move past point estimation, and develop nonparametric confidence intervals for $\theta_G(z)$, i.e., intervals with the following property (for $\alpha \in (0, 1)$):

$$I_\alpha(z) = \left[ \hat{\theta}_-^z(z), \hat{\theta}_+^z(z) \right], \quad \liminf_{n \to \infty} P_G[\theta_G(z) \in I_\alpha(z)] \geq 1 - \alpha \text{ for all } G \in \mathcal{G}. \quad (2)$$

Despite widespread use of empirical Bayes methods, the problem has received surprisingly little attention. In fact, we are not aware of confidence intervals with property (2) beyond two special cases: one proposal by Lord and Cressie [1975] for inference about the posterior mean in the binomial model and another by Robbins [1980] for the same task in the Poisson model.

### 1.1 Motivating application: Predicting automobile insurance claims

To motivate our interest in confidence intervals of the form (2), we revisit the historical work of Bichsel [1964]. Bichsel developed a theoretical framework for assigning automobile insurance premium rates, in a way that accounts for the claims experience of each individual. He analyzed a dataset (Table 2) of claims made in the year 1961 by holders of a Swiss automobile insurance policy. Bichsel posited that $Z_i(t)$, the number of claims made in year $t$ by the $i$-th insurance holder, is distributed as Poisson ($\mu_i$), where $\mu_i$ is $i$’s latent risk. $\mu_i$ is a random draw from a distribution $G$ that captures the heterogeneity of the insurance portfolio. Bichsel further assumed that the number of claims $Z_i(t), Z_i(t')$ in different years $t \neq t'$ are i.i.d. conditionally on $\mu_i$. Given these assumptions, Bichsel sought to estimate the expected number of claims in the next year, among all insurance holders that made $Z_i(1961) = z$ claims in 1961,

$$\theta_G(z) = E_G[Z(1962) \mid Z(1961) = z] = E_G[\mu \mid Z(1961) = z]. \quad (3)$$

Bichsel reasoned, that if $\theta_G(z)$ were known, it could be used by the insurance company for policy decisions, such as increasing or decreasing the premium of a policy holder with $z$ claims in 1961. Since $G$, and consequently $\theta_G(z)$, were not known to Bichsel, he considered an empirical Bayes approach.\(^1\)

\(^1\) $\theta_G(z)$ is a property of $G$, i.e., of the portfolio heterogeneity. The goal is to best assess how many claims will be made across all individuals in the portfolio that made $z$ claims in 1961, and not to reason about the risk $\mu_i$ of any individual policy holder with $Z_i(1961) = z$. The problem of forming intervals containing the true $\mu_i$ (for individuals) is of scientific importance, see e.g., Morris [1983], Laird and Louis [1987], Armstrong, Kolesár, and Plagborg-Møller [2020], Koenker [2020] for some proposals; however it is not the problem we consider in this work.

### Table 1: Example applications for empirical Bayes inference in model (1)

<table>
<thead>
<tr>
<th>Subject</th>
<th>i</th>
<th>$Z_i$</th>
<th>$\mu_i$</th>
<th>$Z_i \mid \mu_i \sim$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actuarial science</td>
<td>Contract</td>
<td>Number of insurance claims</td>
<td>Risk profile</td>
<td>Poisson ($\mu_i$)</td>
</tr>
<tr>
<td>Education</td>
<td>Student</td>
<td>Score in test with $N$ multiple choice questions</td>
<td>Latent ability</td>
<td>Binom($N, \mu_i$)</td>
</tr>
<tr>
<td>Genomics</td>
<td>Gene</td>
<td>t-statistic comparing expression between conditions</td>
<td>Standardized effect size</td>
<td>$N(\mu_i, 1)$</td>
</tr>
</tbody>
</table>
The problem of point estimation for \( \theta_G(z) \) is well understood. One popular nonparametric solution is to first estimate \( G \) through the nonparametric maximum likelihood (NPMLE) of Kiefer and Wolfowitz [1956] and Simar [1976]: one estimates \( \hat{G} \) as the maximizer of the marginal log-likelihood \( \sum_i \log(f_G(Z_i)) \), where \( f_G(z) = \int \exp(-\mu)\mu^z/z!dG(\mu) \), among all possible prior distributions \( G \). Then, with \( \hat{G} \) in hand, one estimates \( \theta_G(z) \) through the plug-in principle, i.e., \( \hat{\theta}_G(z) = \theta_{\hat{G}}(z) \) (shown in the third column of Table 2).

However, in so far as \( \theta_G(z) \) may be used for policy decisions of the insurance company, it is also important to assess the uncertainty in estimating it. In this paper we develop two complementary approaches that address the problem of inference for empirical Bayes estimands, and enable the construction of intervals with the property (2) under the general model (1). The last two columns of Table 2 show the two confidence intervals that we propose for Bichsel’s data. The assumption we make in forming these intervals is that \( G \) is supported on \([0, 5]\). The ‘F-localization’ intervals (fourth column of Table 2) have simultaneous coverage for all \( z \), while the ‘AMARI’ intervals (last column) have pointwise coverage. We next provide a high-level overview of our two constructions.

### 1.2 Empirical Bayes confidence intervals

In our approach the data analyst first specifies (1), i.e., \( \theta_G(z) \), the empirical Bayes estimand of interest (e.g., the posterior mean (3)) and the conditional distribution of \( Z \) given \( \mu \) (e.g., Poisson \((\mu)\)), which we represent by its conditional density \( p(\cdot | \mu) \) with respect to a \( \sigma \)-finite measure \( \lambda \) on a subset of \( \mathbb{R} \) (e.g., the counting measure on \( \mathbb{N}_{\geq 0} \)). We also require the data analyst to specify a convex class of priors \( \mathcal{G} \) such that \( G \in \mathcal{G} \). For example, for our analysis of Bichsel’s data in Table 2 we assumed that \( G \in \mathcal{G} = \mathcal{P}([0, 5]) \), where:

\[
\mathcal{P}(\mathcal{K}) = \{ G \text{ distribution} : \text{support}(G) \subset \mathcal{K} \} \quad \text{for } \mathcal{K} \subset \mathbb{R}.
\]

\[2\]In his work, Bichsel modeled \( G \) parametrically as a Gamma distribution with unknown parameters.

\[3\]We provide more guidance for choosing \( \mathcal{G} \) in Section 8.
1.2.1 F-localization

Our first confidence interval construction is based on the notion of F-localization. The key idea is to construct a confidence set for the marginal distribution of $Z$ and then determine all $G \in \mathcal{G}$ consistent with this confidence set. Let us denote the marginal distribution of $Z$ by $F_G$ and its $d\lambda$-density by $f_G$, i.e.,

$$f_G(z) = \int p(z \mid \mu) dG(\mu), \quad F_G(t) = \mathbb{P}_G [Z \leq t] = \int 1(z \leq t) f_G(z) d\lambda(z). \quad (5)$$

We then define an F-localization as an (asymptotic) $1 - \alpha$ confidence set $\mathcal{F}_n(\alpha)$ of distributions, i.e., a set such that

$$\lim_{n \to \infty} \inf \{ \mathbb{P}_G [F_G \in \mathcal{F}_n(\alpha)] - (1 - \alpha) \} \geq 0. \quad (6)$$

With an F-localization $\mathcal{F}_n(\alpha)$ in hand, and deferring the construction of such to Section 2, we can form confidence intervals $\mathcal{I}_n(\alpha) = [\hat{\theta}_n^-(z), \hat{\theta}_n^+(z)]$ for $\theta_G(z)$ by letting,

$$\hat{\theta}_n^-(z) = \inf \{ \theta_G(z) : G \in \mathcal{G}(\mathcal{F}_n(\alpha)) \}, \quad \hat{\theta}_n^+(z) = \sup \{ \theta_G(z) : G \in \mathcal{G}(\mathcal{F}_n(\alpha)) \}, \quad (7)$$

where $\mathcal{G}(\mathcal{F}) = \{ G \in \mathcal{G} : F_G \in \mathcal{F} \}$. \quad (8)

The intervals (7) satisfy (2), since $\mathbb{P}_G[\theta_G(z) \in [\hat{\theta}_n^-(z), \hat{\theta}_n^+(z)]] \geq \mathbb{P}_G [F_G \in \mathcal{F}_n(\alpha)]$, and the same argument also demonstrates that coverage holds simultaneously over all possible empirical Bayes estimands $\hat{\theta}_G(z) = \mathbb{E}_G \left[ h(\mu) \mid Z = z \right]$, where both $z$ and $h$ can vary. In Section 2 we explain how for common choices of $\mathcal{G}$ and $\mathcal{F}_n(\alpha)$, (7) can be computed by solving two linear programs.

It is interesting to consider the F-localization approach in the context of the dichotomy of Efron [2014, 2019] on F- versus G-modeling. Under model (1), it is typically straightforward to estimate $F_G$, because the observed $Z_i$ are direct measurements from $F_G$. In contrast, estimation of $G$ is a difficult inverse problem. In some cases, the empirical Bayes estimand of interest may be expressed directly in terms of $F_G$: for example, Robbins [1956] proves that the posterior mean in the Poisson model ($p(\cdot \mid \mu) = \text{Poisson}(\mu)$ in (1)) is equal to,

$$\theta_G(z) = \mathbb{E}_G \left[ \mu \mid Z = z \right] = (z + 1) \frac{f_G(z + 1)}{f_G(z)}, \quad f_G(z) = \mathbb{P}_G [Z = z], \quad z \in \mathbb{N}_{\geq 0}. \quad (9)$$

When a formula as (9) is available, it is convenient to proceed by F-modeling, i.e., to estimate $F_G$ and evaluate the corresponding F-formula by the plug-in principle. In the Poisson model, letting $\hat{F}_G$ be the empirical distribution of the $Z_i$, the plug-in principle leads to the estimate $\hat{\theta}_{\text{Robbins}}(z) = (z + 1) \# \{ Z_i = z + 1 \} / \# \{ Z_i = z \}$ for (9). A caveat of F-modeling, however, is that, natural constraints on the empirical Bayes estimand are not enforced. For instance, the posterior mean (9) in the Poisson model is non-decreasing in $z$, but $\hat{\theta}_{\text{Robbins}}(\cdot)$ does not enforce such monotonicity. In contrast, natural constraints such as monotonicity are automatically enforced under G-modeling, that is, if one first estimates $\hat{G}$ and then lets $\hat{\theta}_G(z) = \hat{\theta}_{\hat{G}}(z)$.

For inference using F-localization, the two perspectives are complementary. The data analyst constructs (6), an F-modeling task, and the Bayes structure of the problem is enforced through (7). For example, in the Poisson posterior mean problem the lower bounds of the confidence intervals, $\hat{\theta}_n^-(z)$, are monotonic in $z$, and similarly for the upper bounds $\hat{\theta}_n^+(z)$.
1.2.2 AMARI (Affine Minimax Anderson-Rubin Intervals)

The $F$-localization approach is generic, streamlined to implement and enables simultaneous inference for all empirical Bayes estimands of interest. The $F$-localization intervals for a specific estimand $\theta_G(z)$, however, can be overly wide. Our second construction, AMARI, seeks to do better than $F$-localization, i.e., to provide shorter confidence intervals, by focusing on a specific estimand (compare e.g., columns 3 and 4 of Table 2). The starting point for AMARI is the observation that we can write the empirical Bayes estimand $\theta_G(z)$, as a ratio of two linear functionals of $G$, i.e.,

$$
\theta_G(z) = \frac{\int h(\mu)p(z|\mu)\,dG(\mu)}{\int p(z|\mu)\,dG(\mu)} = \frac{a_G(z)}{f_G(z)},
$$

where $f_G(\cdot)$ is the marginal density of $Z$ and $a_G(z)$ is used to denote the numerator. Hence, by a construction that goes back to at least Fieller [1940], the following two hypothesis tests are equivalent for $c \in \mathbb{R}$,

$$
H_0 : \theta_G(z) = c \iff H_0 : \theta_G^{lin}(z,c) = 0, \text{ where } \theta_G^{lin}(z,c) = a_G(z) - cf_G(z).
$$

By inverting the test for $H_0 : \theta_G(z) = c$ we can form confidence intervals for $\theta_G(z)$. The upshot of (11) then is that it suffices to construct confidence intervals for linear functionals $L(G)$ of $G$, say $L(G) = \theta_G^{lin}(z,c)$. We provide the details of this reduction in Section 4, and proceed to explain our approach to inference for linear functionals of $G$. Our core proposal is to estimate $L(G)$ as an affine estimator, i.e., one of the form

$$
\hat{L} = \bar{L}(G) = \frac{1}{n} \sum_{i=1}^{n} Q(Z_i),
$$

where $Q(\cdot)$ is chosen to optimize a worst-case bias-variance tradeoff depending on the prior class $G$. To form confidence intervals, we first estimate the variance and worst-case bias of (12) as

$$
\hat{V} = \frac{1}{n(n-1)} \left[ \sum_{i=1}^{n} Q^2(Z_i) - \left( \frac{\sum_{i=1}^{n} Q(Z_i)}{n} \right)^2 / n \right],
$$

$$
\hat{B}^2 = \sup_{G \in \mathcal{G}(F_{\alpha})} \{ \text{Bias}_G[Q,L^2] \}, \quad \text{Bias}_G[Q,L] = \int Q(z)f_G(z)d\lambda(z) - L(G).
$$

Here, the worst-case bias is computed with respect to $G(F_{\alpha})$ (8), where $F_{\alpha} = F_{\alpha}(\alpha_n)$ is an $F$-localization at level $\alpha_n \to 0$ as $n \to \infty$. With $\hat{V}, \hat{B}$ in hand, we build bias-aware confidence intervals $I_{\alpha}$ for $L(G)$ [e.g., Armstrong and Kolesár, 2018, Imbens and Manski, 2004, Imbens and Wager, 2019]

$$
I_{\alpha} = \hat{L} \pm t_{\alpha}(\hat{B}, \hat{V}), \quad t_{\alpha}(B,V) = \inf \left\{ t : P \left[ |b + V^{1/2}W| > t \right] < \alpha \text{ for all } |b| \leq B \right\},
$$

where $W \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable. Sections 3 and 4 have formal results establishing asymptotic coverage properties for these intervals.

Conceptually, our AMARI intervals build on recent work by Noack and Rothe [2019], who consider inference of average treatment effects in the fuzzy regression discontinuity design. There, the estimand also takes the form of a ratio of two linear functionals as in (10). Noack and Rothe [2019] name their approach after Anderson and Rubin [1949], who develop confidence intervals in the linear instrumental variable model, and so similarly we acknowledge Anderson and Rubin [1949] as part of the acronym AMARI.
1.3 Related Work

As discussed briefly above, the empirical Bayes principle has spurred considerable interest over several decades. One of the most successful applications of this idea involves compound estimation of a high-dimensional Gaussian mean: we observe \( Z \sim \mathcal{N}(\mu, I) \), and want to recover \( \mu \) under squared error loss. If we assume that the individual \( \mu_i \) are drawn from a prior \( G \), then empirical Bayes estimation provides a principled shrinkage rule [Efron and Morris, 1973, Efron, 2011], whose theoretical properties are well-understood [Brown and Greenshtein, 2009, Jiang and Zhang, 2009]. The compound estimation problem when the individual \( Z_i \) are Poisson, is also reasonably well understood [Brown et al., 2013].

The more general empirical Bayes problem (1) has raised interest in applications [Efron, 2010, 2016, Stephens, 2016, Koenker and Gu, 2017]; however, the accompanying formal results are less comprehensive. Muralidharan [2012] considered compound estimation in (1), when \( p(\cdot | \mu) \) is a one-dimensional exponential family. For \( p(\cdot | \mu) = p(\cdot - \mu) \), a smooth location family, some authors, including Butucea and Comte [2009], Pensky [2017], have considered rate-optimal estimation of linear functionals of \( G \); and their setup covers, for example, the numerator \( a_G(z) \) in (10). The main message of these papers, however, is rather pessimistic: for example, Pensky [2017] shows that for many linear functionals, the minimax rate for estimation in mean squared error over certain Sobolev classes \( G \) is logarithmic (to some negative power) in the sample size.

In this paper, we study a closely related problem but take a different point of view. Even if minimax rates of optimal point estimates \( \hat{\theta}(z) \) may be extremely slow (or even if estimands are only partially identified), we seek confidence intervals for \( \theta_G(z) \) that still achieve accurate coverage in reasonable sample sizes and explicitly account for bias. The results of Butucea and Comte [2009] and Pensky [2017] imply that the length of our confidence intervals must go to zero very slowly in general; but this does not mean that our intervals cannot be useful in finite samples (and, in fact, our real data applications in Section 5 and numerical experiments in Section 6 suggest that they can be).

To the best of our knowledge, with the exception of a handful of special cases, the problem of nonparametric inference in empirical Bayes problems has been left unexplored. Furthermore, practitioners using empirical Bayes ideas typically do not conduct inference and instead only consider point estimates of functionals of the unknown prior \( G \). Among recent empirical Bayes works, Efron [2014, 2016, 2019] has advocated estimating (and reporting) the variance of empirical Bayes estimates \( \hat{\theta}(z) \), and then using these variance estimates for uncertainty quantification. Such intervals, however, do not account for bias and so could only achieve valid coverage via undersmoothing; and it is unclear how to achieve valid undersmoothing in practice, noting the very slow rates of convergence in empirical Bayes problems. Efron [2014, 2016, 2019] himself does not suggest his intervals be combined with undersmoothing, and rather uses them as pure uncertainty quantification tools.

Two notable existing results for inference as in (2) concern the posterior mean in the Binomial [Lord and Cressie, 1975, Lord and Stocking, 1976] and Poisson problems [Robbins, 1980, Karlis et al., 2018]. The \( F \)-localization approach we propose, generalizes the approach of Lord and Cressie [1975] and Lord and Stocking [1976] for inference of the posterior mean in the Binomial problem to the general empirical Bayes problem (1). We provide more details regarding this connection at the end of Section 2.1, and in Section 5.1 we revisit the data application of Lord and Cressie [1975]. The Poisson posterior mean problem is special, and particularly amenable to the task of forming confidence intervals, because of the existence of Robbins’ formula (9), as we elaborate in Section 7.1.
From a methodological perspective, our work relies upon advances in convex programming and is inspired by Koenker and Mizera [2014], who demonstrated that it is fruitful to revisit traditional ideas in empirical Bayes estimation through the lens of modern convex optimization. The F-localization approach requires solving two linear programs (or more generally, quasi-convex programs, cf. Section 2). AMARI, our second approach, builds heavily on the literature on affine minimax estimation of linear functionals in Gaussian problems. Donoho [1994] and related papers [Armstrong and Kolesár, 2018, Cai and Low, 2003, Donoho and Liu, 1991, Low, 1995, Johnstone, 2019] show that there exist affine estimators that achieve quasi-minimax performance and can be efficiently derived via convex programming. In turn, such affine estimators have recently proven useful for statistical inference in a number of settings, such as semiparametrics [Hirshberg and Wager, 2021, Kallus, 2020] and regression discontinuity designs [Armstrong and Kolesár, 2018, Imbens and Wager, 2019, Eckles et al., 2020].

2 Simultaneous confidence intervals through F-localization

In this section we discuss our first approach, namely F-localization confidence intervals. The key idea is to ‘localize’ the marginal distribution \( F_G \) with high probability, i.e., to construct a set \( \mathcal{F}_n(\alpha) \) (6) such that \( F_G \in \mathcal{F}_n(\alpha) \) with (asymptotic) probability at least \( 1 - \alpha \). \( \mathcal{F}_n(\alpha) \) then implies a confidence set \( \{ G \in \mathcal{G} : F_G \in \mathcal{F}_n(\alpha) \} \) for \( G \), which we project to form confidence intervals for \( \theta_G(z) \) as in (7).

A convenient and universal F-localization proceeds by restricting \( F \) to be in a Kolmogorov-Smirnov ball around the empirical distribution function \( \hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1(Z_i \leq t) \),

\[
\mathcal{F}_n^{DKW}(\alpha) = \left\{ F \text{ distribution} : \sup_{t \in \mathbb{R}} \left| F(t) - \hat{F}_n(t) \right| \leq \sqrt{\log \left( \frac{2}{\alpha} \right) / (2n)} \right\}.
\]

By Massart’s tight constant for the Dvoretzky–Kiefer–Wolfowitz (DKW) inequality [Massart, 1990], the above is a finite-sample F-localization, i.e., \( \mathbb{P} \left[ F_G \in \mathcal{F}_n^{DKW}(\alpha) \right] \geq 1 - \alpha \) for all \( n \) and for any choice of \( p(\cdot | \mu) \) in (1).

This construction is rooted in empirical Bayes tradition. In his discussion, Robbins [1956] suggests that one could achieve asymptotically optimal empirical Bayes regret\(^4\) by choosing \( \hat{G} \) such that \( \sup_t | F_G(t) - \hat{F}_n(t) | \leq c_n \), with \( c_n \to 0 \) as \( n \to \infty \) and then using a plug-in estimate of the posterior mean \( \hat{\theta}_G(z) = \hat{\theta}_G(z) \); see also Donoho and Reeves [2013] for a modern refinement and implementation to achieve optimal empirical Bayes regret in the Gaussian problem.\(^5\)

The optimization problem (7) defining \( \hat{\theta}_G^+(z) \) (and similarly for \( \hat{\theta}_G^-(z) \)) can be solved using modern convex optimization solvers, as long as \( \mathcal{G} \) can be efficiently discretized (cf. Supplement D.2). The simplest case occurs when \( \mathcal{G} \) and \( \mathcal{F}_n(\alpha) \) may be represented by linear constraints. In that case, we may use the Charnes and Cooper [1962] transformation for linear-fractional programming and compute \( \hat{\theta}_G^+(z) \) by solving a linear program. As a concrete example (see (21) below for the general case), consider the prior class \( \mathcal{G} = \mathcal{P}(\mathcal{K}) \)

\(^4\)That is, to learn a denoiser \( \hat{\theta}(z) \) such that \( \mathbb{E}[(\mu - \hat{\theta}(Z))^2] \) converges to the mean squared error Bayes risk for estimating \( \mu \) in model (1).

\(^5\)Anderson [1969] suggested to use the DKW band to form confidence intervals for the mean of a \([0, 1] \)-valued random variable, as follows: one takes the minimum, resp. maximum of \( \int zdF(z) \) subject to \( F \in \mathcal{F}_n^{DKW}(\alpha) \) and \( F \) supported on \([0, 1]\); cf. Romano and Wolf [2000].
from (4) with \( \mathcal{K} = \{\mu_1, \ldots, \mu_p\} \) a finite set. Then, we can compute \( \hat{\theta}_n^+ (z) \) by solving the linear program:

\[
\begin{align*}
\text{maximize} \quad & \sum_{j=1}^p h(\mu_j)p(z \mid \mu_j)g_j \\
\text{subject to} \quad & \sum_{j=1}^p g_j = \zeta, \quad \sum_{j=1}^p p(z \mid \mu_j)g_j = 1, \quad g_j \geq 0, \quad j = 1, \ldots, p, \quad \zeta \geq 0, \\
& \sup_{t \in \mathbb{R}} \left| \sum_{j=1}^p g_j \int_{(\zeta, \infty]} p(\tilde{z} \mid \mu_j)d\lambda(\tilde{z}) - \zeta \hat{F}_n(t) \right| \leq \zeta \cdot \sqrt{\log (2/\alpha)/(2n)}.
\end{align*}
\]

The optimization variables \( \zeta, g_j \) in (17) have the interpretation \( \zeta = 1/f_G(z) \), \( g_j = \zeta \cdot P_G \{ \{\mu_j\} \} \) for \( j = 1, \ldots, p \), where \( G \in \mathcal{G} \).

### 2.1 Refined \( F \)-localization

The \( F \)-localization (16) is universal, and so works for any choice of likelihood \( p(\cdot \mid \mu) \) in (1). In some settings, however, it is also possible to construct \( F \)-localizations that are tailored towards properties of a specific choice of likelihood \( p(\cdot \mid \mu) \). We provide two such constructions in this section, the Gauss and \( \chi^2 \)-\( F \)-localizations. Empirically we observe that our tailored \( F \)-localizations outperform (16) in terms of the length of (7) for the empirical Bayes estimands we consider in our data examples (Section 5) and simulations (Section 6).

It is an interesting future theoretical question to determine how one should choose the \( F \)-localization to direct power towards specific empirical Bayes estimands and likelihoods. However such considerations are outside the scope of this work.

**Gauss-\( F \)-localization:** Our first tailored \( F \)-localization is applicable in the Gaussian empirical Bayes problem, i.e., (1) with \( Z \mid \mu \sim \mathcal{N}(\mu, \sigma^2) \) with known noise variance \( \sigma^2 > 0 \). In this case, the marginal density \( f_G \) is the convolution of \( G \) with the Gaussian density, and so is extremely smooth and can be estimated at quasi-parametric rates [Kim, 2014].

Here we build on this observation and seek to construct confidence intervals in terms of \( h(\mu_j)p(z \mid \mu_j)g_j \) and so is extremely smooth and can be estimated at quasi-parametric rates [Kim, 2014].

We form confidence bands using Efron’s multinomial bootstrap [Efron, 1979]. In the \( b \)-th bootstrap resample, we draw \( (W_{1b}, \ldots, W_{nb}) \sim \text{Multinomial}(n, (1/n, \ldots, 1/n)) \) and compute

\[
\hat{\epsilon}_n^b = \left\| \hat{f}_n^K - \hat{f}_n^{K,b} \right\|_{\infty,M}, \quad \hat{f}_n^{K,b}(z) = \frac{1}{nh_n} \sum_{i=1}^n W_i^K K\left( \frac{Z_i - z}{h_n} \right).
\]

The proposition below defines the Gauss-\( F \)-localization and proves its asymptotic validity.
Proposition 1 (Coverage of $\| \cdot \|_{\infty,M}$ localization in the Gaussian empirical Bayes problem). Assume (1) holds with $p(\cdot | \mu) = \mathcal{N}(\mu, \sigma^2)$. Let $\hat{c}_n(\alpha)$ be the $(1 - \alpha)$-quantile of $\hat{c}_n^\alpha$ with respect to the Bootstrap distribution in (19). Then,

$$
\mathcal{F}_n^{\text{Gauss}}(\alpha) = \left\{ F \text{ distribution with Lebesgue density } f : \| f - f_n^\alpha \|_{\infty,M} \leq \hat{c}_n(\alpha) \right\}
$$

asymptotically covers the true distribution at level $1 - \alpha$, in the sense of (6).

Similarly to (16), (20) also enforces linear constraints on $F_G$, $G \in \mathcal{G}$. Hence, if $\mathcal{G}$ may be represented using linear constraints, then an analogous linear program to (17) can be used to compute $\hat{\theta}^+ (\alpha)$. More generally, whenever $\mathcal{G}$ and $\mathcal{F}_n(\alpha)$ may be represented by linear constraints, then, by the Charnes and Cooper [1962] transformation, $\hat{\theta}^+ (\alpha)$ (7) may be computed by the following linear program,

$$
\hat{\theta}^+ (\alpha) = \sup \left\{ a_G(z) : f_G(z) = 1, G \in \mathcal{G}(\mathcal{F}_n(\alpha)), \zeta \geq 0 \right\}, \text{ where}
$$

$$
\zeta \mathcal{G}(\mathcal{F}_n(\alpha)) = \{ \zeta \cdot G : G \in \mathcal{G}(\mathcal{F}_n(\alpha)) \}, \quad f_G(z) = \int p(z | \mu) d\tilde{G}(\mu), \quad a_G(z) = \int h(\mu)p(z | \mu) d\tilde{G}(\mu).
$$

\textbf{\(\chi^2\)-F-localization:} Our second construction pertains to categorical likelihoods, i.e., when $Z_i \in \mathcal{Z}$ and $\mathcal{Z}$ is a finite set with $\# \mathcal{Z} = N + 1$, $N \in \mathbb{N}$. It is based on Pearson’s $\chi^2$ distance,

$$
\mathcal{F}_n^{\chi^2}(\alpha) = \left\{ F \in \mathcal{P}(\mathcal{Z}) \text{ with pmf } f : \sum_{z=0}^{N} \frac{(n f_n(z) - n f(z))^2}{n f(z)} \leq \chi^2_{N,1-\alpha} \right\}, \quad (22)
$$

where $f_n(z) = \# \{ Z_i = z \} / n$ is the empirical probability of $z$ and $\chi^2_{N,1-\alpha}$ is the $1 - \alpha$ quantile of the $\chi^2$ distribution with $N$ degrees of freedom. The validity of (22) in the sense of (6) follows from asymptotics in categorical data analysis [Agresti, 2013] and coverage will (approximately) hold in finite samples as long as $n \cdot f_G(z)$ is sufficiently large for all $z$.

The $\chi^2$-F-localization approach to inference in the empirical Bayes problem is not new. Lord and Cressie [1975] and Lord and Stocking [1976] considered the Binomial problem with $Z_i | \mu_i \sim \text{Binom}(N, \mu_i)$ for $N \in \mathbb{N}$ and $\mathcal{G} = \mathcal{P}([0,1])$. They suggested to form confidence intervals for the posterior mean $\theta_G(z) = \mathbb{E}_G[\mu | Z = z]$ by the F-localization approach (7) with $\mathcal{F}_n(\alpha)$ as in (22).\footnote{It may seem surprising that Lord and Cressie [1975] consider only the case of a Binomial likelihood, $\mathcal{G} = \mathcal{P}([0,1])$ and posterior mean estimands. One reason is that they devise a numerical scheme for computing (7) that relies on these choices, cf. Section 5.1.} For the F-localization intervals in the introductory Poisson example (Table 2), we used the $\chi^2$-F-localization with categories $0, \ldots, 4$ and grouping all observations $Z_i \geq 5$ as a sixth category.

For the $\chi^2$-F-localization, the Charnes-Cooper transformation is not directly applicable, yet the resulting optimization problem is quasi-convex and tractable; see Supplement D.1 for details.

3 Inference for linear functionals of $G$

Our next goal is to develop the AMARI approach for targeted inference about $\theta_G(z)$. However, as a preliminary for this task, we need to develop some general results on inference for
linear functionals $L(G)$ in the empirical Bayes problem; and this will be the focus of this Section. Formally, $L(\cdot)$ is a map from $\mathcal{G} \to \mathbb{R}$, that is linear in $G$, i.e.,

$$L \left( \lambda G + (1 - \lambda) \tilde{G} \right) = \lambda L(G) + (1 - \lambda) L(\tilde{G}) \quad \text{for all } G, \tilde{G} \in \mathcal{G}, \lambda \in [0, 1].$$

(23)

The main reason we are interested in confidence intervals for $L(G)$ is that we will use these as building blocks of the AMARI intervals for $\theta_G(z)$ in Section 4. Nevertheless, the class (23) includes functionals that are interesting in their own right. Some examples of linear functionals of interest include $L(G) = \mathbb{P}_G[\mu = 0]$ (the proportion of null effects), $L(G) = \mathbb{P}_G[\mu \geq 0]$ (the proportion of non-negative effects), and $L(G) = \mathbb{E}_G[\mu^2]$ (the second moment of the prior). Inference for $\mathbb{P}_G[\mu \geq 0]$ has been considered for example by Es and Uh [2005], Dattner et al. [2011], Efron [2016], Greenshtein and Itskov [2018] and Brennan et al. [2020] form confidence intervals for $L(G)$ using constructions that are analogous to the $F$-localization intervals developed in this work.\(^8\) Such $F$-localization intervals have simultaneous coverage over all possible choices of (linear) functionals, but can be overly wide for a specific linear functional $L(G)$. Instead, the intervals we develop in this section are targeted towards a specific linear functional and so can be shorter.

### 3.1 Affine minimax inference for linear functionals

Our key idea is to estimate the linear functional $L(G)$ of $G$ with an affine estimator, i.e., an estimator of the form $\hat{L} = \sum Q(Z_i)/n$ (12). The class of affine estimators is convenient because it enables explicit control of the worst-case bias (14) and it is broad enough to include kernel density estimators as in (18) and the Fourier estimators of Butucea and Comte [2009], Pensky [2017]. The latter provably attain minimax optimal rates for estimation of linear functionals of $G$ in the empirical Bayes problem, when $p(\cdot | \mu)$ is a smooth location family.

We choose $Q(\cdot)$ in a purely computational and data-driven way. To do so, we first construct a pilot $F$-localization $\mathcal{F}_n = \mathcal{F}_n(\alpha_n)$ with $\alpha_n \to 0$ and a pilot estimate $\tilde{f}_n(\cdot)$ of the marginal density $f_G(\cdot)$ (5). $\mathcal{F}_n$ could be, for example, any of the $F$-localizations described in Section 2. For $\tilde{f}_n(\cdot)$ we use the Kolmogorov-Smirnov minimum distance estimator, that was studied in the empirical Bayes problem by Deely and Kruse [1968] and Heinrich and Kahn [2018]:\(^9\)

$$\tilde{f}_n(z) = f_{\hat{G}_n}(z), \quad \hat{G}_n \in \operatorname{argmin}_{G \in \mathcal{G}} \left\{ \sup_{t \in \mathbb{R}} \left| F_G(t) - \hat{F}_n(t) \right| \right\}.$$

(24)

$\mathcal{F}_n, \tilde{f}_n$ enable us to navigate a bias-variance trade-off in choosing $Q(\cdot)$. $\tilde{f}_n$ facilitates estimating the variance of any fixed $Q(\cdot)$, through the quadratic form (in $Q(\cdot)$),

$$\text{Var}_{\tilde{f}_n}[Q] = \int Q^2(z) \tilde{f}_n(z) d\mu(z) - \left( \int Q(z) \tilde{f}_n(z) d\mu(z) \right)^2.$$

(25)

$\mathcal{F}_n$ facilitates computation of the $F$-localized worst-case bias (14) of $Q(\cdot)$ among all priors $G_n \in \mathcal{G}_n := \mathcal{G}(\mathcal{F}_n)$ (8). With these two ingredients, we choose $Q(\cdot) = Q_n(\cdot)$ in a data-driven way, by minimizing the localized worst-case bias, subject to controlling the estimated variance of $Q(\cdot)$:

$$\min_{Q(\cdot) \in \mathbb{R}} \sup_{G_n \in \mathcal{G}_n} \left\{ \text{Bias}_{\mathcal{F}}[Q, L]^2 \right\} \quad \text{s.t.} \quad \frac{1}{n} \text{Var}_{\tilde{f}_n}[Q] \leq \Gamma_n, \quad Q(\cdot) \text{ constant in } \mathbb{R} \setminus [-M, M].$$

(26)

\(^8\)Solving (7) for linear functionals is typically more straightforward compared to ratio functionals (10). For example, the Charnes and Cooper [1962] transformation is not required.

\(^9\)Case-by-case constructions would be possible here too or one could use the NPMLE.
Here, \( M > 0 \) is a (large) constant, and we restrict attention to functions \( Q(\cdot) \) that are constant outside the interval \([-M, M]\) to avoid regularity issues at infinity and so that our inference is not unduly sensitive to outliers. \( \Gamma_n > 0 \) is a hyperparameter that controls the bias-variance trade-off. For smaller values of \( \Gamma_n \), we enforce that the \( Q(\cdot) \) solving (26) takes on smaller values of \( \text{Var}_f_n [Q] \), at the cost of potentially increasing the \( F \)-localized worst-case bias. We explain how we choose \( \Gamma_n \) below, after first outlining how we solve (26).

First, to (formally) enforce \( Q(\cdot) \) to be constant outside \([-M, M]\) as in (26), we (formally) censor \( Z \) in (1) and define

\[
Z^M = Z \text{ if } Z \in [-M, M], \quad Z^M = \varsigma \text{ if } Z < -M, \quad Z^M = \vartheta \text{ if } Z > M.
\]  

(27)

In view of (27), we only need to define \( Q(\cdot) \) on the set \( \{ \varsigma, \vartheta \} \cup [-M, M] \). \( Z^M \) has conditional density \( p^M(z \mid \mu) = p(z \mid \mu) \) for \( z \in [-M, M] \), \( p^M(\varsigma \mid \mu) = \int_{(-\infty,-M)} p(z \mid \mu) d\lambda(z) \) and \( p^M(\vartheta \mid \mu) = \int_{(M,\infty)} p(z \mid \mu) d\lambda(z) \) with respect to the measure \( \lambda^M = \delta_\varsigma + \delta_\vartheta + \lambda_{[-M,M]} \), where \( \delta_x \) is a point mass at \( x \). The marginal density \( f^M_G \) (and estimated density \( \hat{f}^M_n \)) of \( Z^M \) is supported on \( \{ \varsigma, \vartheta \} \cup [-M, M] \) with \( f^M_G(\varsigma) = \int_{(-\infty,-M)} f_G(z) d\lambda(z), f^M_G(\vartheta) = \int_{(M,\infty)} f_G(z) d\lambda(z), \) and similarly for \( \vartheta \) and \( z \in [-M, M] \).

To solve (26) we build upon a construction of Donoho [1994], who formalizes a powerful heuristic due to Charles Stein on hardest one-dimensional subproblems. We refer the interested reader to Supplement B.2 for details and proofs in the context of our application and also to Donoho and Liu [1989], Donoho [1994], Low [1995], Armstrong and Kolesár [2018] and references therein. The consequence of interest here is that to solve (26), it suffices to solve the following surrogate optimization problem:

\[
\begin{align*}
\sup & \left\{ L(G_1) - L(G_{-1}) : G_1, G_{-1} \in \mathcal{G}_n, \int \frac{(f^M_{G_1}(z) - f^M_{G_{-1}}(z))^2}{\hat{f}^M_n(z)} d\lambda^M(z) \leq \frac{\delta^2}{n} \right\} ,
\end{align*}
\]  

(28)

The surrogate optimization problem is parameterized by another hyperparameter \( \delta^{10} \) and it is a second order conic program (SOCP) [Boyd and Vandenberghe, 2004] that is tractable by modern conic optimizers, such as MOSEK [ApS, 2020].\(^{11}\) The value of the supremum in (28) is called the modulus of continuity \( \omega_n(\delta) \) at \( \delta > 0 \). We say that the modulus problem (28) is solvable at \( \delta > 0 \) if there exist \( G^\delta_1, G^\delta_{-1} \in \mathcal{G}_n \) such that \( |L(G^\delta_1)|, |L(G^\delta_{-1})| < \infty \) and

\[
L(G^\delta_1) - L(G^\delta_{-1}) = \omega_n(\delta), \quad n \cdot \int \frac{(f^M_{G^\delta_1}(z) - f^M_{G^\delta_{-1}}(z))^2}{\hat{f}^M_n(z)} d\lambda^M(z) = \delta^2,
\]  

(29)

and we call \( G^\delta_1, G^\delta_{-1} \) solutions of \( \omega_n(\delta) \). \( G^\delta_1, G^\delta_{-1} \) are close observationally, that is their marginal distributions have distance at most \( \delta/\sqrt{n} \) in terms of the \( \text{pseudo} \cdot \chi^2 \)-distance in (28), and they exhibit the largest separation of the linear functional \( L(G) \). These two priors determine the worst-case optimal \( Q(\cdot) \) in (26) at a specific value of \( \Gamma_n \) that depends on \( \delta \), i.e., \( \Gamma_n = \Gamma_n(\delta) \). Let \( G^\delta_0 = (G^\delta_1 + G^\delta_{-1})/2 \) and define \( Q(\cdot ; \delta) = Q(\cdot ; \delta, \omega_n(\delta), G^\delta_1, G^\delta_{-1}) \) as,

\[
\begin{align*}
\frac{n \cdot \omega_n(\delta)}{\delta} & \left\{ \frac{f^M_{G^\delta_1}(\cdot) - f^M_{G^\delta_{-1}}(\cdot)}{\hat{f}^M_n(\cdot)} - \int \frac{f^M_{G^\delta_1}(z) - f^M_{G^\delta_{-1}}(z)}{\hat{f}^M_n(z)} f^M_G(z) d\lambda^M(z) \right\} + L(G^\delta_0). \quad (30)
\end{align*}

\(^{10}\delta \) maps to the hyperparameter \( \Gamma_n \) of (26), see below.
\(^{11}\) See Supplement E for implementation details including discretization considerations.
Algorithm 1: Affine minimax confidence intervals for linear functionals \( L(G) \).

1. Form a pilot estimate \( \hat{f}_n(\cdot) \) of the marginal density \( f_G(\cdot) \) as in (24) and a pilot F-localization (6) \( F_n = \mathcal{F}_n(\alpha_n) \).
2. Choose \( \delta_n \) as in (31).
3. Solve the modulus problem (28) at \( \delta_n \) and use the solution to compute \( Q(\cdot) \) as in (30), as well as its worst-case bias \( \tilde{B} \).
4. Form the estimate \( \hat{L} \) of \( L(G) \) as in (12) and its estimated variance \( \hat{V} \) as in (13).
5. Form bias-aware confidence intervals as in (15).

Theorem 2 (Central limit theorem for affine minimax estimator). Assume that for all \( G \in \mathcal{G} \), the linear functional \( L(G) \) is well-defined with \( \sup_G |L(G)| < \infty \) and that \( f^M_G(\cdot) \in L^2(\lambda^M) \). Furthermore, assume that,

A. For each \( n \), the modulus problem (28) has solutions \( G^1_n, G^2_n \) at \( \delta_n \in [\delta^1, \delta^2] \), where \( \delta^1, \delta^2 \in (0, \infty) \) are fixed (i.e., do not change with \( n \)).

B. \( G \) lies in the convex set of distributions \( \mathcal{G} \) \( (G \in \mathcal{G}) \).

C. There exists \( \eta > 0 \) s.t. \( \inf_{z \in [-\theta, \theta]} \{ f^M_G(z) \} > \eta \).

D. \( \hat{f}_n^M \) is a density (\( f^M_G(z) d\lambda^M(z) = 1 \) and \( f^M_n \geq 0 \)). It holds that \( \mathbb{P}_G[A_n] \to 1 \), where \( A_n \) is the event on which,

\[
\begin{align*}
\left\| f^M_n(\cdot) - f^M_G(\cdot) \right\|_{\infty} &\leq c_n, \\
\left\| f^M_{G^2_n}(\cdot) - f^M_{G^1_n}(\cdot) \right\|_{\infty} &\leq c_n \quad F_G \in \mathcal{F}_n,
\end{align*}
\]

for a deterministic sequence of constants \( c_n \to 0 \) as \( n \to \infty \).

E. \( \mathcal{F}_n \) and \( f^M_n \) are independent of \( Z_1, \ldots, Z_n \).

---

\(^{12}\) That is, \( \omega_n'(\delta) \) satisfies \( \omega_n(\delta) \leq \omega_n(\delta) + \omega_n'(\delta)(\delta - \delta) \) for all \( \delta > 0 \). Such an element exists, because \( \omega_n(\delta) \) is concave in \( \delta > 0 \) [Rockafellar, 1970]. We provide details in Supplement B.1.

\(^{13}\) In our implementation, we use the concrete choice \( \Delta = \Delta_{\text{grid}} := \{0.2, 0.7, \ldots, 6.2, 6.7\} \).
Then, letting $Q(\cdot) = Q(\cdot; \delta_n)$ (30) and $\hat{L}$ (12) the affine estimator of the linear functional $L(G)$, it holds that,

\[
\left( \hat{L} - L(G) - \text{Bias}_G(Q, L) \right) \overset{D}{\to} \mathcal{N}(0, 1), \quad \mathbb{P}_G[|\text{Bias}_G(Q, L)| \leq \hat{B}] \to 1 \text{ as } n \to \infty,
\]

where $\text{Bias}_G(Q, L)$ is defined in (14). It follows that the intervals (15) provide asymptotically correct coverage of the target $L(G)$, i.e., $\liminf_{n \to \infty} \mathbb{P}_G[L(G) \in \mathcal{I}_\alpha] \geq 1 - \alpha.\rfloor^{14}$

We emphasize that $Q(\cdot)$ changes with $n$, and so, the central limit theorem above is that of a triangular array. The statistical assumption driving Theorem 2 is Assumption B, namely that model (1) holds with $G \in G$. Using a good choice of prior class $G$ is critical, and we discuss this choice further in Section 8. The rest of the assumptions are under control of the analyst and may be verified before any data analysis is conducted. Assumption A guarantees that we can solve (26) by convex programming, cf. Supplement B.2, and Assumption C is an overlap condition. Assumption D concerns the quality of the pilot localization $F_n$ and pilot density estimator $f_n$, while Assumption E requires that both $F_n$ and $f_n$ are independent of $Z_1, \ldots, Z_n$. Assumption E holds if we use sample-splitting. Following Hajek [1962] and Bickel [1982] we demonstrate that it suffices to retain an asymptotically vanishing fraction of the $Z_i$ to estimate $F_n, \hat{f}_n$.

**Proposition 3.** Suppose we use $k = k_n$ samples from model (1) to construct $F_n(\alpha_n)$, $\hat{f}_n$ and the remaining $n - k_n$ samples for step 4 of Algorithm 1. Suppose further that $k/n \to 0$, $\alpha_n \to 0$ and $k \cdot \alpha_n \to \infty$ as $n \to \infty$. Then, Assumption D of Theorem 2, is satisfied in the following cases:

1. **Gaussian likelihood:** $Z \mid \mu \sim \mathcal{N}(\mu, \sigma^2)$, $\sigma^2 > 0$, and we use the DKW (16) or the Gauss-F-localization (20) and $\hat{f}_n$ is the minimum distance estimator (24).

2. **Binomial or Poisson likelihood:** $Z \mid \mu \sim \text{Binom}(N, \mu)$ or $\sim \text{Poisson}(\mu)$, $G$ is not degenerate\footnote{Although the result stated only holds elementwise, we can obtain a uniform statement in the sense of, e.g., Robins and Van Der Vaart [2006] by adding slightly more constraints on the class $G$. Specifically, consider $G^\eta = \{G \in G : \inf_{\eta \in \{0,1\}} \liminf_{n \to \infty} \mathbb{P}_G[L(G) \in \mathcal{I}_\alpha] \geq 1 - \alpha, \text{ provided that } \mathbb{P}_G[A_n] \to 1 \text{ uniformly in } G \in G^\eta, \text{ where } A_n \text{ has been defined in the statement of Theorem 2.} \}$ and we use the DKW (16) or the $\chi^2$-F-localization (22) (wherein, in the Poisson case, we treat all observations $> M$ as a single category) and $\hat{f}_n$ is the minimum distance estimator (24).

In practice, we use the full data twice and do not sample-split; we have not observed any overfitting or loss of coverage thereby.

### 4 Pointwise confidence intervals with AMARI

In this section we return to our main task of forming confidence intervals for empirical Bayes estimands $\theta_G(z)$ and discuss our second approach, AMARI (Affine Minimax Anderson–Rubin Intervals). In contrast to the $F$-localization intervals, the AMARI intervals are targeted towards a specific empirical Bayes estimand and have a pointwise (rather than...
simultaneous) coverage guarantee. The upshot is that AMARI intervals can be substantially shorter.

The starting point for AMARI is (10), i.e., the fact that we can write the empirical Bayes estimand \( \theta_G(z) \) as a ratio of linear functionals of \( G, a_G(z)/f_G(z) \). Next, fix \( c \in \mathbb{R} \) and write, \( L(G) := \theta_{G}^{lin}(z;c) := a_G(z) - cf_G(z) \) as in (11). \( L(\cdot) \) is a linear functional of \( G \), as defined in (23). The Affine Minimax component of the AMARI acronym refers to the application of the affine minimax approach of Section 3.1 to form confidence intervals \( I^\alpha_{\theta_G}(z;c) \) for \( \theta_{G}^{lin}(z;c) \), treating the latter as a generic linear functional \( L(G) \). The Anderson-Rubin component of AMARI enables us to construct confidence intervals for \( \theta_G(z) \) by lifting our intervals for \( \theta_{G}^{lin}(z;c) \) following the approach of Noack and Rothe [2019], who in turn build upon Anderson and Rubin [1949] and Fieller [1940, 1954].

The following Corollary captures the basic idea of our approach:

**Corollary 4.** Suppose that for each \( c \in \mathbb{R} \), we construct a confidence interval \( I^\alpha_{\theta_G}(z;c) \) for \( \theta_{G}^{lin}(z;c) \) (11) following Algorithm 1. Suppose furthermore that the assumptions of Theorem 2 hold for the linear functional \( L(G) = \theta_{G}^{lin}(z;c^*) \), where \( c^* = \theta_G(z) \). Then,

\[
\lim_{n \to \infty} \mathbb{P}_G [\theta_G(z) \in \mathcal{S}_\alpha(z)] \geq 1 - \alpha, \quad \mathcal{S}_\alpha(z) = \{ c \in \mathbb{R} : 0 \in I^\alpha_{\theta_G}(z;c) \}.
\]

**Proof.** By definition of \( c^* \) and \( \mathcal{S}_\alpha(z) \), it holds that \( \theta_G(z) \in \mathcal{S}_\alpha(z) \iff 0 \in I^\alpha_{\theta_G}(z;c^*) \). On the other hand, \( \theta_{G}^{lin}(z;c^*) = a_G(z) - c^* f_G(z) = a_G(z) - (a_G(z)/f_G(z)) f_G(z) = 0 \), i.e., \( \mathbb{P}_G [\theta_G(z) \in \mathcal{S}_\alpha(z)] = \mathbb{P}_G [\theta_{G}^{lin}(z;c^*) \in I^\alpha_{\theta_G}(z;c^*)] \). We conclude by Theorem 2. \( \square \)

We provide Corollary 4 for intuition. However, the confidence set \( \mathcal{S}_\alpha(z) \) from (33) has some disadvantages. First, \( \mathcal{S}_\alpha(z) \) will in general not be an interval. Second, computing the interval \( I^\alpha_{\theta_G}(z;c) \), even for a single \( c \), is computationally demanding and requires the solution of (28) along a grid of \( \delta \) values (31), and so, computing \( I^\alpha_{\theta_G}(z;c) \) for ‘all’ values of \( c \) is not computationally tractable. Instead, in our actual implementation of AMARI, which we describe in Section 4.1 below, we use an ‘accelerated’ Anderson-Rubin procedure that is computationally streamlined (we only need to form \( I^\alpha_{\theta_G}(z;c) \) for two values of \( c \)), and leads to a confidence interval \( I_\alpha(z) \) for \( \theta_G(z) \), rather than a confidence set.\(^{16}\)

### 4.1 Implementation of Anderson-Rubin inversion for AMARI

We now describe our actual implementation of AMARI. The key intuition is that we start with a preliminary interval such that \( \theta_G(z) \in [c^\ell, c^u] \) with high probability, and then find the affine minimax \( Q^\ell, Q^u \) (26) for \( \theta_{G}^{lin}(z;c^\ell) \), resp. \( \theta_{G}^{lin}(z;c^u) \). Then, for any other \( c \), say \( c = \kappa c^\ell + (1 - \kappa) c^u, \kappa \in (0, 1) \), instead of resolving (26), we use \( Q^c = \kappa Q^\ell + (1 - \kappa) Q^u \). The variance of \( Q^c \) then can be directly computed from the covariance of \( Q^\ell, Q^u \), and their individual variances, while the worst-case bias of \( Q^c \) can be upper bounded by the convex combination \( \tilde{B}^c = \kappa \tilde{B}^\ell + (1 - \kappa) \tilde{B}^u \) of the worst-case biases of \( Q^\ell \) and \( Q^u \). Algorithm 2 describes all steps of AMARI. Step 4 can be computed efficiently using grid search, since the evaluation of \( I_\alpha(z;c) \) for different values of \( c \in [c^\ell, c^u] \) is fast.

As a consequence of Theorem 2, we can now prove, that the AMARI confidence intervals asymptotically cover the empirical Bayes estimand \( \theta_G(z) \).

\(^{16}\)The ‘accelerated’ Anderson-Rubin approach could be fruitful in other settings; for example it could allow replacing the local linear estimators in the fuzzy regression discontinuity approach of Noack and Rothe [2019] by the affine minimax estimators of Imbens and Wager [2019].
Algorithm 2: AMARI confidence intervals for empirical Bayes estimands $\theta_G(z)$

1. Construct a pilot $F$-localization $F_n = F_n(\alpha_n)$ at level $\alpha_n$, as in Step 1 of Algorithm 1. Let $[c^L, c^U]$ be the $F$-localization interval (7) for $\theta_G(z)$ based on $F_n$.
2. Apply Algorithm 1 to the linear functionals $\theta_G^{\text{lin}}(z; c^L)$ and $\theta_G^{\text{lin}}(z; c^U)$ defined in (11).
   Let $Q^L, Q^U$ be the corresponding affine minimax kernels, $\hat{L}^c, \hat{L}^u$ the point estimates (12), $\hat{B}^c, \hat{B}^u$ the worst-case biases (14), $\hat{V}^L, \hat{V}^u$ the variances (13) and $\text{Cov} \left[ \hat{L}^c, \hat{L}^u \right] = \frac{1}{n(n-1)} \sum_{i=1}^{n} Q^L(Z_i)Q^U(Z_i) - \left( \sum_{i=1}^{n} Q^L(Z_i) \right) \left( \sum_{i=1}^{n} Q^U(Z_i) \right) / n$.
3. For $c = \kappa c^L + (1 - \kappa)c^U$, $\kappa \in [0, 1]$, let $\hat{L}^c = \kappa \hat{L}^L + (1 - \kappa)\hat{L}^U$, $\hat{B}^c = \kappa \hat{B}^L + (1 - \kappa)\hat{B}^U$, $\hat{V}^c = \kappa^2 \hat{V}^L + (1 - \kappa)^2 \hat{V}^U + 2\kappa(1 - \kappa)\hat{Cov} \left[ \hat{L}^c, \hat{L}^c \right]$ and $\hat{C}_\alpha(z; c) = \hat{L}^c + t_\alpha(\hat{B}^c, \hat{V}^c)$ as in (15).
4. Report the interval $I_\alpha(z) = [\inf C, \sup C] \cap [c^L, c^U]$, $C = \{c \in [c^L, c^U]: 0 \in \hat{C}_\alpha(z; c)\}$.

Theorem 5 (Coverage of AMARI). Consider the confidence intervals constructed in Algorithm 2. Suppose the pilot $F$-localization interval endpoints $c^L, c^U$ are finite for all $n$ and let the assumptions of Theorem 2 hold for $L(G) = \theta_G^{\text{lin}}(z; c^L)$ and $L(G) = \theta_G^{\text{lin}}(z; c^U)$.
Furthermore, assume that $Q^L$ and $Q^U$ are not perfectly anticorrelated, i.e., there exists $\varepsilon > 0$, such that, $P_G[\text{Cov}[L^c, L^u]/(V^L V^U)^{1/2} \leq -1 + \varepsilon] \to 1$ as $n \to \infty$. Then, $\lim_{n \to \infty} \inf P_G[\theta_G(z) \in I_\alpha(z)] \geq 1 - \alpha$.

The additional assumption of Theorem 5 on the correlation between $Q^L$ and $Q^U$ is mild and can be verified from the data at hand; in applications we typically find a positive correlation.

5 Empirical applications

In this section we apply the $F$-localization intervals and AMARI in the context of two applications; one in education and one in genomics.

5.1 Predicting student ability in psychometric tests

Lord and Cressie [1975] studied a dataset of scores by $n = 12,990$ students on a psychological test with $N = 20$ multiple choice questions (5 choices per question) and posited that $Z_i | \mu_i \sim \text{Binom}(20, \mu_i)$, where $\mu_i$ is the ‘true-score’ of student $i$ [Lord, 1969]. Lord and Stocking [1976] were working for the Educational Testing Service (ETS) at the time and the following motivation for confidence intervals of the posterior mean $\theta_G(z) = E_G[\mu | Z = z]$ with the property (2) seems plausible: if the ETS were to use an estimate $\hat{\theta}(z)$ for student assessment, then it would be important to account for the uncertainty in estimating the regression function $E_G[\mu | Z]$ due to both variability (which can be substantial even for large sample sizes, e.g., $n = 12,990$ in this example) and partial identification.

The empirical frequencies $\hat{f}_n(z) = \# \{Z_i = z\} / n$ of test scores are shown in Figure 1a). Panel b) shows three 95% confidence intervals for $\theta_G(z)$ that make no assumptions on $G$.

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17 The proof of Theorem 2 uses triangular array asymptotics, and so, the linear functional $L(G)$ may depend on $n$.
18 In the Binomial empirical Bayes problem ($p_i(\cdot | \mu) = \text{Binom}(N, \mu_i), N \in N > 0$) and without further restrictions on $G$, the posterior mean $\theta_G(z) = E_G[\mu | Z = z]$ is only partially identified and cannot be consistently estimated, even as $n \to \infty$. We discuss this issue further in Section 7.2.
i.e., \( G \in \mathcal{G} = \mathcal{P}([0, 1]) \) (4). The \( \chi^2\)-F-localization intervals (22) were developed by Lord and Cressie [1975], Lord and Stocking [1976]. We computed these intervals by grouping the lowest scores \( z = 0 \) and \( z = 1 \) together (to ensure the \( \chi^2 \)-interval (22) has the right coverage) and then used the parametric convex programming approach from Supplement D.1 with the discretization \( \mathcal{P}([0, 1]) \approx \mathcal{P}(\mathcal{K}) \) and \( \mathcal{K} \) an equidistant grid on \([0, 1]\) with 300 points. The intervals agree with the ones reported in Lord and Cressie [1975]. The latter used a numerical optimization routine due to Martha Stocking that leveraged the fact that in the Binomial empirical Bayes problem (\( N = 20 \)) and when \( \theta_G(z) \) is the posterior mean, then the worst-case \( G \) in (7) must be discrete and supported on at most 11 points. We also report intervals based on the DKW-F-localization, as well as the AMARI intervals (with pilot \( \chi^2\)-F-localization using \( \alpha_n = 0.01 \)).

We see that, for low scores \( z \) there is substantial uncertainty. For students with score \( z = 0 \), one could predict their true score as being almost 0, or one could predict their true score as better than random guessing (1/5); and both predictions would be consistent with the data. On the other hand, for intermediate values of \( z \) the intervals become substantially shorter. In this example we also observe that the DKW-F-Localization intervals are overly wide. The AMARI intervals are shorter than the \( \chi^2 \)-intervals for most \( z \); at the cost of not having simultaneous coverage.

5.2 Identifying genes associated with prostate cancer

Our next dataset is the ‘Prostate’ dataset [Efron, 2010, Singh et al., 2002], by now a classic dataset used to illustrate empirical Bayes principles. The dataset consists of Microarray expression levels measurements for \( m = 6033 \) genes of 52 healthy men and 50 men with prostate cancer. For each gene, a t-statistic \( T_i \) is calculated (based on a two-sample equal variance t-test) and z-scores are calculated as \( Z_i = \Phi^{-1}(F_{100}(T_i)) \), where \( \Phi \) is the standard normal CDF and \( F_{100}(\cdot) \) is the CDF of the t-distribution with 100 degrees of freedom. We posit \( Z_i | \mu_i \sim N(\mu_i, 1) \), where \( \mu_i \) is the standardized effect size and is expected to be close to null for most genes [Efron et al., 2001]. We seek to form confidence intervals for two
empirical Bayes estimands.

Our first estimand of interest is the posterior mean, \( \theta_G(z) = \mathbb{E}_G [\mu \mid Z = z] \), which could be used to denoise the noisy measurements \( Z_i \) by \( \theta_G(Z_i) \) and to provide estimates of \( \mu_i \) that are (nearly) immune to selection bias [Efron, 2011]. The standard empirical Bayes approach provides point estimates of these oracle quantities by sharing information across genes, but the empirical Bayes estimation error may be rather opaque and so it is not clear to what extent the estimates \( \hat{\theta} [\mu_i \mid Z_i] \) eliminate selection bias. Our confidence intervals attach a measure of uncertainty to the estimation of \( \theta_G(z) \).

Second, we consider the local false sign rate \( \theta_G(z) = \mathbb{P}_G [\mu z \leq 0 \mid Z = z] \), which measures the posterior probability that the sign of an observed signal \( Z_i \) disagrees with the sign of the true effect \( \mu_i \). Local false sign rates provide a principled approach to multiple testing without assuming that the distribution \( G \) of the effect sizes \( \mu_i \) is spiked at 0, and form an attractive alternative to the local false discovery rate, \( \text{lfdr}(z) = \mathbb{P}_G [\mu = 0 \mid Z = z] \), without requiring a sharp null hypothesis [Stephens, 2016, Zhu et al., 2018]. Inferential emphasis is thus placed on whether we can reliably detect the direction of an effect. Below, for ease of visualization, we report results on the (substantively equivalent) quantity \( \theta_G(z) = \mathbb{P}_G [\mu \geq 0 \mid Z = z] \) (instead of \( \mathbb{P}_G [\mu z \leq 0 \mid Z = z] \)) so that the resulting confidence bands are monotonic in \( z \).

While both the posterior mean and the local false sign rate are often reported in the analysis of genomics datasets [Stephens, 2016, Zhu et al., 2018], the statistical difficulty of estimating them in the Gaussian empirical Bayes model is vastly different. The posterior mean can be estimated at the quasi-parametric rate \( \log(n)^{3/4}/\sqrt{n} \) over the class \( \mathcal{G} \) of priors with Lebesgue density and finite first moment [Matias and Taupin, 2004]. Meanwhile, minimax point estimates for the local false sign rate over Sobolev classes of priors converge at extremely slow rates, e.g., polynomial in \( 1/\log(n) \) [Butucea and Comte, 2009, Pensky, 2017]. Consequently, we expect that our confidence intervals for the posterior mean will be substantially shorter than the ones for the local false sign rate.

We form 95% confidence intervals using 3 \( \times \) 2 methods, namely the DKW (16) and Gauss (20) (with \( M = 3 \)) \text{F-localization} methods, as well as AMARI (with Gauss-\text{F-localization} pilot, \( \alpha_n = 0.01 \)), each applied based on two specifications for \( G \). First, we consider the Gaussian location mixture

\[
\mathcal{LN}(\tau^2, \mathcal{K}) = \left\{ G \text{ distribution} : \frac{dG(\mu)}{d\lambda_{\text{Leb}}} = \int \frac{1}{\tau} \varphi \left( \frac{\mu - u}{\tau} \right) d\Pi(u), \Pi \in \mathcal{P}(\mathcal{K}) \right\},
\]

where \( \tau > 0, \mathcal{K} \subset \mathbb{R} \), \( \lambda_{\text{Leb}} \) is the Lebesgue measure and \( \varphi \) is the standard Gaussian density. This is a natural choice of smooth priors in the Gaussian empirical Bayes problem [Magdal and Zeger, 1996, Cordy and Thomas, 1997] and the noise level \( \tau \) provides an interpretable way of specifying the smoothness of the priors. Here we make the concrete choice \( \mathcal{G} = \mathcal{LN}(0.25^2, [-3.3]) \).

Second, we consider Gaussian scale mixtures with mode at zero, discretized as suggested by Stephens [2016], i.e., for \( 0 < \tau_e < \tau_u \) and \( \eta > 1 \):

\[
\mathcal{SN}(\tau_e, \tau_u, \eta) = \left\{ G \text{ dbn} : \frac{dG(\mu)}{d\lambda_{\text{Leb}}} = \int \varphi \left( \frac{\mu}{\tau} \right) d\Pi(\tau), \Pi \in \mathcal{P}(\{\tau_e, \eta \cdot \tau_e, \ldots, \tau_u\}) \right\}.
\]

We take \( \tau_e = 0.1, \tau_u = 10.7 > (\max_i \{ Z_i^2 - 1 \})^{1/2} \) and \( \eta = 1.1 \). Stephens [2016] argues that the unimodal Gaussian scale mixture leads to more accurate inference provided that it holds; our intervals allow a quantitative assessment of this claim for any analyzed dataset.

\[\text{We discretize it as } \mathcal{LN}(0.25^2, \mathcal{K}) \text{ with } \mathcal{K} \text{ an equidistant grid on } [-3, -3] \text{ of step size equal to } 0.05.\]

\[\text{Stephens [2016] uses a coarser grid with } \eta = \sqrt{2}.\]
Figure 2: Empirical Bayes inference for the Prostate dataset [Efron, 2010, Singh et al., 2002]. a) Empirical distribution of the $Z_i$ and DKW-F-localization band. b) Histogram of the $Z_i$ and Gauss-F-localization. c) 95% confidence intervals for the posterior mean $E_G [\mu | Z = z]$ assuming $G$ is a Gaussian location, resp. d) scale mixture. e) 95% confidence intervals for the local false sign rate $E_G [\mu \geq 0 | Z = z]$ assuming $G$ is a Gaussian location, resp. f) scale mixture.

Figure 2 shows the results of the analysis. For the posterior mean, we observe that all intervals suggest that many effects are close to null and so there is substantial shrinkage towards zero. $E_G [\mu | Z = z]$ is almost flat in the interval $[-1.5, 1.5]$; and we can say so with confidence. All four $F$-localization bands are quite similar, while AMARI leads to substantially shorter intervals, and the improvement is more noticeable for $G = \mathcal{LN}(0.25^2, [-3.3])$. For the local false sign rate, the intervals are long when assuming $G \in \mathcal{LN}(0.25^2, [-3.3])$. The DKW-F-localization intervals perform worst, while the AMARI and Gauss-F-localization intervals perform comparably (with AMARI leading to shorter intervals only for more extreme values of $z$). As explained above, long confidence intervals are expected in this case. On the other hand, if we are willing to assume that $G$ is a Gaussian scale mixture with mode at 0, then inference for the local false sign rate is much more precise, exactly as argued by Stephens [2016]. However, the assumption is strong, and for example it implies that the local false sign rate at 0 is equal to 1/2; all intervals proposed here have vanishing length in that case.
6 Simulations

The setting of our simulations is similar to the Prostate data example in Section 5.2. We consider model (1) with $Z \mid \mu \sim \mathcal{N}(\mu, 1)$, $n = 5000$, and two different data-generating priors,

$$G_{\text{Spiky}} = 0.4\mathcal{N}(0, 0.25^2) + 0.2\mathcal{N}(0, 0.5^2) + 0.2\mathcal{N}(0, 1) + 0.2\mathcal{N}(0, 2^2),$$
$$G_{\text{NegSpiky}} = 0.8\mathcal{N}(-0.25, 0.25^2) + 0.2\mathcal{N}(0, 1).$$

$G_{\text{Spiky}}$ was used in the simulations of Stephens [2016] and is a unimodal symmetric prior centered at 0, while $G_{\text{NegSpiky}}$ is a prior with strong peak just to the left of zero, reflecting many slightly negative effects. Figure 3 shows the Lebesgue densities of the priors and the induced marginal densities $f_G(z)$.

We seek to form 95% confidence intervals for the posterior mean and the local false sign rate using the DKW-$F$-localization, Gauss-$F$-localization ($M = 4$) and AMARI approaches (with pilot Gauss-$F$-localization, $\alpha_n = 0.01$) and $G$ equal to the Gaussian location mixture class $\mathcal{LN}(0.25^2, [-4, 4])$.

We also consider an additional plug-in baseline [Efron, 2016, Narasimhan and Efron, 2020]. We estimate $\hat{G}$ by (penalized) maximum likelihood over a flexible exponential family with a natural spline (5 degrees of freedoms) as the sufficient statistic and base measure $U[-4, 4]$. We then obtain $\hat{\theta}(z)$ by applying Bayes rule with prior $\hat{G}$. As is standard in the literature, this baseline constructs confidence intervals for $\theta_G(z)$ using the delta method, which captures the variance of $\theta(z)$ but not its bias. Such confidence intervals are only guaranteed to cover $\theta_G(z)$ in the presence of undersmoothing or if the parametric specification is correct. See Supplement F for implementation details of this plug-in baseline.

Figure 4 shows the results of the simulations for the posterior mean, averaged over 400 Monte Carlo replicates. The length of the different confidence intervals is qualitatively similar to what we observed in Figure 2. The log-spline intervals are shortest; however they do not achieve nominal coverage, while all other methods do. The pointwise coverage of the $F$-localization intervals is close to 100%, while the coverage of AMARI is closer to the nominal 95%. The simultaneous coverage of AMARI for $E_G[\mu \mid Z = z]$ as $z$ varies in Figure 4 is 75% for $G_{\text{Spiky}}$ and 70% for $G_{\text{NegSpiky}}$. The $F$-localization methods have simultaneous coverage above 95%.

Figure 5 shows the simulation results for the local false sign rate. Most conclusions are similar to the ones we made for the posterior mean. However, here the Gauss-$F$-localization
Figure 4: Simulation results: Inference for the posterior mean in the Gaussian empirical Bayes problem. a) Expected confidence intervals in the simulation with the prior $G^{\text{Spiky}}$ (36). 4 different inference methods are shown, as well as the ground truth as a function of $z$. b) Coverage of the above confidence intervals as a function of $z$. c, d) Inference results in the simulation with the prior $G^{\text{NegSpiky}}$ (36).

leads to shorter intervals compared to the DKW-$F$-localization. Furthermore, in this case, both $F$-localization intervals and AMARI have pointwise coverage close to 100%; the reason is that the worst-case bias is substantial, and so bias-aware intervals lead to conservative inference for most $G \in \mathcal{G}$. In fact, AMARI has simultaneous coverage above 95% for $P [\mu \geq 0 \mid Z = z]$ as $z$ varies in Figure 5.

**Gaussian scale mixture $G$:** We next repeat our simulations with the same settings, but using a different choice of $\mathcal{G}$, namely the Gaussian scale mixture class $\mathcal{SN}(0,1,15.6,1.1)$. The scale mixture class $\mathcal{SN}(0,1,15.6,1.1)$ is strongly misspecified for $G^{\text{NegSpiky}}$. This was detected by our proposed methods, as the intersection of $\{F_G : G \in \mathcal{SN}(0,1,15.6,1.1)\}$ and $F$-localizations $\mathcal{F}_n$ was empty. Thus, in Figure 6 we report the results of our simulations only for $G^{\text{Spiky}}$. We observe that the assumption that $G$ is a scale mixture centered at 0, instead of a location mixture, leads to substantially more precise inference, and especially so for the local false sign rate.

**Degrees of freedom for the logspline approach:** One might at this point wonder whether one can reduce the bias of the plug-in logspline approach and achieve nominal
coverage by increasing the degrees of freedom of the spline; we explore this in Figure 7 for the above simulation with the prior $G_{NegSpiky}$. In general, coverage indeed improves as the degrees of freedom increase; however, with many degrees of freedom, the variance can be so large that the resulting confidence intervals are longer than the intervals proposed in this work. More importantly it is not clear a-priori, i.e., without knowing the ground truth, how to properly undersmooth the plug-in estimation and choose a number of degrees of freedom that provides good coverage. Efron [2016] does not suggest undersmoothing, and instead, acknowledges that using a low-dimensional parametric family induces ‘definitional bias’ in point estimates, ‘the pay-off being reduced variability’. On the other hand, as this example highlights, if we want confidence intervals that cover the true local false sign rate, it is important to explicitly account for bias.

7 On the asymptotic power of $F$-Localization and AMARI

Our goal in this work is to provide a unified approach for constructing intervals with the coverage property (2) that lead to useful confidence statements in applied situations (cf. Sections 5 and 6). Given the generality of (1), we suspect it may be difficult to develop a unified theory of optimality. Nevertheless in this section we consider the issue of optimality and asymptotic relative efficiency in two concrete settings to provide the following conceptual insights. First, we describe a situation in which AMARI is asymptotically efficient and
Figure 6: Simulation results: Inference in the Gaussian empirical Bayes problem with $G = SN$ a Gaussian scale mixture. a,b) Inference for the posterior mean (same simulation setting as Figure 4a,b). c,d) Inference for the local false sign rate (same simulation setting as Figure 5a,b).

Figure 7: Coverage versus expected length of confidence intervals: Here the simulation setting is the same as that of Figure 5, panels c,d) but we only consider inference at $z = 2$, i.e., for $\theta_G(2) = P_G [\mu \geq 0 \mid Z = 2]$. We apply the exponential family plug-in estimator for a range of degrees of freedom (from 2 to 12 shown by the number as well as progressively darker blue color), while in Figure 5 only the estimator with 5 degrees is considered.
outperforms the $F$-localization approach. Second, we illustrate the form of the $Q(\cdot)$ that solves the worst-case bias-variance problem (26) in a familiar context. Third, we elaborate on the issue of partial identification.

7.1 Asymptotic relative efficiency in the Poisson model

Consider the Poisson model in which we seek to conduct inference for the posterior mean $\theta_G(z) = \mathbb{E}_G[\mu \mid Z = z]$. In view of Robbins’ formula (9), $\theta_G(z)$ can be estimated at the parametric $1/\sqrt{n}$ rate and so we can compare confidence intervals and estimators in terms of their asymptotic relative efficiency. Before studying $\theta_G(z)$, we first discuss inference for the linear functional $L(G) = f_G(z) = \mathbb{P}_G[Z = z]$ for a fixed $z$. A consistent estimator in this case is given by the sample proportion $\hat{f}(z) = \# \{Z_i = z\} / n$, which has the limiting distribution,

$$\sqrt{n} \left( \hat{f}(z) - f_G(z) \right) \xrightarrow{D} \mathcal{N}(0, f_G(z)(1 - f_G(z)))$$

and so we can build an asymptotic $1 - \alpha$ confidence interval with asymptotic length equal to $2q_{1-\alpha/2} \sqrt{f_G(z)(1 - f_G(z))}/\sqrt{n}$, where $q_{1-\alpha/2}$ is the $1 - \alpha/2$ quantile of the standard normal distribution. Tierney and Lambert [1984] prove that among a class of regular estimators, the estimator in (37) is asymptotically efficient and so, asymptotically, the information that $F_G$ is a Poisson mixture, is not helpful for inference of $L(G)$. In this setting, it turns out that AMARI\(^{21}\) matches the efficient confidence interval based on (37) and is shorter than the DKW-$F$-Localization interval.

**Proposition 6.** We consider inference for $L(G) = f_G(z)$, in the Poisson model at level $\alpha \in (0,1)$ for a fixed $z \in \mathbb{N}_{\geq 1}$. We treat the observations as right-censored for $Z_i > M$ as in (27), for fixed $M \geq z + 1$, and consider the prior class $G = \mathcal{P}([0,1])$, $0 \leq a < b < \infty$. If $G \in \mathcal{G}$ and $G$ is supported on at least $M + 2$ points, then as $n \to \infty$ it holds that: AMARI has asymptotically the same length as the confidence interval constructed using (37), namely $2q_{1-\alpha/2} \sqrt{f_G(z)(1 - f_G(z))}/\sqrt{n}(1 + o_{P_G}(1))$. The DKW-$F$-Localization interval has asymptotic length $2\sqrt{2\log(2/\alpha)/n} (1 + o_{P_G}(1))$.

The key argument in the proof of the above proposition is that, in this setting, the optimal $Q(\cdot)$ solving (26) is with high probability equal to $1(\cdot = z)$ for $n$ large enough, in which case, $L = \# \{Z_i = z\} / n$. In finite samples, however, the optimal $Q(\cdot)$ takes the form of a kernel smoother that upweights $Z_i$ in a neighborhood of $z$. To illustrate, we simulate from the Poisson empirical Bayes model (1) with $G = U[0, 2]$ and let $n$ vary. We specify $G = \mathcal{P}([0,4])$. The optimal $Q(\cdot)$ of AMARI for different values of $n$ is shown in Figure 8a.

Figure 8b) shows the expected length of the AMARI and DKW-$F$-Localization confidence intervals, as well as the asymptotic lengths from Proposition 6. As expected, AMARI has shorter length than the DKW-$F$-localization intervals. The information that $F_G$ is a Poisson mixture is not helpful to both approaches for $n$ large, however, both AMARI and DKW-$F$-Localization can use this information for smaller $n$ to provide sharper inference.

The next proposition and Figure 8c) pertain to the posterior mean $\theta_G(z) = \mathbb{E}_G[\mu \mid Z = z]$ and are analogous to Proposition 6 and Figure 8b). Our findings are similar; AMARI outperforms the DKW-$F$-localization intervals and for small $n$, both methods perform better than predicted by the asymptotic limit.

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\(^{21}\)We slightly abuse terminology in this section, and use the term AMARI also for Algorithm 1, i.e., for our inference approach for linear functionals. The pilot estimates for AMARI are chosen as in Proposition 3.
We consider inference for \( \theta \) as per Proposition 6. c) Carlo replicates. The panel also shows the asymptotic interval length for the two methods as per Proposition 6. c) Analogous to panel b) for inference of the posterior mean \( \theta_G(3) = E_G [\mu \mid Z = 3] \). Asymptotic lengths are derived in Proposition 7.

\[
\sqrt{n} |I_{\alpha}^{AMARI}(z)| \xrightarrow{p} 2(z + 1)q_{1 - \alpha/2} \sqrt{f_G(z + 1)(1 + f_G(z + 1)/f_G(z))} / f_G(z), \\
\sqrt{n} |I_{\alpha}^{DKW-F-Loc}(z)| \xrightarrow{p} 2(z + 1)\sqrt{2\log(2/\alpha) \cdot (1 + f_G(z + 1)/f_G(z))} / f_G(z).
\]

In this case, AMARI is asymptotically equivalent to the intervals constructed by Robbins [1980] and Karlis et al. [2018]. The confidence intervals of Robbins [1980] are based on the joint central limit theorem for \( \# \{Z_i = z\} / n \) and \( \# \{Z_i = z + 1\} / n \), the delta method and his formula (9).

### 7.2 Sharp partial identification in the Bernoulli model

Above we compared methods in a problem in which parametric rates are attainable. Here we compare methods under partial identification, i.e., when confidence intervals will not shrink to a point mass, even as \( n \to \infty \). We consider model (1) with \( Z_i \mid \mu_i \sim \text{Bernoulli}(\mu_i) \), i.e., the Binomial model with a single (\( N = 1 \)) trial. Furthermore, we do not impose additional structure on \( G \), i.e., we assume that \( G \in \mathcal{G} = \mathcal{P}([0,1]) \) (4). \( Z_i \) is supported on \{0,1\} and we can take \( \lambda = \delta_0 + \delta_1 \) to be the counting measure on \{0,1\} and \( p(z \mid \mu) = \mu^z (1 - \mu)^{1-z} \). The marginal distribution \( F_G \) is fully determined by \( f_G(1) \).

We first consider inference for the second moment \( L(G) = \int \mu^2 \, dG(\mu) \), which is a linear functional of \( G \). The distribution of \( Z_i \), however, does not point identify \( L(G) \), unless \( f_G(1) \in \{0,1\} \), and the partial identification interval for \( L(G) \) is equal to \([f_G(1)^2, f_G(1)]\).\(^{22}\)

The posterior mean, \( \theta_G(1) = E_G [\mu \mid Z = 1] = \int \mu^2 \, dG(\mu) / f_G(1) \) is also not identified and its partial identification interval is equal to \([f_G(1), 1] \). The next proposition shows that, as

\(^{22}\)The right bound is attained by the prior on \([0,1] \) with \( F_G([0]) = f_G(0), F_G([1]) = f_G(1), \) and the left bound by the point mass prior with \( F_G([f_G(1)]) = 1 \). Both of these priors induce the same marginal distribution \( F_G \).
Proposition 8. Consider inference for \(L(G) = \int \mu^2 \, dG(\mu)\) in the Bernoulli model with the DKW-F-localization (22), the \(\chi^2\)-F-localization (22), or AMARI (using any of the above as a pilot F-localization). We use \(G = \mathcal{P}([0,1])\) and suppose that \(0 < f_G(1) < 1\). The length of all of these confidence intervals asymptotically converges to the length of the partial identification interval, i.e., \(|\mathcal{I}_n| / (f_G(1)(1 - f_G(1)))\). Similarly, all of the above confidence intervals for \(\theta_G(1) = \mathbb{E}_{G|Z=1} \mu|Z=1\) asymptotically match the partial identification intervals, i.e., \(|\mathcal{I}_n(z)| / (1 - f_G(1))\).

8 On the choice of \(G\)

Throughout this paper, we have taken the choice of \(G\) for granted and suggested some choices such as (4), (34) and (35) in our numerical examples. There are two difficulties regarding this choice: first, \(G\) needs to capture the true \(G\) and second, if \(G\) is infinite-dimensional, then it has to be suitably discretized to numerically solve optimization problems such as (7) or (28). Recent successful applications of empirical Bayes for point estimation use a discretized convex class \(G\). For example, Koenker and Mizera [2014] and Koenker and Gu [2017] use the nonparametric maximum likelihood estimator (NPMLE) for a plethora of different likelihoods in (1) with \(\tilde{G} \in \mathcal{P}(\mathcal{K})\) and \(\mathcal{K}\) a finite set (an equidistant discretization of a compact interval). The above classes \(G\) are typically discretized so densely that a nonparametric approach to forming confidence intervals, as pursued in this work, is warranted. In cases where a choice of \(G\) has been used for estimation, we suggest that the point estimates be accompanied by confidence intervals using the same choice of prior class.

8.1 Sensitivity analysis for \(G\)

The choice of \(G\) is not innocuous and the sensitivity of our intervals to the nonparametric specification of \(G\) is an important consideration for their practical adoption. One could hope to choose \(G\) on the basis of goodness-of-fit testing. However, goodness-of-fit tests are only able to rule out \(G\) that are inconsistent with the data, but there may be many choices of \(G\) that are consistent with the data, and for each of these, depending on the target of inference, the length of the confidence intervals may vary substantially or remain relatively stable. To illustrate these ideas further, we suppose that \(G\) is location mixture of Gaussians as in (34). These classes are nested as \(\mathcal{LN}(\tilde{\tau}^2, \mathbb{R}) \supset \mathcal{LN}(\tau^2, \mathbb{R})\) for \(\tilde{\tau} < \tau\). If we define,

\[\tau^*(G) = \sup \{ \tau > 0 : \mathcal{G} \in \mathcal{LN}(\tau^2, \mathbb{R}) \},\]

then inference using our methods will be valid with \(\mathcal{G} = \mathcal{LN}(\tau^2, \mathbb{R})\) for any \(\tau \leq \tau^*(G)\), and will be more conservative, the smaller \(\tau\) is. Ideally we would like to use \(\tau = \tau^*(G)\), however, \(\tau^*(G)\) is a one-sided discontinuous statistical functional in the sense of Donoho [1988], so that it is impossible to derive a non-trivial data-driven lower bound on it without making additional assumptions; see also Donoho and Reeves [2013]. On the other hand, it

\[^{24}\]We provide guidance for the numerical discretization of infinite-dimensional \(G\) in Supplement D.2.

\[^{25}\]Suppose \(G \in \mathcal{LN}(\tau^2, \mathbb{R})\), i.e., \(G = \mathcal{N}(0, \tau^2) \ast H\) for \(\tau > 0\) and a distribution \(H\), where \(\ast\) denotes convolution. For any \(0 < \tilde{\tau} < \tau\), we can write \(G = \mathcal{N}(0, \tilde{\tau}^2) \ast \tilde{H}\) with \(\tilde{H} = \mathcal{N}(0, \tau^2 - \tilde{\tau}^2) \ast H\), and so \(G \in \mathcal{LN}(\tilde{\tau}^2, \mathbb{R})\).
is possible to derive upper bounds on \( \tau^*(G) \).\(^{25}\) Hence, a goodness-of-fit test may be able to reject values of \( \tau \) that are too large, however it cannot disambiguate between choices of small \( \tau \).

Thus, the only way to obtain practically meaningful results is by the analyst choosing a range of \( \tau \)'s that appear to be plausible. The analyst can further interpret the results and evaluate how pessimistic a choice of \( \tau \) (or more generally, of \( G \)) may be by inspecting the worst-case priors that determine the confidence interval for a given estimand (the worst-case priors in (7) for \( F \)-Localization, and the worst-case priors in (28) for the affine minimax approach). In Supplement G, we explore these issues in the context of the Prostate data analysis of Section 5.2 using the split-likelihood-ratio of Wasserman et al. [2020] for goodness-of-fit testing.

While conducting the suggested sensitivity analysis, it is important to recall that the minimax estimation error for \( \theta_G(z) \) decays extremely slowly (often poly-logarithmically) with sample size [Butucea and Comte, 2009, Pensky, 2017] for some of the problems we consider (e.g., local false sign rate in the Gaussian empirical Bayes problem). In this case, unlike in classical estimation problems, we cannot expect to make our confidence intervals meaningfully shorter by, say, collecting 100 times more data than we have now. From this perspective, the amount of assumptions (smoothness, unimodality and so forth) we are willing to impose on \( G \) determines the accuracy with which we can ever hope to learn \( \theta_G(z) \), and the sensitivity analysis discussed above is closely aligned with recommendations for applications with partially identified parameters [Armstrong and Kolesár, 2018, Imbens and Wager, 2019, Rosenbaum, 2002].

9 Discussion

We have presented two general approaches towards building confidence intervals for empirical Bayes estimands in model (1) that work for any choice of \( h(\cdot) \), convex class of priors \( G \) and likelihood \( p(\cdot|\mu) \). Our methods are computationally intensive and require repeatedly solving non-trivial convex optimization problems. Nevertheless, in light of an available software implementation, our confidence intervals are practical and can accompany non-parametric empirical Bayes point estimates in applied work. As in Koenker and Mizera [2014], our implementation is facilitated by recent advances in convex optimization.

Here, we focused on inference for empirical Bayes estimands of the form \( \mathbb{E}_G[h(\mu) | Z = z] \) in model (1); our approach, however, can also handle other empirical Bayes estimands. As explained in Section 3, simpler versions of our methods can be used to form confidence intervals for linear functionals of \( G \) such as \( \mathbb{P}_G[\mu \geq 0] \). Our methods are also directly applicable to tail (rather than local) empirical Bayes quantities, such as the tail (marginal) false sign rate \( \mathbb{P}_G[\mu \cdot Z \leq 0 | Z \geq z] \) as considered in, e.g., Yu and Hoff [2019]. Another important class of estimands consists of posterior quantiles \( \theta_{G}^{p}(z;p) = \inf \{ t : \mathbb{P}_G[\mu \leq t | Z = z] \geq p \} \) for \( p \in (0,1) \). \( F \)-Localization can be used to conduct inference for posterior quantiles by inverting simultaneous confidence intervals for \( \mathbb{P}_G[\mu \leq t | Z = z] \), \( t \in \mathbb{R} \). However, it seems more challenging to generalize AMARI to posterior quantiles.

A further direction for future work is to handle generalizations of model (1). In some applications, it would be important to allow for unknown structural parameters or global parameters, such as the variance parameter \( \sigma^2 \) in the Gaussian model or covariate effects [Gu and Koenker, 2017]. (1) could also be extended to higher-dimensions, e.g., \( \mu_i \in \mathbb{R}^d \), and to

\(^{25}\)For example, \( \tau^*(G)^2 \leq \text{Var}_G[|Z|] - \sigma^2 \), when \( \mu \sim G \) and \( Z | \mu \sim \mathcal{N}(\mu, \sigma^2) \).
heteroskedastic problems in which the likelihood \( p_i(\cdot \mid \mu_i) \) can vary across \( i \), for example, the Gaussian location model with per-observation noise standard deviation \( \sigma_i \), so that \( p_i(\cdot \mid \mu_i) = N(\mu_i, \sigma_i^2) \) [Stephens, 2016, Weinstein et al., 2018]. It would also be useful in applications to handle prior classes \( \mathcal{G} \) that are unions of convex classes, e.g., Gaussian scale mixtures as in (35) with unknown mode [Stephens, 2016].

Software

All numerical results in this paper can be reproduced with the code available on the Github repository [https://github.com/nignatiadis/empirical-bayes-confidence-intervals-paper](https://github.com/nignatiadis/empirical-bayes-confidence-intervals-paper). There we provide an implementation of the proposed methods as a package in the Julia programming language [Bezanson et al., 2017] that depends, among others, on JuMP.jl [Dunning et al., 2017] and Distributions.jl [Besançon et al., 2021].

Supplemental materials

The supplemental materials contain proofs for all formal results in this paper, computational details, a brief description of the exponential family approach of Efron [2016], and the empirical sensitivity analysis mentioned in Section 8.1.

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References


A Gaussian $F$-localization: Proof of Proposition 1

Throughout this section we assume that model (1) holds with $p(\cdot | \mu) = N(\mu, \sigma^2)$. Furthermore, without loss of generality, we assume that $\sigma^2 = 1$. The key idea of the proof is the following: we first use the smoothness of $f_G(z)$ in the Gaussian empirical Bayes problem to verify that $\hat{f}(z) = \hat{f}_G^k(z)$ (18) has bias of order $O(1/\sqrt{n \log(n)})$. Thus the dominant error in $\sup_{z \in [-M,M]} |\hat{f}(z) - f_G(z)|$ is stochastic and equal to $\sup_{z \in [-M,M]} |\hat{f}(z) - \mathbb{E}_G[f(z)]|$. We then study the variance of the stochastic term (pointwise) and use results of Chernozhukov et al. [2014, 2016] to verify the accuracy of the bootstrap approximation.

Notation: We often omit the dependence on $n$, for example, we write $h$ for $h_n = 1/\sqrt{\log(n)}$. All integrals in this section are computed with respect to the Lebesgue measure.

A.1 Bias of KDE

For a function $\psi : \mathbb{R} \to \mathbb{R}$, we write $\psi^*(t)$ for its Fourier transform, i.e., $\psi^*(t) = \int \exp(itx)\psi(x)dx$, assuming it exists. The crucial property of the kernel $K(\cdot)$ in (18) that we will use to control bias, is that $K^*$ is equal to 1 on $[-1, 1]$ [Politis and Romano, 1993]:

$$K^*(t) = \begin{cases} 1, & \text{if } t \in [-1, 1] \\ 0, & \text{if } |t| \geq 1.1 \\ 11 - 10 \cdot |t|, & \text{if } |t| \in [1.1] \\ \end{cases}$$

(38)

We are ready to state our result on the bias.

**Proposition 9.** Consider estimating the marginal density $f_G$ (for some effect size distribution $G$) with the KDE (18). Then, for some constant $C$ it holds that,

$$\text{Bias}_G[\hat{f}(z), f_G(z)]^2 = \left( \mathbb{E}_G \left[ \hat{f}(z) \right] - f_G(z) \right)^2 \leq C \frac{1}{n \log(n)} \text{ for all } z \in \mathbb{R}.$$

**Proof.**

$$|\text{Bias}_G[\hat{f}(z), f_G(z)]| = \left| f_G(z) - \frac{1}{h} \mathbb{E}_G \left[ K \left( \frac{Z_i - z}{h} \right) \right] \right|$$

$$= \left| f_G(z) - \int \frac{1}{h} K \left( \frac{u - z}{h} \right) f_G(u)du \right|$$

$$= \frac{1}{2\pi} \left| \int \exp(-itz)f_G^*(t)dt - \int f_G^*(t) \left( \frac{1}{h} K \left( \frac{\cdot - z}{h} \right) \right)^*(t)dt \right|$$

$$= \frac{1}{2\pi} \left| \int \exp(-itz)f_G^*(t)dt - \int f_G^*(t) \exp(-itz)K^*(th)dt \right|$$

$$\leq \frac{1}{2\pi} \int \left| f_G^*(t) \right| dt$$

$$\leq \frac{1}{2\pi} \int \left| t \geq \frac{1}{2} \right| \exp(-t^2/2)$$

$$\leq \frac{1}{\pi} h \exp \left( -\frac{1}{2h^2} \right)$$

$$= \frac{1}{\pi} \frac{1}{\sqrt{n \log(n)}}$$

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In the 3rd line we used the Fourier inversion formula, as well as the Plancherel isometry [Meister, 2009, Theorem A.4]. We also used the facts that $K$ is square integrable, $K$ is even, $K^*(t) = 1$ on $[-1, 1]$, $|K^*(t)| \leq 1$ outside $[-1, 1]$ and that $|f_G^*(t)| = |\int \exp(\im t\mu - t^2/2)dG(\mu)| \leq \exp(-t^2/2)$. Finally, we used the Gaussian tail inequality.

We note that Taupin [2001, Section 3.2] also sketches the above argument.

A.2 Variance of KDE

We next study the variance term.

**Proposition 10.** There exist constants $c, C > 0$ and $n_0 \in \mathbb{N}$, such that,

$$\frac{c}{n} \leq \text{Var}_G \left[ \hat{f}(z) \right] \leq \frac{C}{n}$$

for all $z \in [-M, M]$, $n \geq n_0$

**Proof.** A consequence of Proposition 9 is that,

$$\mathbb{E}_G \left[ K \left( \frac{Z_i - z}{h} \right) \right] = h (f_G(z) + o(1)).$$

For the second moment, we get:

$$\mathbb{E}_G \left[ K^2 \left( \frac{Z_i - z}{h} \right) \right] = \int K^2 \left( \frac{u - z}{h} \right) f_G(u)du$$

$$= h \int K^2(u)f_G(uh + z)du$$

$$= h \left( f_G(z) \int K^2(u)du + o(1) \right).$$

Combining these two results, we find that,

$$\text{Var}_G \left[ \frac{1}{h} K \left( \frac{Z_i - z}{h} \right) \right] = \frac{f_G(z) \int K^2(u)du + o(1)}{h} - (f_G(z) + o(1))^2.$$

We conclude after noting that the $o(1)$ terms are uniform in $z \in [-M, M]$ and that

$$0 < \inf_{z \in [-M, M]} f_G(z) \leq \sup_{z \in [-M, M]} f_G(z) < \infty.$$ 

A.3 Validity of bootstrap approximation

Let us define the following suprema,

$$\hat{W} = \sqrt{n}h \sup_{z \in [-M, M]} \left| f(z) - \hat{f}(z) \right|$$

$$W = \sqrt{n}h \sup_{z \in [-M, M]} \left| \mathbb{E} \left[ \hat{f}(z) \right] - \hat{f}(z) \right|$$

$$\beta = \sqrt{n}h \sup_{z \in [-M, M]} \left| \mathbb{E} \left[ \hat{f}(z) \right] - f(z) \right|$$

$$W^* = \sqrt{n}h \sup_{z \in [-M, M]} \left| \hat{f}^*(z) - \hat{f}(z) \right|.$$
where \( \hat{f}^*(z) \) is a single bootstrap evaluation of the KDE as in (19). Our high-level strategy is to argue that the distribution of \( W^* \) conditionally on the data \( \mathbf{Z} = (Z_1, \ldots, Z_n) \) is close to the unconditional distribution of \( W \) and that \( \hat{W} \) and \( W \) are essentially indistinguishable (compared to the fluctuations of the above suprema), because \( \beta \), i.e. the worst case bias over \( z \), is small (Proposition 9).

We first record the following fact: Define the class \( \mathcal{H} \) of all functions that are dilations and translations of the kernel \( K(\cdot) \), i.e.,

\[
\mathcal{H} = \{ K(a \cdot + b) : a > 0, b \in \mathbb{R} \}. \tag{43}
\]

This class has envelope function \( \|K(\cdot)\|_\infty \), it is pointwise measurable and is of VC type, i.e., there exist constants \( A \geq e, v \geq 1 \) such that for any finitely discrete probability measures \( Q \) and any \( \varepsilon \in (0, \|K(\cdot)\|_\infty) \), it holds that

\[
N(\mathcal{H}, \mathcal{L}^2(Q), \varepsilon) \leq \left( \frac{A}{\varepsilon} \right)^v, \tag{44}
\]

where \( N(\mathcal{H}, \mathcal{L}^2(Q), \varepsilon) \) is the \( \varepsilon \)-covering number of \( \mathcal{H} \) with respect to the \( \mathcal{L}^2(Q) \) norm. This follows directly from Giné and Nickl [2016, Proposition 3.6.12], since the kernel \( K(\cdot) \) is of bounded variation.

The fact that \( \mathcal{H} \) is VC type will allow us to construct a coupling \( W \) and \( \tilde{W} \), where \( \tilde{W} \) is the supremum of a Gaussian process. Concretely, let \( \mathcal{G} \) be a Gaussian process indexed by \([-M, M]\) with mean 0 and covariance,

\[
\text{Cov}[\mathcal{G}(s), \mathcal{G}(t)] = \frac{1}{h} \cdot \text{Cov}\left[K\left(\frac{Z-t}{h}\right), K\left(\frac{Z-s}{h}\right)\right]. \tag{45}
\]

Then the following holds,

**Proposition 11.** \( \mathcal{G} \) is a tight Gaussian process on \( \ell^\infty([-M, M]) \). Furthermore, there exists a coupling \( \tilde{W}, W \) (with \( W \) defined in (40)) and \( r_1, r_2 \) such that

\[
\tilde{W} \overset{D}{=} \sup_{z \in [-M, M]} \mathcal{G}(t)
\]

and,

\[
\mathbb{P}\left[|W - \tilde{W}| > r_1\right] \leq r_2, \quad r_1 = O\left((nh)^{-1/6} \log(n)\right), \quad r_2 = O(1/\log(n)).
\]

Similarly, we can construct a conditional coupling of the bootstrap statistic \( W^* \) and \( \tilde{W}^* \), where \( \tilde{W}^* \) conditionally on \( \mathbf{Z} \) has the same distribution as (the unconditional law) of \( \tilde{W} \).

**Proposition 12.** There exists a coupling \( \tilde{W}^*, W^* \) (with \( W^* \) defined in (42)), such that

\[
(W^* \mid \mathbf{Z}) \overset{D}{=} \sup_{z \in [-M, M]} |\mathcal{G}(z)|
\]

and such that there exists an event \( \mathcal{E} \) with \( \mathbb{P}[\mathcal{E}^c] = O(1/\sqrt{\log(n)}) \) on which

\[
\mathbb{P}\left[|W^* - \tilde{W}^*| > r_1^* \mid \mathbf{Z}\right] \leq r_2^*, \quad r_1^* = O\left((nh)^{-1/6} \log(n)\right), \quad r_2^* = O(1/\sqrt{\log(n)}).
\]
Before proceeding with the proof of Proposition 1, we need one final ingredient. We define Levy’s function for \( \tilde{W} \) as
\[
\epsilon(r) = \sup_{t \in \mathbb{R}} \mathbb{P} \left( |\tilde{W} - t| \leq r \right).
\] (46)
The following Proposition holds as a consequence of Chernozhukov et al. [2014, Lemma A.1].

**Proposition 13.** The Levy concentration function (46) satisfies:
\[
r \cdot \sqrt{\log(\log(n))} \to 0 \text{ as } n \to \infty \implies \epsilon(r) \to 0 \text{ as } n \to \infty.
\]

We postpone the proof of the above three Propositions to the end of this section and proceed with the main argument.

**Proof of Proposition 1:** Write \( c^* = \tilde{c}_n(\alpha) \cdot \sqrt{n} \) for the \( 1 - \alpha \) conditional quantile of \( W^* \), i.e.,
\[
c^* = \inf \{ t : \mathbb{P} [W^* \leq t \mid Z] \geq 1 - \alpha \}.
\]

It holds that,
\[
\mathbb{P} [F \in \mathcal{F}_n] = \mathbb{P} \left[ \tilde{W} \leq c^* \right] \\
\geq \mathbb{P} [W \leq c^* - \beta] \\
\geq \mathbb{P} \left[ \tilde{W} \leq c^* - \beta \right] - \epsilon(r_1) - r_2 \\
\geq \mathbb{P} \left[ \tilde{W} \leq c^* \right] - \epsilon(\beta) - \epsilon(r_1) - r_2 \\
\geq 1 - \alpha - \mathbb{P} [\mathcal{E}] - r_1^* - \epsilon(r_1) - \epsilon(\beta) - r_2 \\
\equiv 1 - \alpha - o(1)
\]

We justify the individual steps \((i) - (v)\).

\( (i) \) follows by the triangle inequality, since \( \tilde{W} \leq W + \beta \).

\( (ii) \) follows from Proposition 11, along with the definition of the Levy function (46). In more detail:
\[
\mathbb{P} \left[ \tilde{W} \leq c^* - \beta \right] \leq \mathbb{P} \left[ \tilde{W} \leq c^* - \beta, |\tilde{W} - W| \leq r_1 \right] + \mathbb{P} \left[ |\tilde{W} - W| > r_1 \right] \\
\leq \mathbb{P} \left[ \tilde{W} \leq c^* - \beta, |\tilde{W} - W| \leq r_1, W \leq c^* - \beta \right] + \mathbb{P} \left[ 0 \leq W - c^* + \beta \leq r_1 \right] + r_2 \\
\leq \mathbb{P} [W \leq c^* - \beta] + \epsilon(r_1) + r_2.
\]

\( (iii) \) follows from the definition of the Levy function (46).

\( (iv) \) follows by properties of the Bootstrap approximation. First, note that by definition of \( c^* \), it holds
\[
\mathbb{P} [W^* \leq c^* \mid Z] \geq 1 - \alpha
\]
On the other hand, consider the event $\mathcal{E}$. On that event,

$$\Pr\left[ \hat{W}^* \leq c^* \mid \mathcal{Z} \right] \geq \Pr\left[ W^* \leq c^* \mid \mathcal{Z} \right] - \epsilon(r_1^*) - r_2^* \geq 1 - \frac{1}{n} - \epsilon(r_1^*) - r_2^*,$$

where the first inequality follows from Proposition 12 and the definition of the Levy function (46) and the second inequality follows by definition of $c^*$. Hence,

$$\Pr\left[ \hat{W}^* \leq c^* \right] \geq \mathbb{E}\left[ \Pr\left[ \hat{W}^* \leq c^* \mid \mathcal{Z} \right] \mid \mathcal{E} \right] \geq 1 - \frac{1}{n} - \epsilon(r_1^*) - r_2^* - \Pr[\mathcal{E}^c].$$

(v) From Propositions 11 and 12 it follows that $\Pr[\mathcal{E}^c], r_2, r_2^* = o(1)$, so it remains to argue about the Levy terms. From the same propositions, it also holds that $r_1, r_1^* = O(1/\sqrt{\log(n)})$ and $\beta = O((1/\log(n))^{3/4})$ (Proposition 9). Thus, applying Proposition 13, it also follows that the terms $\epsilon(r_1^*), \epsilon(r_1)$ and $\epsilon(\beta)$ are $o(1)$. \qed

### A.4 Proofs of intermediate results

**Proof of Proposition 11.** This follows directly from Chernozhukov et al. [2014, Proposition 3.1.] by noting that the first term, i.e., $O((nh)^{-1/6} \log(n))$ is dominant for the choice $h = h_n = 1/\sqrt{\log(n)}$. Assumptions (B1), (B4), (B5) of Chernozhukov et al. [2014] hold trivially, (B2) holds by (44) and (B3), i.e., that $Z$ has a bounded Lebesgue density on $\mathbb{R}$ follows from the fact that $f_G(z) \leq 1/\sqrt{2\pi}$ for all $z$, since $f_G$ is the convolution of $G$ with a standard Gaussian pdf. Finally, we note that Chernozhukov et al. [2014, Proposition 3.1.] is stated in a slightly different form than Proposition 11, however the result directly follows by inspecting the proof, which is based on Chernozhukov et al. [2014, Corollary 2.2]. \qed

**Proof of Proposition 12.** We seek to apply Chernozhukov et al. [2016, Theorem 2.3]. To this end, we will study the rescaled supremum $U^* = \sqrt{h} \cdot W^*$ and rescaled Gaussian process $\mathcal{B} = \sqrt{h} \cdot \mathcal{G}$. Then, note that as in the proof of Proposition 11, Assumptions (A)-(C) of Chernozhukov et al. [2016, Theorem 2.3] are satisfied. Furthermore, recalling (44), in the notation of that paper, we can apply the result for $\eta \to 0$, $K_n = O(\log(n))$, $q = 4$, $b$ bounded, $\sigma^2 = O(h)$ and $\gamma = 1/\log(n)$. It then follows that there exists a coupling

$$\langle \bar{U}^* \mid \mathcal{Z} \rangle \overset{D}{=} \sup_{z \in [-M, M]} |U(z)|,$$

such that

$$\Pr\left[ \left| U^* - \bar{U}^* \right| \geq \bar{r}^*_1 \right] \leq \bar{r}^*_2,$$

and $\bar{r}^*_2 = O(1/\log(n))$, while

$$\bar{r}^*_1 = O\left( \frac{\log(n)^{9/4}}{n^{1/4}} + \frac{\log(n)h^{1/3}}{n^{1/6}} + \frac{\log(n)^2 h^{1/4}}{n^{1/4}} \right) = O\left( \frac{\log(n)h^{1/3}}{n^{1/6}} \right).$$

By applying Markov’s inequality, we get that with probability at least $1 - 1/\sqrt{\log(n)}$, it holds that,

$$\Pr\left[ \left| U^* - \bar{U}^* \right| \geq \bar{r}^*_1 \mid \mathcal{Z} \right] \leq \bar{r}^*_2 \cdot \sqrt{\log(n)}.$$
This is the event $E$ in the statement of Proposition 12 and we can take $r_2^* = \tilde{r}_2^* \cdot \sqrt{\log(n)} = 1/\sqrt{\log(n)}$. Recalling that $W^* = U^*/\sqrt{h}$ and defining $\tilde{W}^* = \tilde{U}^*/\sqrt{h}$, we get,

$$
r_1^* = \tilde{r}_1^* = O\left(\frac{\log(n)}{(nh)^{1/6}}\right).$$

Proof of Proposition 13. First note that by Proposition 10, we have that there exist $\bar{\sigma}, \bar{\sigma}, n_0$ such that the following holds for the Gaussian process $G$ of Proposition 11 (with covariance (45)):

$$
\bar{\sigma}^2 \leq \text{Var} \left[ G(z) \right] \leq \bar{\sigma}^2,
$$

for all $z \in [-M, M], n \geq n_0$.

Hence, by Chernozhukov et al. [2014, Lemma A.1.], we have for a constant $C$ (that depends on $\bar{\sigma}, \bar{\sigma}$) that:

$$
\epsilon(r) \leq C \cdot r \cdot \left\{ \mathbb{E} \left[ \tilde{W} \right] + \sqrt{\max\{1, \log(\bar{\sigma}/r)\}} \right\}.
$$

We next bound, $\mathbb{E}[\tilde{W}] = \mathbb{E}[\sup_{z \in [-M, M]} |G(z)|]$. To this end, we make the following observations: first, $G$ is a Gaussian process, and so in particular it is a sub-Gaussian process with respect to $d_2^2(z, z') = \mathbb{E}[|G(z) - G(z')|^2]$. Applying Proposition 10 again, we find that $
\sup_{z, z' \in [-M, M]} d(z, z') \leq D < \infty$ is finite (and $D$ can be chosen the same for all $n$). In addition, by (44), we find that for some $A', v' \geq e$,

$$
N([-M, M], d, \epsilon) \leq \left(\frac{A'}{\epsilon \cdot \sqrt{h}}\right)^{v'}.
$$

and so,

$$
\int_0^D \sqrt{\log(N([-M, M], d, \epsilon))} d\epsilon = O\left(\sqrt{\log(1/h)}\right).
$$

By Dudley’s Entropy integral [Van Der Vaart and Wellner, 1996, Corollary 2.2.8], we thus also get that,

$$
\mathbb{E} \left[ \tilde{W} \right] = O\left(\sqrt{\log(1/h)}\right).
$$

Combing the above with (47), we find that $\epsilon(r) \to 0$ for $r = o(1/\sqrt{\log(\log(n))})$. □

B Proofs for AMARI inference

**Notation:** Throughout this supplement we omit the $M$ superscript, e.g., we write $f_G$ instead of $f_G^M$, $\lambda$ instead of $\lambda^M$ and so forth.

**B.1 Properties of the modulus of continuity**

**Proposition 14.** Assume $G_n$ is convex, $\inf_z \bar{f}(z) > 0$ and $\sup_{G \in G_n} |L(G)| < \infty$. Then, the modulus $\omega_n(\cdot)$ defined in (28), as a function of $\delta > 0$, has the following properties:

(a) It is non-decreasing.

(b) It is bounded and nonnegative.

(c) It is concave.
(d) For $\delta > 0$, there exists an element $\omega'_n(\delta) \geq 0$ in the superdifferential of $\omega_n(\cdot)$ at $\delta$, i.e., $\omega'_n(\delta)$ satisfies the property defined in Footnote 12. It holds that $\omega_n(\delta) \geq \delta \cdot \omega'_n(\delta)$.

Proof. (a) and (b) follow directly by the definition of $\omega_n(\cdot)$. For (c), let us take $\delta_n, \delta_b > 0$, $\lambda \in (0, 1)$ and let $(G_1^{\delta_n}, G_{-1}^{\delta_n}), (G_1^{\delta_b}, G_{-1}^{\delta_b})$ solve the corresponding modulus problems. If solutions for either of these do not exist, we may take an approximate minimizer and use standard approximation arguments. Now for, $\lambda \in (0, 1)$ and $\delta(\lambda) = \lambda\delta_a + (1 - \lambda)\delta_b$, consider $G_i^{\delta(\lambda)} = \lambda G_i^{\delta_n} + (1 - \lambda) G_i^{\delta_b}$ with $i \in \{-1, 1\}$. Then $G_i^{\delta(\lambda)} \in \mathcal{G}_n$ by convexity of $\mathcal{G}_n$ and furthermore by the triangle inequality,

$$\left\{ n \cdot \int \left( f_{G_1^{\delta(\lambda)}}(z) - f_{G_{-1}^{\delta(\lambda)}}(z) \right)^2 / \tilde{f}(z) d\lambda(z) \right\}^{1/2} \leq \lambda\delta_a + (1 - \lambda)\delta_b = \delta(\lambda).$$

Hence:

$$\omega_n(\delta(\lambda)) \geq L(G_1^{\delta(\lambda)}) - L(G_{-1}^{\delta(\lambda)}) = \lambda\omega_n(\delta_a) + (1 - \lambda)\omega_n(\delta_b).$$

To check (d), we note that the existence of $\omega'_n(\delta)$ follows from (b,c) and results from convex analysis [Rockafellar, 1970]. $\omega'_n(\delta)$ satisfies the property defined in Footnote 12, or equivalently,

$$\omega_n(\delta) - \omega_n(\delta) \geq \omega'_n(\delta)(\delta - \delta) \text{ for all } \delta > 0. \quad (48)$$

Suppose $\omega'_n(\delta) < 0$, then e.g., letting $\delta = 2\delta$ in (48), it would follow that $\omega_n(\delta) - \omega_n(2\delta) > 0$, which would be a contradiction to part (a). Thus $\omega'_n(\delta) \geq 0$. Finally, by nonnegativity of $\omega_n(\cdot)$, it follows that $\omega_n(\delta) \geq \omega_n(\delta) - \omega_n(\delta)$, and taking $\delta \to 0$ in (48), we deduce that $\omega_n(\delta) \geq \delta \cdot \omega'_n(\delta)$. 

\[\square\]

### B.2 Stein’s heuristic

In this Section we provide more details regarding optimization problem (26) and the modulus of continuity problem (28) and provide rigorous arguments for the ideas sketched at the beginning of Section 3.1

As already mentioned, at first sight, it is not obvious how to solve optimization problem (26), since the problem is not concave in $G$, hence standard min-max results for convex-concave problems are not applicable. Nevertheless, Donoho [1994] provides a solution to this optimization problem by formalizing a powerful heuristic that goes back to Charles Stein.

The key steps are as follows:

1. We search for the hardest 1-dimensional subfamily, i.e., we find $G_1, G_{-1} \in \mathcal{G}_n$, such that solving problem (26) over ConvexHull$(G_1, G_{-1})$ (instead of over all of $\mathcal{G}_n$) is as hard as possible. The precise definition of “hardest” is given by the modulus problem (28).

2. We find the minimax optimal estimator of problem (26) over the hardest 1-dimensional subfamily.

3. We then find that this solution is in fact optimal over all of $\mathcal{G}_n$.

To implement Step 1 of the heuristic, we solve the modulus problem (28) at $\delta$ (and assume it is solvable) and let $G_1^{\delta}, G_{-1}^{\delta}$ be solutions and $\omega'_n(\delta)$ an element of the superdifferential. Then $Q$ defined in (30) solves the minimax problem (26) over ConvexHull$(G_1^{\delta}, G_{-1}^{\delta})$ for $\Gamma_n = \Gamma_n(\delta) = \omega'_n(\delta)^2$ (Step 2). In fact, it solves this minimax problem over all of $\mathcal{G}_n$ (Step 3), as can be verified by the proposition below, and so (26) can be computed by solving the modulus problem (28) (we postpone computational details to Supplement E).
Proposition 15 (Properties of \(Q\) in (30)). Assume \(\mathcal{G}_n\) is convex, \(\sup_{\hat{G} \in \mathcal{G}_n} |L(\hat{G})| < \infty\) and that \(f_{\hat{G}}(\cdot) \in C^2(\lambda)\) for all \(\hat{G} \in \mathcal{G}_n\). Furthermore, assume that exist \(G^+_1, G^-_1 \in \mathcal{G}_n\) that solve the modulus problem at \(\delta > 0\), i.e., are such that (29) holds and that \(\inf_z f(z) > 0\). Then:

(a) \(Q\) defined by (30), achieves its worst case positive bias over \(\mathcal{G}_n\) for estimating \(L(G)\) at \(G^+\) and negative bias at \(G^-\), i.e., letting \(\text{Bias}_{\mathcal{G}_1}[Q, L] = \int Q(z)f_{G_1}(z)d\lambda(z) - L(G)\), it holds that,

\[
\sup_{G \in \mathcal{G}_n} \text{Bias}_{\mathcal{G}_1}[Q, L] = -\text{Bias}_{G^-_1}[Q, L] = -\inf_{G \in \mathcal{G}_n} \text{Bias}_{\mathcal{G}_1}[Q, L].
\]

(b) For \(Q : Z \rightarrow \mathbb{R}\), write \(\text{Var}_f[Q] = \int Q^2(z)f(z)d\lambda(z) - \left(\int Q(z)f(z)d\lambda(z)\right)^2\). Let \(\Gamma_n = \text{Var}_f[Q]/n\), then for any other function \(\tilde{Q}\) with \(\text{Var}_f[\tilde{Q}] \leq \Gamma_n \cdot n\), it holds that:

\[
\sup_{G \in \mathcal{G}_n} \text{Bias}_{\mathcal{G}_1}[\tilde{Q}, L]^2 \geq \sup_{G \in \mathcal{G}_n} \text{Bias}_{\mathcal{G}_1}[Q, L]^2.
\]

(c) \(\Gamma_n\) and the worst case bias have explicit expressions in terms of the modulus \(\omega_n(\delta)\) and its superdifferential \(\omega'_n(\delta)\):

\[
\sup_{G \in \mathcal{G}_n} \text{Bias}_{\mathcal{G}_1}[Q, L] = \frac{1}{2} [\omega_n(\delta) - \delta \omega'_n(\delta)], \quad \Gamma_n = \omega'_n(\delta)^2.
\]

Proof. The arguments in this proof are well-known and appear in different forms for example in [Donoho, 1994, Low, 1995, Armstrong and Kolesár, 2018]. However, the statements there are provided in the context of Gaussian mean estimation and therefore we give a simplified, self-contained exposition.

(a) Below for notational convenience we will write \(G_1, G_{-1}\) for \(G_1^+\) and \(G_{-1}^-\). First let us check what the bias is at \(G_1\):

\[
\text{Bias}_{G_1}[Q, L] = \int Q(z)f_{G_1}(z)d\lambda(z) - L(G_1)
\]

\[
= -\frac{1}{2} (L(G_1^+ - L(G_{-1}))
\]

\[
+ \frac{n \omega'_n(\delta)}{\delta} \left\{ \int f_{G_1}(z) - f_{G_{-1}}(z) f_{G_1}(z)d\lambda(z) - \int \frac{f_{G_1}(z) - f_{G_{-1}}(z)}{f(z)} f_{G_n}(z) d\lambda(z) \right\}
\]

\[
= -\frac{1}{2} \omega_n(\delta) + \frac{n \omega'_n(\delta)}{2\delta} \int \frac{(f_{G_1}(z) - f_{G_{-1}}(z))^2}{f(z)} d\lambda(z)
\]

\[
= -\frac{1}{2} \omega_n(\delta) + \frac{\omega'_n(\delta)}{2\delta} \delta^2
\]

\[
= -\frac{1}{2} [\omega_n(\delta) - \delta \omega'_n(\delta)]
\]

Similarly, we get that: \(\text{Bias}_{G_{-1}}[Q, L] = \frac{1}{2} [\omega_n(\delta) - \delta \omega'_n(\delta)]\). Let us now show that the worst case positive bias over \(\mathcal{G}_n\) is indeed obtained at \(G_{-1}\). To this end take any other \(G \in \mathcal{G}_n\),

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By the above we conclude that:

\[ \Delta(\lambda) = \left( n \cdot \int \frac{\left[ f_{G_1}(z) - ((1 - \lambda)f_{G_{-1}}(z) + \lambda f_G(z)) \right]^2}{f(z)} \, d\lambda(z) \right)^{1/2} \]

\[ J(\lambda) = L(G_1) - L((1 - \lambda)G_{-1} + \lambda G) - \omega'_{n}(\delta)\Delta(\lambda) \]

Observe that for any \( \lambda \geq 0 \):

\[ J(\lambda) \leq \omega_n(\Delta(\lambda)) - \omega'_n(\delta)\Delta(\lambda) \]

\[ \leq \sup_{\delta \geq 0} \left\{ \omega_n(\delta) - \omega'_n(\delta)\delta \right\} \]

\[ \leq \omega_n(\delta) - \omega'_n(\delta)\delta \]

\[ = J(0) \]

(i) follows by definition of the modulus \( \omega_n \) and (ii) by noting that \( \delta \mapsto \omega_n(\delta) - \omega'_n(\delta)\delta \) is concave and its superdifferential at \( \delta \) includes the element \( \omega'_n(\delta) = \omega'_n(\delta) = 0 \) (also compare to (48)). The last equality holds by definition of \( J(\lambda) \).

Continuing, by the chain rule and dominated convergence, it holds that \( J(\lambda) \) is differentiable at 0 and so \( J'(0) \leq 0 \). Furthermore,

\[ J'(0) = L(G_{-1}) - L(G) + \frac{n}{\Delta(0)} \int \left( \frac{f_{G_1}(z) - f_{G_{-1}}(z)}{f(z)}(f_G(z) - f_{G_{-1}}(z)) \right) \, d\lambda(z) \]

And now also note that \( \Delta(0) = \delta \) and:

\[ \text{Bias}_{G}[Q, L] - \text{Bias}_{G_{-1}}[Q, L] = \left( \int Q(z) f_G(z) \, d\lambda(z) - L(G) \right) - \left( \int Q(z) f_{G_{-1}}(z) \, d\lambda(z) - L(G_{-1}) \right) \]

\[ = L(G_{-1}) - L(G) + \frac{n}{\delta} \int \left( \frac{f_{G_1}(z) - f_{G_{-1}}(z)}{f(z)}(f_G(z) - f_{G_{-1}}(z)) \right) \, d\lambda(z) \]

\[ = J'(0) \]

\[ \leq 0. \]

By the above we conclude that:

\[ \text{Bias}_{G}[Q, L] \leq \text{Bias}_{G_{-1}}[Q, L]. \]

Finally, by repeating the same argument

\[ \text{Bias}_{G}[Q, L] \geq \text{Bias}_{G_1}[Q, L]. \]

(b) First let us write \( \Gamma_n \) in terms of \( \omega'_n(\delta) \). Note that \( \int Q(z)f(z)\lambda(z) = 0 \), since

\[ \int \frac{f_{G_1}(z) - f_{G_{-1}}(z)}{f(z)} \, f(z) \, d\lambda(z) = \int (f_{G_1}(z) - f_{G_{-1}}(z)) \, d\lambda(z) = 1 - 1 = 0, \]
and so,

\[
\widetilde{\text{Var}}_f[Q] = \frac{n^2 \cdot \omega''(\delta)^2}{\delta^2} \cdot \int \left( \frac{f_{G_1}(z) - f_{G_{-1}}(z)}{f(z)} \right)^2 f(z) d\lambda(z)
\]

\[
= \frac{n^2 \cdot \omega''(\delta)^2}{\delta^2} \cdot \int \frac{(f_{G_1}(z) - f_{G_{-1}}(z))^2}{f(z)} d\lambda(z)
\]

\[
= \frac{n^2 \cdot \omega''(\delta)^2}{\delta^2} \cdot \frac{\delta^2}{n}
\]

\[= n \cdot \omega''(\delta)^2.\]

Thus, \(\Gamma_n = \omega''(\delta)^2\).

Now take any other function \(\bar{Q}(\cdot)\) and decompose it as, \(\bar{Q}(\cdot) = \bar{Q}_0 + \bar{Q}_1(\cdot)\), where \(\bar{Q}_0 = \int \bar{Q}(z) f(z) d\lambda(z)\), so that,

\[
\widetilde{\text{Var}}_f[\bar{Q}] = \int \bar{Q}_1(z)^2 f(z) d\lambda(z) \leq \Gamma_n \cdot n = n \cdot \omega''(\delta)^2.
\]

Then:

\[
\text{Bias}_{G_{-1}}[\bar{Q}, L] - \text{Bias}_{G_1}[\bar{Q}, L]
\]

\[
= \left( \int \bar{Q}_1(z) f_{G_{-1}}(z) d\lambda(z) - L(G_{-1}) \right) - \left( \int \bar{Q}_1(z) f_{G_1}(z) d\lambda(z) - L(G_1) \right)
\]

\[
= L(G_1) - L(G_{-1}) + \int \bar{Q}_1(z) \left( f_{G_{-1}}(z) - f_{G_1}(z) \right) d\lambda(z)
\]

\[
= \omega_n(\delta) + \int \bar{Q}_1(z) \bar{f}(z)^{1/2} \frac{f_{G_{-1}}(z) - f_{G_1}(z)}{f(z)^{1/2}} d\lambda(z)
\]

\[
\geq \omega_n(\delta) - \left( \int \bar{Q}_1(z)^2 \bar{f}(z) d\lambda(z) \right)^{1/2} \left( \int \left( \frac{f_{G_1}(z) - f_{G_{-1}}(z)}{f(z)} \right)^2 \bar{f}(z) d\lambda(z) \right)^{1/2}
\]

\[
\geq \omega_n(\delta) - \left( n \cdot \omega''(\delta)^2 \right)^{1/2} \left( \delta^2 / n \right)^{1/2}
\]

\[
= \omega_n(\delta) - \omega''(\delta) \delta.
\]

\[
= 2 \sup_{G \in \tilde{G}_n} \{ |\text{Bias}_G[Q, L]| \}.
\]

Above we used properties of \(\omega_n(\cdot)\) shown in Proposition 14. Next,

\[
\sup_{G \in \tilde{G}_n} \{ |\text{Bias}_G[Q, L]| \} \geq \max \{ |\text{Bias}_{G_{-1}}[\bar{Q}, L]|, |\text{Bias}_{G_1}[\bar{Q}, L]| \}
\]

\[
\geq \left( |\text{Bias}_{G_{-1}}[\bar{Q}, L]| + |\text{Bias}_{G_1}[\bar{Q}, L]| \right) / 2
\]

\[
\geq \sup_{G \in \tilde{G}_n} \{ |\text{Bias}_G[Q, L]| \}.
\]

(c) We already proved these statements as intermediate steps while proving (a) and (b).

\[\square\]

### B.3 Proof of Theorem 2

**Proof.** A word on notation: We drop the dependence on \(n, M\) and \(\delta_n\), whenever this does not cause confusion, for example we write \(G_1\) instead of \(G_1^n\) and so forth. Furthermore, we...
write \( \mathbb{E} [\cdot] \) for conditional expectations with respect to \( f, \mathcal{F}_n \), and \( \mathbb{P} [\cdot], \text{Var} [\cdot] \) for conditional probabilities, resp. variances.

Before embarking on the formal argument, we briefly sketch our proof strategy. Our proof makes heavy use of the representation of \( \bar{Q}(\cdot) \) in \((30)\). As a consequence of \((30)\), it suffices to verify a central limit theorem for \( \sum_{i=1}^{n} \bar{Q}(Z_i) \), where,

\[
\bar{Q}(\cdot) = \frac{fG_1(\cdot) - fG_{-1}(\cdot)}{f(\cdot)}.
\]

In other words, we drop the additive and multiplicative constants in front of \( \bar{Q}(\cdot) \) that appear in the expression of \( Q(\cdot) \) in \((30)\). A central limit theorem (CLT) for \( \bar{Q} \) directly implies a CLT for \( \bar{Q} \) and thus also for \( L \). To prove the CLT for the sum of the \( \bar{Q}(Z_i) \), we note that \( \bar{Q}(Z_1), \ldots, \bar{Q}(Z_n) \) are i.i.d. conditionally on \( f, \mathcal{F}_n \), and so it suffices to verify Lindeberg’s condition conditionally.

All our calculations of conditional expectations happen on the event \( A_n \), which has asymptotic probability equal to 1. By definition of the event \( A_n \) in the statement of the theorem, there furthermore exists deterministic \( n_0 \) such that for all \( n \geq n_0 \), we also have that \( 4c_n \leq \eta \) and that \( \{ \inf f(z) > \eta/2, f_G(z)/f(z) \in [1/2, 2] \} \subset A_n \). We assume \( n \geq n_0 \) henceforth.

We start by studying the (conditional) moments of \( \bar{Q}(Z_i) \). For the first moment, we want to argue that its square is negligible compared to the second moment. Our argument crucially depends on the following cancellation:

\[
\int \frac{fG_1(z) - fG_{-1}(z)}{f(z)} \bar{f}(z) d\lambda(z) = \int fG_1(z) d\lambda(z) - \int fG_{-1}(z) d\lambda(z) = 1 - 1 = 0.
\]

Using this cancellation, we get:

\[
|\mathbb{E}_G [\bar{Q}(Z_i)]| = \left| \int \frac{fG_1(z) - fG_{-1}(z)}{f(z)} f_G(z) d\lambda(z) \right|
= \left| \int \frac{(fG_1(z) - fG_{-1}(z))}{f(z)} (f_G(z) - \bar{f}(z)) d\lambda(z) \right|
= \left| \int \frac{(fG_1(z) - fG_{-1}(z))}{f(z)} \frac{(f_G(z) - \bar{f}(z))}{f_G(z)} f_G(z) d\lambda(z) \right|
\leq \left| \int \frac{|fG_1(z) - fG_{-1}(z)|}{f(z)} \frac{|f_G(z) - \bar{f}(z)|}{f_G(z)} f_G(z) d\lambda(z) \right|
\leq \frac{c_n}{\eta} \int \frac{|fG_1(z) - fG_{-1}(z)|}{f(z)} f_G(z) d\lambda(z).
\]

We next turn to lower bound the second moment. Observe that almost surely, by Jensen’s inequality:

\[
\mathbb{E}_G [\bar{Q}(Z_i)^2] \geq \mathbb{E}_G [\bar{Q}(Z_i)]^2 = \left( \int \frac{|fG_1(z) - fG_{-1}(z)|}{f(z)} f_G(z) d\lambda(z) \right)^2.
\]

These two displays together imply that:

\[
\mathbb{E}_G [\bar{Q}(Z_i)]^2 \leq \frac{c^2_n}{\eta^2} \mathbb{E}_G [\bar{Q}(Z_i)^2] \leq \frac{1}{16} \mathbb{E}_G [\bar{Q}(Z_i)^2].
\]
Next,
\[
\mathbb{E}_G \left[ \tilde{Q}(Z_i)^2 \right] = \int \left( \frac{f_{G_1}(z) - f_{G_{-1}}(z)}{f(z)} \right)^2 \frac{f_G(z)}{f(z)} \, d\lambda(z) \geq \frac{1}{2} \int \left( \frac{f_{G_1}(z) - f_{G_{-1}}(z)}{f(z)} \right)^2 \, d\lambda(z) = \frac{\delta_n^2}{2n}.
\]

Hence,
\[
\Var_G \left[ \tilde{Q}(Z_i) \right] = \mathbb{E}_G \left[ \tilde{Q}(Z_i)^2 \right] - \mathbb{E}_G \left[ \tilde{Q}(Z_i) \right]^2 \geq \frac{15}{16} \mathbb{E}_G \left[ \tilde{Q}(Z_i)^2 \right] \geq \frac{15 \delta_n^2}{32 n} \geq \frac{\delta_n^2}{4n}.
\]

Furthermore,
\[
\left\| \tilde{Q}(\cdot) - \mathbb{E}_G \left[ \tilde{Q}(Z_i) \right] \right\|_{\infty} \leq 2 \left\| \tilde{Q}(\cdot) \right\|_{\infty} \leq \frac{2 \left\| f_{G_1}(z) - f_{G_{-1}}(z) \right\|_{\infty}}{\inf f(z)} \leq \frac{4c_n}{\eta}.
\]

So:
\[
\frac{\mathbb{E}_G \left[ \left| \tilde{Q}(Z_i) - \mathbb{E}_G \left[ \tilde{Q}(Z_i) \right] \right|^3 \right]}{\Var_G \left[ \tilde{Q}(Z_i)^{3/2} n^{1/2} \right]} \leq \frac{\left\| \tilde{Q}(\cdot) - \mathbb{E}_G \left[ \tilde{Q}(Z_i) \right] \right\|_{\infty}}{\Var_G \left[ \tilde{Q}(Z_i)^{1/2} n^{1/2} \right]} \leq \frac{8}{\eta} \frac{c_n}{n^{1/2} (\delta_n / n^{1/2})} \leq \frac{8c_n}{\eta \delta^2}. \tag{54}
\]

As argued in the beginning of the proof, this also implies the same bound for \( Q(\cdot), \)
\[
\frac{\mathbb{E}_G \left[ \left| Q(Z_i) - \mathbb{E}_G [Q(Z_i)] \right|^3 \right]}{\Var_G \left[ Q(Z_i)^{3/2} n^{1/2} \right]} \leq \frac{8c_n}{\eta \delta^2}. \tag{55}
\]

We could in principle conclude now by applying the Lyapunov/Lindeberg CLT conditionally on \( F_n, \tilde{f} \) along with Slutsky. To explain why the coverage of our intervals is uniform (under the conditions stated in the footnote of the theorem), we instead apply the Berry-Esseen bound (conditionally on \( F_n, \tilde{f} \)) for Student’s statistic [Bentkus and Götzte, 1996, Theorem 1.1.]. Recall the definition of \( \tilde{L} \) in (12), and define
\[
T = \frac{1}{n} \sum_{i=1}^{n} (Q(Z_i) - \mathbb{E}_G [Q(Z_i)]) / \tilde{V}^{1/2} = \left( \tilde{L} - \text{Bias}_G [Q, L] \right) / \tilde{V}^{1/2},
\]
and let \( \Phi \) be the standard Normal CDF. Then, there exists a constant \( C > 0, \) such that on the event \( A_n, \) and for \( n \) sufficiently large,\(^{26}\)
\[
\sup_{t \in \mathbb{R}} \left| \mathbb{E}_G [T \leq t] - \Phi(t) \right| \leq \min \left\{ 1, \frac{C}{\eta} \frac{c_n}{\delta^2} \right\}.
\]

\(^{26}\)Note that Bentkus and Götzte [1996], do not apply the \((n - 1)\) correction to the sample variance, in contrast to the definition of \( \tilde{V} \) in (13). The additional error introduced due to this discrepancy is negligible and may be absorbed into \( C. \)
It follows (unconditionally) that,
\[
\sup_{t \in \mathbb{R}} |P_G \{ T \leq t \} - \Phi(t) | = \sup_{t \in \mathbb{R}} \left| E_G \left[ \bar{P}_G \{ T \leq t \} - \Phi(t) \right] \right|
\leq \sup_{t \in \mathbb{R}} \left| E_G \left[ 1(A_n) \cdot \left( \bar{P}_G \{ T \leq t \} - \Phi(t) \right) \right] \right| + P_G \{ A_n^c \}
\leq C \frac{c_n}{\eta \delta^t} + P_G \{ A_n^c \}
\]
The first part of the Theorem follows, since \( c_n \to 0 \) and \( P_G \{ A_n^c \} \to 0 \) as \( n \to \infty \), and so,
\[
\left( \hat{L} - L(G) - \text{Bias}_G(Q, L) \right) / \sqrt{V} \overset{D}{\rightarrow} \mathcal{N}(0, 1).
\]
From Proposition 15, we know that \( \sup_{G \in \mathcal{G}_n} |\text{Bias}_G(Q, L)| = \hat{B} \), and so, on the event \( \{ G \in \mathcal{G}_n \} \subset A_n \) it also holds that \( |\text{Bias}_G(Q, L)| \leq \hat{B} \), i.e.,
\[
P_G \left[ |\text{Bias}_G(Q, L)| \leq \hat{B} \right] \geq P_G \{ A_n \} = 1 - o(1).
\]
It remains to prove coverage. Let \( \hat{t}_n = t_\alpha(\hat{B}/\sqrt{V^{1/2}}, 1) \) and \( \hat{b} = \text{Bias}_G(Q, L)/\sqrt{V^{1/2}} \), then,
\[
P_G \{ L(G) \in \mathcal{I}_\alpha \} = P_G \left[ \hat{L} - L(G) \leq t_\alpha(\hat{B}, \sqrt{V}) \right]
= P_G \left[ \left( \hat{L} - L(G) - \text{Bias}_G(Q, L) \right) / \sqrt{V^{1/2}} \leq \hat{t} - \hat{b} \right]
= P_G \left[ -\hat{t} - \hat{b} \leq \left( \hat{L} - L(G) - \text{Bias}_G(Q, L) \right) / \sqrt{V^{1/2}} \leq \hat{t} - \hat{b} \right]
= E_G \left[ \Phi(\hat{t} - \hat{b}) - \Phi(-\hat{t} - \hat{b}) \right] - o(1)
\geq E_G \left[ \left( \Phi(\hat{t} - \hat{b}) - \Phi(-\hat{t} - \hat{b}) \right) \cdot 1 \left( |\hat{b}| \leq \hat{B}/\sqrt{V^{1/2}} \right) \right] - o(1)
\overset{(i)}{\geq} (1 - \alpha) P_G \left[ |\text{Bias}_G(Q, L)| \leq \hat{B} \right] - o(1)
\overset{(ii)}{=} 1 - \alpha - o(1)
\]
In (i) we used the definition of \( t_\alpha(\cdot, \cdot) \) from (15) and in (ii) we used the bound on the bias we derived above.

**B.4 Proof of Proposition 3**

**Proof.** Throughout the proof we take \( c_n = k^{-1/4} \log(k) \) with \( k = k_n \). There are three things we need to check (for each case).

(i) First we check the quality of the pilot \( \hat{f} \). The key to our argument here is that
\[
\sup_{t \in \mathbb{R}} \left| \hat{F}_n(t) - F_G(t) \right| = O_P(1/\sqrt{k}),
\]
see e.g., below (16) for the justification (and note that here we use \( k \) instead of \( n \) samples).
Hence,

\[ t_n := \sup_{t \in \mathbb{R}} |F_{\tilde{G}_n}(t) - F_G(t)| \]

\[ \leq \sup_{t \in \mathbb{R}} |F_{\tilde{G}_n}(t) - \hat{F}_n(t)| + \sup_{t \in \mathbb{R}} |\hat{F}_n(t) - F_G(t)| \]

\[ \leq 2 \sup_{t \in \mathbb{R}} |\hat{F}_n(t) - F_G(t)| \]

\[ = O_P(1/\sqrt{k}). \]

We note that (*) follows by the definition of \( \hat{G}_n \) as the minimum distance estimator (24). In the remainder of the proof we seek to bound \( \sup_z |f_G(z) - \hat{f}(z)| \) in terms of \( t_n \).

For the discrete examples from part (b), we only use the fact that \( \lambda \) is the counting measure on (a subset of) \( \mathbb{N}_{\geq 0} \), so that e.g., \( f_G(0) = F_G(0) \) and \( f_G(z) = F_G(z) - F_G(z-1) \) for \( z > 0 \). Hence, recalling that \( \hat{f}(z) = f_{\tilde{G}_n}(z) \), and by the triangle inequality, we conclude that,

\[ \sup_z |f_G(z) - \hat{f}(z)| \leq 2 \sup_{t \in \mathbb{R}} |F_{\tilde{G}_n}(t) - F_G(t)| = 2t_n = O_P(1/\sqrt{k}). \]

Let us now turn to the Gaussian example from part (a), say with \( \sigma = 1 \) (without loss of generality). We handle the density at \( z = <, \) resp. \( z = > \) as in the discrete examples and focus now on the Lebesgue density at \( z \in [-M,M] \). We record the following fact. We have that,

\[ f_G(z) = \mathbb{E}_G [\varphi(z - \mu)], \]

where \( \varphi \) is the standard Gaussian pdf. Hence,

\[ f_G'(z) = \mathbb{E}_G [\varphi'(z - \mu)]. \]  

(56)

Consequently \( f_G'(z) \) is bounded, uniformly over all priors \( G \) and all \( z \). Thus, by Taylor’s theorem, there exists a constant \( C > 0 \) such that for all \( G, z \) and \( h > 0 \):

\[ \left| f_G(z) - \frac{1}{h} (F_G(z+h) - F_G(z)) \right| \leq C \cdot h. \]

Arguing by the triangle inequality, we have that

\[ \sup_z |f_G(z) - \hat{f}(z)| \leq \frac{2t_n}{h} + 2Ch, \]

and by choosing \( h = \sqrt{t_n} \), we conclude that:

\[ \sup_z |f_G(z) - \hat{f}(z)| \leq O_P(\sqrt{t_n}) = O_P(k^{-1/4}). \]

(ii) Second we check that the localizations indeed include \( F_G \), i.e., that \( \mathbb{P}[F_G \in \mathcal{F}_n] \to 1 \). Here we use the fact, that all \( F \)-localizations considered in this proposition are nested in \( \alpha \), i.e., \( \mathcal{F}_n(\alpha) \subset \mathcal{F}_n(\alpha') \) for \( \alpha' \leq \alpha \).

Fix \( \varepsilon > 0 \). Let \( n_0 \) be such that \( \alpha_n < \varepsilon/2 \) for all \( n \geq n_0 \) (recall that \( \alpha_n \to 0 \)) and \( n_1 \) be such that,

\[ \mathbb{P}_G[F_G \in \mathcal{F}_n(\varepsilon/2)] \geq 1 - \varepsilon/2 - \varepsilon/2 = 1 - \varepsilon \quad \text{for all} \quad n \geq n_1. \]
Such $n_1$ exists since all $F$-localizations are asymptotically valid (at a fixed confidence level) in the sense of (6). Thus, using the nestedness property, for $n \geq \max \{n_0, n_1\}$, we have that,

$$\mathbb{P}_G [F_G \in \mathcal{F}_n(\alpha_n)] \geq \mathbb{P}_G [F_G \in \mathcal{F}_n(\varepsilon/2)] \geq 1 - \varepsilon.$$  

Since $\varepsilon > 0$ was arbitrary, we conclude.

(iii) It remains to check that $\|f_{G_1}(\cdot) - f_{G_{-1}}(\cdot)\|_\infty \leq c_0$ with probability tending to 1, where $G_1, G_{-1}$ are the solutions to the modulus problem. Note that by definition, $F_{G_1}, F_{G_{-1}} \in \mathcal{F}_n(\alpha_n)$. We consider each $F$-localization separately.

**DKW-$F$-localization:** By (16) (with the sample size $n$ replaced by $k$),

$$\sup_t |F_{G_i}(t) - F_{G_{-1}}(t)| \leq \sup_t |F_{G_i}(t) - \hat{F}_n(t)| + \sup_t |F_{G_{-1}}(t) - \hat{F}_n(t)| \leq 2\sqrt{\log(2/\alpha_n)/(2k)}.$$  

Then, we can argue as in part (i) of this proof that this implies bounds on $\|f_{G_1}(\cdot) - f_{G_{-1}}(\cdot)\|_\infty$ in all cases (note that in the Gaussian case we use the bounded second derivative argument for $z \in [-M, M]$ and handle $\triangleleft, \triangleright$ as discrete).

**$\chi^2$-$F$-localization:** Consider the event $C_n = \{\inf_z \bar{f}(z) > \eta/2\}$, where $\eta > 0$ is such that $\inf_z f_G(z) > \eta$. $C_n$ has probability tending to 1 as $n \to \infty$ (as follows from the proof of (i) above). Note that, on the event $C_n$, for any $z'$,

$$(f_{G_{-1}}(z') - f_{G_{-1}}(z'))^2 \leq \frac{2}{\eta} \sum_i (f_{G_{-1}}(z) - f_{G_{-1}}(z))^2 \leq \frac{2}{\eta} \frac{(\delta^u)^2}{n}.$$  

Thus,

$$\|f_{G_1}(\cdot) - f_{G_{-1}}(\cdot)\|_\infty = O(1/\sqrt{n}) = o(1/\sqrt{k}).$$

**Gauss-$F$-localization:** Again all our calculations assume that the event $\{\inf_z \bar{f}(z) > \eta/2\}$ has occurred. Let us first treat $\triangleleft$ and $\triangleright$ separately. Arguing as in the case for the $\chi^2$-$F$-localization, we find that,

$$(f_{G_1}(z) - f_{G_{-1}}(z))^2 \leq \frac{2}{\eta} \frac{(\delta^u)^2}{n},$$

and similarly for $\triangleright$. Next, for $z \in [-M, M]$, we use the smoothness of the convolved densities. Namely, suppose,

$$|f_{G_1}(z) - f_{G_{-1}}(z)| =: \varepsilon > 0.$$  

Then, $|f_{G_1}(z) - f_{G_{-1}}(z)| > \varepsilon/2$ in an interval of length $c \cdot \varepsilon$, for a small constant $c > 0$, as follows by the boundedness of the derivative (56). This means that,

$$\frac{(\delta^u)^2}{n} \geq \int_{[-M,M]} \frac{(f_{G_1}(z) - f_{G_{-1}}(z))^2}{f(z)} d\LambdaLeb(z) \geq c' \varepsilon^3,$$

for another constant $c'$. Note that we also used the fact that $\bar{f}(z)$ is uniformly bounded. By rearranging, we find that,

$$|f_{G_1}(z) - f_{G_{-1}}(z)| = O_p(n^{-1/3}) = o_p(k^{-1/3}).$$

\[\square\]
B.5 Proof of Theorem 5

Proof. This proof is a continuation of the proof of Theorem 2, and so we also use the notation used therein. In particular, all calculations take place on the event $A_n$ (which has probability tending to 1 as $n \to \infty$). Furthermore, we write $\text{Cov} \{ \cdot \}$ for the covariance conditionally on $F_n, f$. Let $c^* = \theta_G(z)$ and $\kappa^*$ be such that $c^* = \kappa^* c^l + (1 - \kappa^*) c^u$ and $Q^c = \kappa^* Q^l + (1 - \kappa^*) Q^u$. Note that $\kappa^* \in [0, 1]$, since on the event $A_n$, $F_G \in F_n$, and so $c^l \leq c^* \leq c^u$.

We first note that our algorithm estimates the bias conservatively. To see this, note that:

$$\text{Bias}_G[Q^c, L] = \int Q^c(z) f_G(z) d\lambda(z) - L(G)$$

$$= \int (\kappa^* Q^l(z) + (1 - \kappa^*) Q^u(z)) f_G(z) d\lambda(z) - L(G)$$

$$= \kappa^* \text{Bias}_G[Q^l, L] + (1 - \kappa^*) \text{Bias}_G[Q^u, L].$$

Thus,

$$\sup_G \left| \text{Bias}_G[Q^c, L] \right| \leq \kappa^* \sup_G \left| \text{Bias}_G[Q^l, L] \right| + (1 - \kappa^*) \sup_G \left| \text{Bias}_G[Q^u, L] \right|,$$

and the RHS is precisely our bound for the worst-case bias.

We next seek to prove that the Lyapunov/Lindeberg bound (55) that holds for $Q^l, Q^u$ on the event $A_n$ also applies to $Q^c$ (with a larger constant) on a smaller event, that however also has asymptotic probability equal to 1 (just as $A_n$ does). The result will follow, using the Berry-Esseen bound of Bentkus and Götze [1996] (as in the Proof of Theorem 2) and the argument in the proof of Corollary 4.

All our conditional calculations occur on the event $A_n$ of Theorem 2, and on the event,

$$\tilde{A}_n = \left\{ \text{Cov}_G \{ Q^l(Z_i), Q^u(Z_i) \} \geq (-1 + \varepsilon/2) \cdot \text{Var}_G \{ Q^l(Z_i) \} \cdot \text{Var}_G \{ Q^u(Z_i) \} \right\}.$$

We will show below that $P_G[\tilde{A}_n] \to 1$ and also $P_G[\tilde{A}_n \cap A_n] \to 1$ as $n \to \infty$. For now we seek to provide a Lyapunov/Lindeberg bound (55) for $Q^c$. To this end, first note that on $\tilde{A}_n \cap A_n$,

$$\text{Var}_G \{ Q^c(Z_i) \}$$

$$= \text{Var}_G \left[ \kappa^* Q^l(Z_i) + (1 - \kappa^*) Q^u(Z_i) \right]$$

$$= (\kappa^*)^2 \text{Var}_G \left[ Q^l(Z_i) \right] + (1 - \kappa^*)^2 \text{Var}_G \left[ Q^u(Z_i) \right] + 2 \kappa^* (1 - \kappa^*) \text{Cov}_G \left[ Q^l(Z_i), Q^u(Z_i) \right]$$

$$\geq (\kappa^*)^2 \text{Var}_G \left[ Q^l(Z_i) \right] + (1 - \kappa^*)^2 \text{Var}_G \left[ Q^u(Z_i) \right] - 2 \kappa^* (1 - \kappa^*) (1 - \varepsilon/2) \cdot \text{Var}_G \left[ Q^l(Z_i) \right] \cdot \text{Var}_G \left[ Q^u(Z_i) \right]$$

In the last step we used the inequality $2ab \leq a^2 + b^2$. On the other hand,

$$\tilde{E}_G \left[ \left| Q^c(Z_i) - \tilde{E}_G \left[ Q^c(Z_i) \right] \right|^3 \right]$$

$$= \tilde{E}_G \left[ \left| \kappa^* \left( Q^l(Z_i) - \tilde{E}_G \left[ Q^l(Z_i) \right] \right) + (1 - \kappa^*) \left( Q^u(Z_i) - \tilde{E}_G \left[ Q^u(Z_i) \right] \right) \right|^3 \right]$$

$$\leq 8 (\kappa^*)^3 \tilde{E}_G \left[ \left| Q^l(Z_i) - \tilde{E}_G \left[ Q^l(Z_i) \right] \right|^3 \right] + 8 (1 - \kappa^*)^3 \tilde{E}_G \left[ \left| Q^u(Z_i) - \tilde{E}_G \left[ Q^u(Z_i) \right] \right|^3 \right].$$
Thus we now combine the two aforementioned inequalities,

\[
\frac{\mathbb{E}_G \left[ (Q^c(Z_i) - \mathbb{E}_G [Q^c(Z_i)])^3 \right]}{\text{Var}_G [Q^c(Z_i)]^{3/2} n^{1/2}} \leq \frac{8 (\kappa^*)^3 \mathbb{E}_G \left[ (Q^\ell(Z_i) - \mathbb{E}_G [Q^\ell(Z_i)])^3 \right] + 8 (1 - \kappa^*)^3 \mathbb{E}_G \left[ (Q^\ell(Z_i) - \mathbb{E}_G [Q^\ell(Z_i)])^3 \right]}{\left( (\kappa^*)^2 \text{Var}_G [Q^\ell(Z_i)] + (1 - \kappa^*)^2 \text{Var}_G [Q^u(Z_i)] \right)^{3/2} (\epsilon/2)^{3/2} n^{1/2}}
\]

\[
\leq \frac{C}{\epsilon^{3/2}} \left\{ \frac{\mathbb{E}_G \left[ (Q^\ell(Z_i) - \mathbb{E}_G [Q^\ell(Z_i)])^3 \right]}{\text{Var}_G [Q^\ell(Z_i)]^{3/2} n^{1/2}} + \frac{\mathbb{E}_G \left[ (Q^u(Z_i) - \mathbb{E}_G [Q^u(Z_i)])^3 \right]}{\text{Var}_G [Q^u(Z_i)]^{3/2} n^{1/2}} \right\}
\]

\[
\leq C' \frac{c_n}{\epsilon^{3/2} \eta \delta^2}.
\]

Here \(C, C' > 0\) are some constants. The last step follows by applying (55) that holds for \(Q^\ell, Q^u\) by the proof of Theorem 2. It remains to prove that \(A_n\) has asymptotic probability tending to 1. To this end, let \(\hat{\epsilon} > 0\), and also let,

\[
W_n = \frac{1}{n} \sum_{i=1}^{n} \left( Q^\ell(Z_i) - \mathbb{E}_G [Q^\ell(Z_i)] \right) \left( Q^u(Z_i) - \mathbb{E}_G [Q^u(Z_i)] \right).
\]

Then:

\[
\mathbb{P}_G \left[ W_n - \text{Cov}_G \left[ Q^\ell(Z_i), Q^u(Z_i) \right] \geq \hat{\epsilon} \text{Var}_G [Q^\ell(Z_i)]^{1/2} \text{Var}_G [Q^u(Z_i)]^{1/2} \right] \leq \mathbb{E}_G \left[ \left( Q^\ell(Z_i) - \mathbb{E}_G [Q^\ell(Z_i)] \right)^2 \left( Q^u(Z_i) - \mathbb{E}_G [Q^u(Z_i)] \right)^2 \right] / \left( n\hat{\epsilon}^2 \text{Var}_G [Q^\ell(Z_i)] \text{Var}_G [Q^u(Z_i)] \right)
\]

\[
\leq \frac{C c_n^2}{n\hat{\epsilon}^2 \text{Var}_G [Q^\ell(Z_i)] \text{Var}_G [Q^u(Z_i)]}
\]

for another constant \(C > 0\). The argument for the last line is analogous to the argument that led up to (55). This also means that unconditionally,

\[
\mathbb{P}_G \left[ W_n - \text{Cov}_G \left[ Q^\ell(Z_i), Q^u(Z_i) \right] \geq \hat{\epsilon} \text{Var}_G [Q^\ell(Z_i)]^{1/2} \text{Var}_G [Q^u(Z_i)]^{1/2} \right] \leq \frac{C c_n^2}{(\hat{\epsilon} \eta \delta)^2} + \mathbb{P}_G [A_n^c] \to 0 \text{ as } n \to \infty.
\]

By a similar argument we can prove that \(n \hat{V}^{\ell} / \text{Var}_G [Q^\ell(Z_i)] = 1 + o_p(1), n \hat{V}^u / \text{Var}_G [Q^u(Z_i)] = 1 + o_p(1)\) and that

\[
\frac{1}{n} \sum_{i=1}^{n} \left( Q^\ell(Z_i) - \mathbb{E}_G [Q^\ell(Z_i)] \right) / \text{Var}_G [Q^\ell(Z_i)]^{1/2} = o_p(1),
\]

and similarly for \(Q^u\). Combining the above results, with the assumption of the Theorem, it follows that \(\mathbb{P}_G [A_n \cap A_n] \to 1\). \qed
C Proofs for Section 7 on asymptotic power

A word on notation: We drop the dependence on \( n \) and \( M \), whenever this does not cause confusion. For example we may write \( F_G \) instead of \( F_G^M \). For the results for AMARI, we follow Proposition 3 and assume that the pilot quantities \( \bar{f} \) and \( F_n \) are constructed based on \( k \) samples with \( k \to \infty \), \( k/n \to 0 \) and \( k \cdot \alpha_n \to \infty \) as \( n \to \infty \). To keep the notation lighter we suppose that we compute \( \bar{f} \) and \( F_n \) based on \( k \) fresh samples from model (1) that are independent from \( Z_1, \ldots, Z_n \). The asymptotic confidence interval lengths remain the same as under the sample-splitting of Proposition 3, because \((n - k)/n \to 1\) as \( n \to \infty \).

C.1 Poisson model (Section 7.1)

C.1.1 Moment space calculations

We start with some preliminary definitions and a lemma that will be needed for the proofs of the theoretical results of Section 7.1. For any measure \( H \) supported on \([a, b]\), we write:

\[
m_k(H) := \int_a^b \mu^k dH(\mu), \quad k = 0, \ldots, M, \tag{57}
\]

for its moments.\(^{27}\) We define the moment space:

\[
\mathcal{M} := \left\{ (m_0(H), \ldots, m_M(H)) \in \mathbb{R}^{M+1} : H \text{ measure on } [a, b], \int \exp(\mu) dH(\mu) = 1 \right\}. \tag{58}
\]

The key lemma in this section is the following:

Lemma 16 (Open in Moment space). Let \( H \) be a measure supported on \([a, b]\), \(-\infty < a < b < \infty\) with at least \( M + 2 \) points of support, such that \( \int \exp(\mu) dH(\mu) = 1 \). Then, \((m_0(H), \ldots, m_M(H))\) is an element of the interior of \( \mathcal{M} \).

Proof. We will show at the end of the proof, that we may assume without loss of generality that there exist points \( a \leq \xi_1 < \xi_2 < \ldots < \xi_{M+2} \leq b \) such that \( \min_{j=1}^{M+2} H(\{\xi_j\}) > \zeta \) for some \( \zeta > 0 \).

Take \( \varepsilon > 0 \) (which we will specify later). Let \((m'_0, \ldots, m'_M)\) be such that \( |\Delta_k| = m'_k - m_k(H) \) satisfies \(|\Delta_k| < \varepsilon\) for all \( k = 0, \ldots, M \). We want to show that \((m'_0, \ldots, m'_M)\) is an element of \( \mathcal{M} \). To this end, we will consider perturbations of \( H \) of the following form. For \( a \in \mathbb{R}^{M+2} \), we consider:

\[
H_a = H + \sum_{j=1}^{M+2} a_j \delta_{\xi_j},
\]

where \( \delta_{\xi} \) is the Dirac point mass at \( \xi \). Our goal is to pick \( a \) by solving the following linear system:

\[
\sum_{j=1}^{M+2} a_j \exp(\xi_j) = 0 \quad \sum_{j=1}^{M+2} a_j \xi_j^k = \Delta_k \quad \text{for} \quad k = 0, \ldots, M.
\]

\(^{27}\)\( m_0(H) \) need not be 1, since we do not only consider probability measures.
Call $Ξ$ the matrix of this linear system, then $Ξa = (0, Δ_0, \ldots, Δ_M)^T$, where

$$Ξ = \begin{pmatrix}
\exp(ξ_1) & \exp(ξ_2) & \exp(ξ_3) & \ldots & \exp(ξ_{M+1}) & \exp(ξ_{M+2}) \\
1 & 1 & 1 & \ldots & 1 & 1 \\
ξ_1 & ξ_2 & ξ_3 & \ldots & ξ_{M+1} & ξ_{M+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
ξ_1^M & ξ_2^M & ξ_3^M & \ldots & ξ_{M+1}^M & ξ_{M+2}^M
\end{pmatrix}.$$ 

We will prove below that $Ξ$ is invertible. We pick $ε > 0$ small enough, so that:

$$|Δ_k| < ε \text{ for } k = 0, \ldots, M \implies \||Ξ^{-1}(0, Δ_0, \ldots, Δ_M)^T||_∞ < ζ/2.$$ 

With $a$ picked as above, it then holds that:

$$H_a(\{ξ_j\}) = H(\{ξ_j\}) + a_j ≥ ζ/2 > 0.$$ 

$$\int \exp(μ)dH_a(μ) = \int \exp(μ)dH(μ) + \sum_{j=1}^{M+2} a_j \exp(ξ_j) = 1 + 0 = 1.$$ 

Hence $H_a$ is a candidate measure with moments:

$$m_k(H_a) = m_k(H) + \sum_{j=1}^{M+2} a_j ξ_j^k = m_k(H) + Δ_k = m'_k.$$ 

Thus $(m'_0, \ldots, m'_M) ∈ M$, and so $(m_0(H), \ldots, m_M(H))$ lies in the interior of $M$. We still need to prove the invertibility of $Ξ$. To this end, we define the function

$$D(ξ) = \begin{pmatrix}
\exp(ξ_1) & \exp(ξ_2) & \exp(ξ_3) & \ldots & \exp(ξ_{M+1}) & \exp(ξ) \\
1 & 1 & 1 & \ldots & 1 & 1 \\
ξ_1 & ξ_2 & ξ_3 & \ldots & ξ_{M+1} & ξ \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
ξ_1^M & ξ_2^M & ξ_3^M & \ldots & ξ_{M+1}^M & ξ^M
\end{pmatrix}$$

and want to prove that $D(ξ_{M+2}) = |Ξ| \neq 0$. Suppose otherwise. Then $ξ_{M+2}$ is a root of $D(·)$, and so are $ξ_1, \ldots, ξ_{M+1}$. On the other hand, we can write $D(ξ)$ as:

$$D(ξ) = b_{-1} \exp(ξ) + \sum_{j=0}^{M} b_j ξ^j,$$

for some $b_{-1}, \ldots, b_M ∈ \mathbb{R}$. If $b_{-1} = 0$, then $D(·)$ is a polynomial of degree $M$ and can have at most $M$ roots, which is a contradiction. If $b_{-1} \neq 0$, then by applying Rolle’s theorem multiple times, we find that $b_j \exp(·)$ must have a root, which is also a contradiction. Thus $D(ξ_{M+2}) \neq 0$ and $Ξ$ is invertible.

We also need to justify why we could assume in the beginning of the proof that there exist points $a ≤ ξ_1 < ξ_2 < \ldots < ξ_{M+2} ≤ b$ such that $\min_{j=1}^{M+2} H(\{ξ_j\}) > ζ$ for some $ζ > 0$. We follow the proof idea of Pinelis [2017]. By assumption, the support set of $H$ consists of at least $M + 2$ points in $[a, b]$. This means that there exist pairwise disjoint closed intervals
Proof. We divide the proof into three steps. Throughout we write $C.1.2$ Proof for Proposition 6: DKW-

$\alpha, b$ \checkmark{Tchebycheff system \cite[Definition 1.1, Chapter I]{KarlinStudden1966}} on every closed, $\xi$ that the proof of the invertibility of $\Xi$ also demonstrates that $\mu \mapsto (1, \mu, \ldots, M^\alpha)$, $\exp(\mu)$ is a Tchebycheff system \cite[Theorem 1.2, Chapter II]{KarlinStudden1966}, Definition 1.1, Chapter I\n
on every closed, nonempty interval that is a subset of $[a, b]$. Since $H_j(I_j) = H(I_j) > 0$, there exists $\xi_j \in I_j$ such that $H_j(\{\xi_j\}) > 0$.

Consider the measure $H$ that is defined on Borel sets $A \subset [a, b]$ as follows:

$$H(A) = H\left(A \setminus \bigcup_{j=1}^{M+2} I_j\right) + \sum_{j=1}^{M+2} H_j(A \cap I_j).$$

Then $H$ is also a measure supported on $[a, b]$ and:

$$\int_a^b \exp(\mu)dH(\mu) = \int_a^b \exp(\mu)dH(\mu), \quad m_k(H) = m_k(H), \quad k = 1, \ldots, M.$$ We may now repeat the argument of this proof with $H$ replacing $H$ to arrive at the conclusion of the Lemma.

\textbf{C.1.2 Proof for Proposition 6: DKW-$F$-Localization.}

In this section we provide the proof of the statement for the DKW-$F$-Localization.

Proof. We divide the proof into three steps. Throughout we write $c_n = \sqrt{\log(2/\alpha)/(2n)}$ and $|\mathcal{Z}|$ for the length of the DKW-$F$-localization interval.

- **Step 1**: We first prove that $|\mathcal{Z}| \leq 4c_n$ almost surely.
- **Step 2**: Let $A_n$ be the event on which there exist distributions $F^M_\ell$, $F^M_u$ on $\{0, 1, \ldots, M - 1, M, \mathcal{G}\}$ with $F^M_\ell$, $F^M_u \in \mathcal{F}^{\alpha\text{DKW}}_n$ that make the inequalities used in Step 1 tight. We show that $\mathbb{P}_G[A_n] \to 1$ as $n \to \infty$.
- **Step 3**: In Step 2, we ignored the fact that the distributions we constructed need not be marginal distributions in the empirical Bayes problem. Let $B_n^\alpha$ be the event that the distribution $F^M_u$ from Step 2 may be represented as $F^M_u = F^M_G$ as in (5), where $G \in \mathcal{G}$. $B_n^\alpha$ is defined similarly for $F^M_\ell$. We then prove that also $\mathbb{P}_G[B_n^\alpha \cap B_n^\alpha] \to 1$.

Using the results from Steps 2 and 3, we see that with probability tending to 1, it holds that $|\mathcal{Z}| \geq 4c_n$, and so it also follows that $|\mathcal{Z}|/(4c_n) = 1 + o_G(1)$.

- **Step 1**: Take any distribution $F \in \mathcal{F}^{\alpha\text{DKW}}_n$. Then the following holds for its density at $z \in \{1, \ldots, M\}$.

$$f(z) = F(z) - F(z - 1) = \left(\tilde{F}_n(z) - \tilde{F}_n(z - 1)\right) + \left(F(z) - \tilde{F}_n(z)\right) - \left(F(z - 1) - \tilde{F}_n(z - 1)\right) = \tilde{f}_n(z) + \left(F(z) - \tilde{F}_n(z)\right) - \left(F(z - 1) - \tilde{F}_n(z - 1)\right) \leq \tilde{f}_n(z) + 2c_n. \tag{59}$$
In the last inequality we used the definition of the DKW band and \( \hat{f}_n(z) = \# \{ Z_i = z \} / n \). Similarly, we may conclude that \( f(z) \geq \hat{f}_n(z) - 2c_n \). Combining these two results, we see that the DKW-F-Localization band for \( L(G) = \hat{f}_G(z) \) must satisfy,
\[
I \subset [\hat{f}_n(z) - 2c_n, \hat{f}_n(z) + 2c_n],
\]
and so its length can be at most \( 4c_n \).

**Step 2:** We define \( F_u \) as follows:
\[
F_u(z') = \hat{F}_n(z') \quad \text{for} \quad z' \notin \{ z - 1, z \}, \quad F_u(z - 1) = \hat{F}_n(z - 1) - c_n, \quad F_u(z) = \hat{F}_n(z) + c_n.
\]

\( F_u \) is tight for the inequality in (59), since:
\[
\left( F_u(z) - \hat{F}_n(z) \right) - \left( F_u(z - 1) - \hat{F}_n(z - 1) \right) = 2c_n.
\]

\( F_u \) satisfies the constraints of the DKW-band, however \( F_u \) is not necessarily a distribution function. Let us define \( A_n^* \), as the event on which \( \hat{F}_n(z - 1) - c_n > \hat{F}_n(z - 2) \) (or \( 0 > 0 \) if \( z = 1 \)) and \( \hat{F}_n(z) + c_n < \hat{F}_n(z + 1) \). Since the true distribution is a Poisson mixture \( \hat{F}_G \) and \( G \) is not only supported on the point 0, it holds that \( F_G(z - 2) < F_G(z - 1) \) (if \( z \geq 2 \) and \( F_G(z - 1) > 0 \)) if \( z = 1 \) and that \( F_G(z) < F_G(z + 1) \). Since \( c_n \to 0 \) and by the Glivenko-Cantelli Theorem, we conclude that \( \mathbb{P}_G(A_n^*) \to 1 \). We may similarly define and argue for \( F_t \) (with a corresponding event \( A_n^t \)). Since \( A_n \supseteq A_n^* \cap A_n^t \), we conclude.

**Step 3:** We define \( H_G \) as the measure that is absolutely continuous w.r.t. \( G \) with Radon-Nikodym derivative \( dH_G/dG(\mu) = \exp(-\mu) \). Recall the definition of moments \( m_k(\cdot) \) in (57) and the moment space \( \mathcal{M} \) (58). For the measure \( H_G \) it holds that
\[
m_k(H_G) = \int_a^b \mu^k dH_G(\mu) = \int_a^b \mu^k \exp(-\mu) dG(\mu) = k! f_G(k).
\]

Furthermore, since \( G \) is supported on at least \( M + 2 \) points, so is \( H_G \). By Lemma 16 (\( m_0(H_G), \ldots, m_M(H_G) \)) lies in the interior of the moment space \( \mathcal{M} \), i.e., there exists an open set \( U \subset \mathbb{R}^{M+1} \) such that \( (m_0(H_G), \ldots, m_M(H_G)) \in U \) and \( U \subset \mathcal{M} \). We define the bijective mapping
\[
T = (T_0, \ldots, T_M) : \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}, \quad T_k(u_0, \ldots, u_M) = \sum_{j=0}^k \frac{u_j}{j!}, \quad k = 0, \ldots, M.
\]

Then, for two elements \( u = (u_0, \ldots, u_M), v = (v_0, \ldots, v_M) \in \mathbb{R}^{M+1} \), we let:
\[
d(u, v) = \max_{k=0}^M |T_k(u) - T_k(v)|.
\]

\( d(\cdot, \cdot) \) is a distance for \( \mathbb{R}^{M+1} \) that metrizes the standard topology. Thus there exists \( \varepsilon > 0 \) so that:
\[
u \in U \quad \text{for all} \quad u \quad \text{with} \quad d(u, (m_0(H_G), \ldots, m_M(H_G))) < \varepsilon.
\]

Observing that \( T_k(m_0(H_G), \ldots, m_M(H_G)) = F_G(k) \) for \( k = 0, \ldots, M \), we conclude that:
\[
d(T^{-1}(F_u(0), \ldots, F_u(M)), (m_0(H_G), \ldots, m_M(H_G)))
= \max_{k=0}^M |F_u(k) - F_G(k)| = \sup_t |F_u^M(t) - F_G^M(t)|.
\]

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By construction of $F_u$ in Step 2 and the Glivenko-Cantelli theorem, we see that the RHS above converges almost surely to 0. In turn, this means that

$$\mathbb{P} \left[ T^{-1}(F_u(0), \ldots, F_u(M)) \in \mathcal{U} \right] \to 1.$$ 

On the event inside the probability above, we can find a measure $H_u$ supported on $[a, b]$, such that:

$$T(m_0(H_u), \ldots, m_M(H_u)) = (F_u(0), \ldots, F_u(M)).$$

Finally, let $G_u$ be the measure that is absolutely continuous w.r.t. $H_u$ with Radon-Nikodym derivative $dG_u(\mu)/dH_u(\mu) = \exp(\mu)$. $G_u$ is a probability measure, since:

$$\int dG_u(\mu) = \int \exp(\mu) dH_u(\mu) = 1,$$

by definition of the moment space $\mathcal{M}$. Furthermore, it holds that $F_u = F_{G_u}$ on $\{0, 1, \ldots, M, \triangleright\}$.

We conclude after arguing analogously for $F_\ell$. 

\vspace{0.5cm}

C.1.3 Proof for Proposition 6: AMARI.

The modulus problem (28) at $\delta > 0$ takes the form:

$$\max_{G_{\triangleright}, G_{\triangleright-1} \in \mathfrak{G}_n} f_{G_{\triangleright}}(z) - f_{G_{\triangleright-1}}(z) \quad \text{s.t.} \quad \sum_{z' \in \{0, \ldots, M\} \cup \{\triangleright\}} \frac{(f_{G_{\triangleright}}(z') - f_{G_{\triangleright-1}}(z'))^2}{f(z')} \leq \frac{\delta^2}{n}. \quad (60)$$

We first solve a relaxation of the above optimization problem (we will show that the relaxation is tight later), in which we introduce variables $h_j$ that formally correspond to $f_{G_{\triangleright}}(j) - f_{G_{\triangleright-1}}(j)$:

$$\max_{h_0, \ldots, h_M, h_\triangleright} h_z \quad \text{subject to} \quad \sum_j \frac{h_j^2}{f(j)} \leq \frac{\delta^2}{n}, \quad \sum_j h_j = 0. \quad (61)$$

This is a convex optimization problem. Consider the Lagrangian with dual variables $\zeta \in \mathbb{R}$ and $\xi < 0$ (the dual objective is unbounded for $\xi = 0$):

$$\mathcal{L}(h; \xi, \zeta) = h_z + \xi \left( \sum_j \frac{h_j^2}{f(j)} - \frac{\delta^2}{n} \right) + \zeta \sum_j h_j.$$ 

The derivative with respect to $h_z$ is equal to,

$$\frac{\partial \mathcal{L}(h; \xi, \zeta)}{\partial h_z} = 1 + 2\xi \frac{h_z}{f(z)} + \zeta,$$

and with respect to $h_j, j \neq z$:

$$\frac{\partial \mathcal{L}(h; \xi, \zeta)}{\partial h_j} = 2\xi \frac{h_j}{f(j)} + \zeta.$$
By the first order optimality conditions, we conclude that 
\[ h_j / \bar{f}(j) = h_{j'} / \bar{f}(j') \] for all \( j, j' \neq z \), and so there exists a constant \( t \) such that \( h_j = t \bar{f}(j) \) for all \( j \neq z \). Furthermore,

\[ h_z = - \sum_{j \neq z} h_j = - t \sum_{j \neq z} \bar{f}(j) = - t(1 - \bar{f}(z)). \]

Thus \( t = -h_z / (1 - \bar{f}(z)) \) and we only need to optimize in (61) with respect to a single parameter \( h_z \), which we seek to maximize. The pseudo-\( \chi^2 \) constraint takes the form:

\[
\sum_j \frac{h_j^2}{\bar{f}(j)} = \frac{h_z^2}{\bar{f}(z)} + \sum_{j \neq z} t^2 \bar{f}(j) = \frac{h_z^2}{\bar{f}(z)} + \frac{h_z^2}{1 - \bar{f}(z)} = \frac{h_z^2}{\bar{f}(z)(1 - \bar{f}(z))}.
\]

We want the above to be equal to \( \delta^2 / n \), and so to maximize \( h_z \) subject to the above constraint, we get

\[
h_z = \delta \sqrt{n} / \sqrt{\bar{f}(z)(1 - \bar{f}(z))}, \tag{62}
\]

as the optimal value of the relaxed modulus problem (61). To argue that the relaxation is tight (with probability tending to 1), we need to exhibit priors \( G_1, G_{-1} \in \mathcal{G}_n \) such that \( f_{G_1}(j) - f_{G_{-1}}(j) = h_j \), where \( (h_j)_{j} \) is the maximizer of (61) derived above. To do so, we proceed as follows. Let \( \hat{f}_{DKW}^n(z) \) be the frequency of \( z \) in the sample used to construct the pilot DKW-F-localization. Then define the pmf \( f_n \) on \( \{0, \ldots, M, \triangleright\} \) as:

\[
f_n(z) = \hat{f}_{DKW}^n(z) + \frac{h_z}{2}, \quad f_n(z') = \hat{f}_{DKW}^n(z') - \frac{h_z \bar{f}(z')}{2(1 - \bar{f}(z))} \quad \text{for } z' \in \{0, \ldots, M, \triangleright\} \setminus \{z\}.
\]

We make the following observation. First, \( h_z \) in (62) is of order \( O_p(1/\sqrt{n}) \), which is of smaller order than the width of the DKW band \( O_p(\sqrt{\log(2/\alpha_n)}/\sqrt{n}) \). Thus, arguing as in Steps 2 and 3 of Supplement C.1.2, we can prove that the following event has probability tending to 1: there exists a prior \( G_1 = G_{1,n} \in \mathcal{G} \) such that \( f_{G_1} \in \mathcal{F}_n \) and such that \( f_{G_1}(z') = f_n(z') \) for all \( z' \in \{0, \ldots, M, \triangleright\} \). The same argument also applies to \( f_{\ell} \) defined as \( f_{\ell}(z') = f_n(z') - h_z \) for \( z' \in \{0, \ldots, M, \triangleright\} \). We thus conclude that the optimal value of the modulus problem (60) is equal to (making the dependence of \( \bar{f}(z) \) on \( n \) explicit):

\[
\omega_n(\delta) = \delta \sqrt{n} / \sqrt{\bar{f}_n(z)(1 - \bar{f}_n(z))}.
\]

\( \omega_n(\delta) \) is differentiable in \( \delta \) with derivative

\[
\omega_n'(\delta) = \frac{1}{\sqrt{n}} \cdot \frac{\bar{f}_n(z)(1 - \bar{f}_n(z))}{\sqrt{\bar{f}_n(z)(1 - \bar{f}_n(z))}} = \omega_n(\delta) / \delta.
\]
Let us plug the above into (30) to find the optimal $Q(\cdot)$. First, we consider the part of $Q$ that is a function of $z$. For $j \neq z$ we get

$$\frac{n \cdot \omega_n'(\delta)}{\delta} \cdot \frac{f_{G_1}(j) - f_{G_{-1}}(j)}{f_n(j)} = - \frac{n \cdot \omega_n'(\delta)}{\delta} \cdot \frac{h_z}{1 - f_n(z)} - \frac{n \cdot \omega_n'(\delta)}{\delta} \cdot \frac{\omega_n(\delta)}{1 - f_n(z)} - \frac{n \omega_n(\delta)^2}{\delta^2 \delta f_n(z)} = - \bar{f}_n(z).$$

For $z$ we get:

$$\frac{n \cdot \omega_n'(\delta)}{\delta} \cdot \frac{f_{G_1}(z) - f_{G_{-1}}(z)}{f_n(z)} = \frac{n \cdot \omega_n'(\delta)}{\delta} \cdot \frac{\omega_n(\delta)}{f_n(z)} = \frac{n \omega_n(\delta)^2}{\delta^2 f_n(z)} = \bar{f}_n(z).$$

It remains to evaluate the additive component in (30) that does not depend on $z$. Let $G_0 = (G_1 + G_{-1})/2$. We note that by construction $L(G_0) = f_{G_0}(z)$. Hence the constant term is equal to:

$$- \frac{n \cdot \omega_n'(\delta)}{\delta} \sum_j \left( \frac{f_{G_1}(j) - f_{G_{-1}}(j)}{f_n(j)} \right) f_{G_0}(j) = L(G_0)$$

$$= - \frac{n \cdot \omega_n(\delta)^2}{\delta^2} \cdot \left( \frac{f_{G_0}(z)}{f_n(z)} \cdot \frac{1 - f_{G_0}(z)}{1 - f_n(z)} \right) + L(G_0)$$

$$= \bar{f}_n(z)(1 - f_{G_0}(z)) - (1 - \bar{f}_n(z)) f_{G_0}(z) + L(G_0)$$

$$= \bar{f}_n(z).$$

We conclude that for $j \neq z$, $Q(j) = - \bar{f}_n(z) + \bar{f}_n(z) = 0$ and for $z$, $Q(z) = 1 - \bar{f}_n(z) + \bar{f}_n(z) = 1$. Thus $Q(\cdot) = 1(\cdot = z)$. Hence $\hat{L} = f_n(z) = \#\{Z_i = z\}/n$. The worst case absolute bias of $\hat{L}$ is given by:

$$\hat{B} = \frac{1}{2}(\omega_n(\delta) - \delta \omega_n'(\delta)) = 0.$$ 

The confidence intervals of AMARI in (15) ($\hat{V}$ as in (13)) hence take the form:

$$\hat{L} \pm q_{1-\alpha/2} \cdot \sqrt{\hat{V}},$$

with $q_{1-\alpha/2}$ the $1-\alpha/2$ quantile of the standard Normal distribution. Since $n\hat{V} \rightarrow f_G(z)(1 - f_G(z))$, we conclude.

C.1.4 Proof for Proposition 7: DKW-F-Localization

**Proof.** The proof will be structured very similarly to the proof in Supplement C.1.2 that concerned inference for $f_G(z)$. In particular, we follow the same three steps as in that proof.

**Step 1:** Take any distribution $F \in F_n^{DKW}(\alpha)$. Write $\theta(z) = (z+1)f(z+1)/f(z)$, we seek to provide lower and upper bounds on it. Note that when $F = F_G$, then by (9), we have that $\theta(z) = \mathbb{E}_G [\mu \mid Z = z]$. 

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\begin{align*}
(z + 1) \frac{f(z + 1)}{f(z)} &= (z + 1) \frac{F(z + 1) - F(z)}{F(z) - F(z - 1)} \\
&= (z + 1) \frac{\hat{f}_n(z + 1) + \left( F(z + 1) - \hat{F}_n(z + 1) \right) - \left( F(z) - \hat{F}_n(z) \right)}{\hat{f}_n(z) + \left( F(z) - \hat{F}_n(z) \right) - \left( F(z - 1) - \hat{F}_n(z - 1) \right)} \\
&\leq (z + 1) \frac{\hat{f}_n(z + 1) + 2c_n}{\hat{f}_n(z) - 2c_n}.
\end{align*}

In the last inequality we used the definition of the DKW band. Similarly, we may conclude that
\begin{equation}
(z + 1) \frac{f(z + 1)}{f(z)} \geq (z + 1) \frac{\hat{f}_n(z + 1) - 2c_n}{\hat{f}_n(z) + 2c_n}.
\end{equation}

Combining these two results, we see that the DKW-F-Localization band for $L(G) = f_G(z)$ must satisfy,
\begin{equation*}
\mathcal{I} \subset \mathcal{I}' := \left[ (z + 1) \frac{\hat{f}_n(z + 1) - 2c_n}{\hat{f}_n(z) + 2c_n}, (z + 1) \frac{\hat{f}_n(z + 1) + 2c_n}{\hat{f}_n(z) - 2c_n} \right].
\end{equation*}

We will prove that the above inclusion is in fact an equality below (with high probability and for large $n$). For now, we verify that $\mathcal{I}'$ has the claimed asymptotic length.
\begin{equation}
\sqrt{n} |\mathcal{I}'| = \sqrt{n}(z + 1) \frac{4c_n \left( \hat{f}_n(z + 1) + \hat{f}_n(z) \right)}{\left( \hat{f}_n(z) - 2c_n \right) \left( \hat{f}_n(z) + 2c_n \right)} = 2(z + 1) \sqrt{2 \log(2/\alpha) \frac{f_G(z) + f_G(z + 1)}{f_G(z)^2}} + o_{P_G}(1).
\end{equation}

**Steps 2 and 3:** We define $F_u$ as follows:
\begin{align*}
F_u(z') &= \hat{F}_n(z') \text{ for } z' \notin \{z - 1, z, z + 1\}, \\
F_u(z - 1) &= \hat{F}_n(z - 1) + c_n, \quad F_u(z) = \hat{F}_n(z) - c_n, \quad F_u(z + 1) = \hat{F}_n(z + 1) + c_n.
\end{align*}

Plugging $F_u$ into (63), we see that the last inequality is an equality. Furthermore, arguing as in Steps 2 and 3 of Supplement C.1.2, we can prove that the following event has probability tending to 1: There exists a prior $G = G_n \in \mathcal{G}$ such that $F_G \in \mathcal{F}_n$ and such that $F_G(z') = F_u(z')$ for all $z' \in \{0, \ldots, M, z\}$. We may define $F_t$ that makes (64) tight analogously. Hence, on an event that has probability tending to 1, it holds that $\mathcal{I} = \mathcal{I}'$. Thus the asymptotic length of $\mathcal{I}$ is equal to the asymptotic length of $\mathcal{I}'$ computed in (65). \qed

**C.1.5 Proof for Proposition 7: AMARI.**

*Proof.* In Supplement C.1.3 we solved the modulus problem in the Poisson problem, when $L(G) = f_G(z)$, and proved that the optimal $Q(\cdot)$ in (30) takes the form $Q(\cdot) = 1(\cdot = z)$ on an event with asymptotic probability 1. Here we will start by proving a generalization of the above result.
Concretely, we will fix \( a = (a_0, \ldots, a_M, a_\Delta) \) and we will consider the linear functional
\[
L(G) = \sum_{z' \in \{0, \ldots, M\} \cup \{\Delta\}} a_{z'} f_G(z')\]
For notational convenience (and with some abuse of notation), we identify \( f_G \) with the vector \( (f_G(0), \ldots, f_G(M), f_G(\Delta)) \). We also define the matrix \( \bar{D} = (f_G(0), \ldots, f_G(M), f_G(\Delta)) \) and write \( h = (h_0, \ldots, h_M, h_\Delta) \). The relaxed modulus problem (compare to (61)) takes the form:
\[
\begin{align*}
\text{maximize} & \quad a^\top h \\
\text{subject to} & \quad \sum_j h_j^2 \leq \frac{\delta^2}{n} \\
& \quad \sum_j h_j = 0.
\end{align*}
\]
Instead of the relaxed modulus problem, we consider the relaxed inverse modulus problem,\(^{28}\) which is parameterized by \( t > 0 \):
\[
\begin{align*}
\text{minimize} & \quad h^\top \bar{D}^{-1} h \\
\text{subject to} & \quad a^\top h = t \\
& \quad 1^\top h = 0.
\end{align*}
\]
(67) will enable us to also solve (66) and then to compute \( \omega_n(\delta) \). (67) is also a convex problem, so we introduce the Lagrangian (with dual variables \( \xi, \zeta \in \mathbb{R} \))
\[
\mathcal{L}(h; \xi, \zeta) = h^\top \bar{D}^{-1} h + 2\xi(a^\top h - t) + 2\zeta 1^\top h.
\]
We multiply the dual variables by 2 only for convenience. By the first order optimality conditions, we see that:
\[
\bar{D}^{-1} h = -\xi a - \zeta \mathbf{1} \implies h = -\bar{D}(\xi a + \zeta \mathbf{1}).
\]
\( \xi \) and \( \zeta \) are determined by a system of two linear equations. Namely from \( a^\top h = t, 1^\top h = 0 \).
\[
\begin{align*}
\xi a^\top \bar{D} a + \zeta a^\top \bar{D} \mathbf{1} & = -t \\
\xi a^\top \bar{D} \mathbf{1} + \zeta & = 0.
\end{align*}
\]
In deriving the second of the above inequalities, we used the fact that \( 1^\top \bar{D} \mathbf{1} = 1 \). It follows that:
\[
\xi = t / \left\{ (a^\top \bar{D} \mathbf{1})^2 - a^\top \bar{D} a \right\}, \quad \zeta = -\xi a^\top \bar{D} \mathbf{1}.
\]
The objective value of (67) is then equal to:
\[
\begin{align*}
h^\top \bar{D}^{-1} h & = \xi^2 a^\top \bar{D} a + \zeta^2 + 2\xi a \bar{D} \mathbf{1} \\
& = \xi^2 \cdot \left\{ a^\top \bar{D} a - (a^\top \bar{D} \mathbf{1})^2 \right\} \\
& = t^2 / \left\{ a^\top \bar{D} a - (a^\top \bar{D} \mathbf{1})^2 \right\}.
\end{align*}
\]
\(^{28}\)The proof of Theorem 3 in Cai and Low [2003] uses a similar proof technique using the inverse modulus of continuity.
We have solved the relaxed inverse modulus problem. This yields the solution to the relaxed modulus problem \((66)\) by choosing \(t\) so that \(h^\top D^{-1} h = \delta^2/n\), and so, the optimal value of \((66)\) is equal to:

\[
t^2 = \frac{\delta^2}{n} \left\{ a^\top Da - (a^\top D1)^2 \right\}.
\]

Arguing as in Supplement C.1.3, we find that the relaxed modulus problem is tight for the modulus problem (with probability tending to 1) and on the latter event:

\[
\omega_n(\delta) = \frac{\delta^2}{n} \left\{ a^\top Da - (a^\top D1)^2 \right\}.
\]

Consequently, \(\omega_n\) is differentiable at \(\delta > 0\) with \(\omega'_n(\delta) = \omega_n(\delta)/\delta\). We can continue as in the proof in Section C.1.3 by plugging the above into (30). The constant additive part of \(Q(\cdot)\) is equal to \(a^\top D1\). Hence, identifying \(Q\) with the vector \((Q(0), \ldots, Q(M), Q(\v))\), we find that:

\[
Q = \frac{n \cdot \omega'_n(\delta)}{\delta} \cdot D^{-1} h + (a^\top D1) 1
\]

\[
= -\frac{n \omega_n(\delta)}{\delta^2} (\xi a + \zeta 1) + (a^\top D1) 1
\]

\[
= -\frac{n \omega_n(\delta)}{\delta^2} \xi \{ a - (a^\top D1) 1 \} + (a^\top D1) 1
\]

\[
= a
\]

In the last step, we used the fact that \(-\frac{n \omega_n(\delta)}{\delta^2} \xi = 1\), since:

\[
-\frac{n \omega_n(\delta)}{\delta^2} \xi = -\frac{n \omega_n(\delta)}{\delta^2} - \omega_n(\delta) \left\{ (a^\top D1)^2 - a^\top Da \right\}
\]

\[
= -\frac{n}{\delta^2} \cdot \frac{\delta^2}{n} \left\{ a^\top Da - (a^\top D1)^2 \right\} \left\{ (a^\top D1)^2 - a^\top Da \right\} = 1.
\]

We note that the resulting estimator for \(L(G)\) is unbiased, i.e., the worst case bias in this case is equal to 0.

We are ready to return to the study of Algorithm 2. Let \([c^\ell, c^u]\) be the pilot \(F\)-localization intervals for \(\theta_G(z)\). By construction \(P_G \left[ \theta_G(z) \in [c^\ell, c^u] \right] \rightarrow 1\), and furthermore, by the proof in Supplement C.1.4, we also have that \(c^\ell = \theta_G(z) + \alpha F_G(1)\) and \(c^u = \theta_G(z) + \alpha F_G(1)\). By the preceding argument, we get for \(z' \in \{0, \ldots, M, \v\}:

\[
Q'(z') = (z+1)1(z' = z+1) - c^\ell 1(z' = z+1), \quad Q'^u(\cdot) = (z+1)1(z' = z+1) - c^u 1(z' = z+1).
\]

In particular \(\tilde{L}^\ell = (z+1)\tilde{f}_n(z+1) - c^\ell \tilde{f}_n(z), \quad \tilde{L}^u = (z+1)\tilde{f}_n(z+1) - c^u \tilde{f}_n(z), \quad \text{and for any } c \in [c^\ell, c^u], \quad \tilde{L}^c = (z+1)\tilde{f}_n(z+1) - c \tilde{f}_n(z).

Next note that \(\tilde{B}^c = 0\) and since \(t_\alpha(0, V) = q_{1-\alpha/2} \sqrt{V} \) in (15), to determine the AMARI confidence interval for \(\theta_G(z)\), we need to determine all \(c \in [c^\ell, c^u]\) such that:

\[
0 \in \tilde{L}_\alpha(z; c) = (z + 1)\tilde{f}_n(z+1) - c\tilde{f}_n(z) \pm q_{1-\alpha/2} \sqrt{V}.
\]

To do this it will be furthermore convenient to express \(\tilde{V}^c\) in Algorithm 2 in a slightly different form, namely

\[
\tilde{V}^c = (z+1)^2\tilde{V}_2 + c^2\tilde{V}_1 - 2c(z+1)\tilde{V}_{12}, \quad \tilde{V}_1 = -\frac{1}{n-1}\tilde{f}_n(z)(1 - \tilde{f}_n(z)),
\]

\[
\tilde{V}_2 = \frac{1}{n-1}\tilde{f}_n(z+1)(1 - \tilde{f}_n(z+1)), \quad \tilde{V}_{12} = -\frac{1}{n-1}\tilde{f}_n(z+1)\tilde{f}_n(z).
\]
Having rewritten \( \hat{V}^c \) as above and shortening \( q = q^{2-\alpha/2} \), we see that:

\[
0 \in \tilde{I}_n(z;c) \iff \left( (z+1)\hat{f}_n(z+1) - q\hat{f}_n(z) \right)^2 \leq q^2 \left[ (z+1)^2 \hat{V}_2 + c^2 \hat{V}_1 - 2c(z+1)\hat{V}_{12} \right].
\]

The latter condition is a quadratic inequality in \( c \), that we may rearrange as:

\[
\left( (z+1)^2 - q^2 \hat{V}_1 \right) c^2 + 2(z+1) \left( q^2 \hat{V}_{12} - \hat{f}_n(z+1)\hat{f}_n(z) \right) c + (z+1)^2 \left( \hat{f}_n(z+1)^2 - q^2 \hat{V}_2 \right) \leq 0.
\]

We make the observation that \( c = (z+1)\hat{f}_n(z+1)/\hat{f}_n(z) \) is an interior point of the above inequality. Furthermore, on the event \( \{ \hat{f}_n(z)^2 > q^2 \hat{V}_1 \} \) the above is a convex quadratic, and so the set of \( c \) satisfying the inequality must be a closed interval. Since \( \mathbb{P}_G(\hat{f}_n(z)^2 > q^2 \hat{V}_1) \to 1 \) as \( n \to \infty \), we restrict attention to that event. On that event, the distance between the two roots of the quadratic is equal to:

\[
\frac{2(z+1)q}{\hat{f}_n(z)^2 - q^2 \hat{V}_1} \sqrt{q^2 \left( \hat{V}_2^2 - \hat{V}_1 \hat{V}_2 \right) + \left( \hat{V}_1 \hat{f}_n(z+1)^2 + \hat{V}_2 \hat{f}_n(z)^2 - 2\hat{f}_n(z) \hat{f}_n(z+1) \hat{V}_{12} \right)}.
\]

Noting that \( n\hat{V}_1 = f_G(z)(1 - f_G(z)) + o_{\mathbb{P}_G}(1) \), \( n\hat{V}_2 = f_G(z+1)(1 - f_G(z+1)) + o_{\mathbb{P}_G}(1) \) and \( n\hat{V}_{12} = -f_G(z)f_G(z+1) + o_{\mathbb{P}_G}(1) \), we conclude that the above is asymptotically equal to:

\[
\frac{2(z+1)q}{\sqrt{n}f(z)^2} \sqrt{f(z)(1 - f(z))f(z+1)^2 + f(z+1)(1 - f(z+1))f(z)^2 + 2f(z)^2f(z+1)^2(1 + o_{\mathbb{P}_G}(1))}
\]

\[
= \frac{2(z+1)q}{\sqrt{n}f(z)^2} \sqrt{f(z)f(z+1)(f(z) + f(z+1))}(1 + o_{\mathbb{P}_G}(1)).
\]

This is the confidence interval length claimed in the statement of the Proposition.

\[ \square \]

### C.2 Bernoulli model (Section 7.2)

In this section we consider model (1) with \( Z_i \mid \mu_i \sim \text{Bernoulli}(\mu_i) \), i.e., the Binomial model with a single \( (N = 1) \) trial. Furthermore, we do not impose additional structure on \( \mathcal{G} \), i.e., we assume that \( G \in \mathcal{G} = \mathcal{P}([0,1]) \). Under the above model, \( Z_i \) is supported on \( \{0,1\} \) and we can take \( \lambda = \delta_0 + \delta_1 \) to be the counting measure on \( \{0,1\} \) and \( p(z \mid \mu) = \mu^z(1-\mu)^{1-z} \). The marginal distribution \( F_G \) is fully determined by \( f_G(1) \), since \( F_G(0) = 1 - f_G(1) \). In this case, the \( F \)-localizations we consider take the following simplified form. First, the DKW \( F \)-localization (16) is equal to:

\[
\mathcal{F}_n^{\text{DKW}}(\alpha) = \left\{ \mathcal{F} \in \mathcal{P}([0,1]) \text{ with pmf } f : \left| f(1) - \hat{f}_n(1) \right| \leq \sqrt{\log(2/\alpha)/(2n)} \right\}.
\] (68)

Second, for the \( \chi^2 \)-\( F \)-localization (22), write \( \tau^2 = \chi^2_{1-\alpha} \), then:

\[
\mathcal{F}_n^{\chi^2}(\alpha) = \left\{ \mathcal{F} \in \mathcal{P}([0,1]) \text{ with pmf } f : \left| f(1) - \hat{f}_n(1) + \tau^2/(2n) \right| \leq \frac{\sqrt{\tau^2/n}}{1 + \tau^2/n} \cdot \sqrt{\hat{f}_n(1)(1 - \hat{f}_n(1)) + \tau^2/(4n)} \right\},
\] (69)

An important observation that we will use throughout the following proofs, is that any distribution \( F \in \mathcal{P}([0,1]) \) can be represented as \( F_G \) for some \( G \in \mathcal{P}([0,1]) \) in model (1) with the Bernoulli likelihood.
C.2.1 Proof of Proposition 8: Second moment

Proof. We study the second moment of the prior. As already mentioned in the main text, this is an example of a linear functional that is partially identified. We discuss the partial identification aspect first. Suppose we know the marginal distribution of \( Z_i \). Notice that \( E_G [\mu] = \int \mu \, dG(\mu) = f_G(1) \). Then, the partial identification interval for \( L(G) \) is the following:

\[
L(G) \in \left[ \left( \int \mu \, dG(\mu) \right)^2, \int \mu \, dG(\mu) \right] = [f_G(1)^2, \ f_G(1)].
\] (70)

For example, when \( f_G(1) = 1/2 \), then \( L(G) \in [1/4, 1/2] \). Why is the above the partial identification interval? First note that \( \int \mu^2 \, dG(\mu) \leq \int \mu \, dG(\mu) \) holds since \( G \) is supported on \([0,1]\) and \( \int \mu^2 \, dG(\mu) \geq (\int \mu \, dG(\mu))^2 \) holds by Jensen’s inequality. Furthermore, there exist choices of \( G \) that make both inequalities tight. In particular, if \( G = \delta_{\hat{\mu}} \) for some \( \hat{\mu} \) then \( \int \mu \, dG(\mu) = \hat{\mu}, \int \mu^2 \, dG(\mu) = \hat{\mu}^2 \), while for \( G = (1 - \hat{\mu}) \delta_0 + \hat{\mu} \delta_1 \), it holds that \( \int \mu \, dG(\mu) = \int \mu^2 \, dG(\mu) = \hat{\mu} \).

We seek to determine the (asymptotic) length of the different confidence intervals we consider in this work. We start with the \( F \)-localization approaches.

\textbf{\textit{F-localization:}} By the above discussion on partial identification, we find that the \( F \)-localization intervals take the form \([\inf_G \{ f_G(1)^2 \}, \sup_G \{ f_G(1) \}] \) where the extrema are taken over all \( G \) such that \( F_G \in \mathcal{F}_n(\alpha) \). We further restrict attention to the event wherein \( \inf_G \{ f_G(1) \} \in (0,1) \) and \( \sup_G \{ f_G(1) \} \in (0,1) \). Since \( f_G(1) \in (0,1) \) under the assumptions of Proposition 3, this event will occur with asymptotic probability 1 for both \( F \)-localizations. For the DKW-\( F \)-localization, in view of (68), we then get the interval:

\[
\mathcal{I}^{\text{DKW}} = \left[ \hat{f}_n(1) - \sqrt{\log (2/\alpha) / (2n)}, \hat{f}_n(1) + \sqrt{\log (2/\alpha) / (2n)} \right].
\]

Similarly, for the \( \chi^2 \)-\( F \)-localization (69) we get the interval (with \( \tau^2 = \chi^2_{1, \alpha} \)):

\[
\mathcal{I}^\chi^2 = \left[ \left( \hat{f}_n(1) + \frac{\tau^2/(2n)}{1 + \tau^2/n} - \frac{\sqrt{\tau^2/n}}{1 + \tau^2/n} \cdot \sqrt{\hat{f}_n(1)(1 - \hat{f}_n(1)) + \tau^2/(4n)} \right)^2, \right. \\
\left. \hat{f}_n(1) + \frac{\tau^2/(2n)}{1 + \tau^2/n} + \frac{\sqrt{\tau^2/n}}{1 + \tau^2/n} \cdot \sqrt{\hat{f}_n(1)(1 - \hat{f}_n(1)) + \tau^2/(4n)} \right].
\]

Since \( \hat{f}_n(1) = f_G(1) + o_P(1) \), as \( n \to \infty \), it follows for both \( \mathcal{I} = \mathcal{I}^{\text{DKW}} \) and \( \mathcal{I} = \mathcal{I}^\chi^2 \), that:

\[
|\mathcal{I}| = f_G(1) - f_G(1)^2 + o_P(1) = f_G(1)(1 - f_G(1)) + o_P(1),
\]

as claimed.

\textbf{AMARI:} The modulus problem (28) at \( \delta > 0 \) takes the form:

\[
\text{maximize}_{G_1, G_{-1} \in \mathcal{G}_n} \ L(G_1) - L(G_{-1}) \ \text{s.t.} \ |f_{G_1}(1) - f_{G_{-1}}(1)| \leq \frac{\delta}{\sqrt{n}} \cdot \sqrt{f_n(1)(1 - \hat{f}_n(1))}. \] (71)
By (70), it may be simplified as:

$$\maximize_{G_1, G_{-1} \in \mathcal{G}_n} \ f_{G_1}(1) - f_{G_{-1}}(1)^2 \, \text{s.t.} \ |f_{G_1}(1) - f_{G_{-1}}(1)| \leq \frac{\delta}{\sqrt{n}} \cdot \sqrt{f_n(1)(1 - f_n(1))}. \quad (72)$$

We write \( p = f_{G_{-1}}(1) \) and \( f_{G_1}(1) = p + \varepsilon \), for a choice of \( \varepsilon \geq 0 \) that we will make below. Then we seek to find (feasible choices of \( p, \varepsilon \) so that \( (p + \varepsilon) - p^2 \) is maximized. We seek to solve this problem for some \( \delta = \delta_n \geq \delta_0 > 0 \). Throughout the rest of the proof we assume that \( f_{G_1}(1) \in [1/2, 1) \); the case \( f_{G_1}(1) \in (0, 1/2) \) being analogous. We define:

$$p_n = \max \left\{ 1/2, \inf \left\{ f_{G_1}(1) \mid \tilde{G} \in \mathcal{G}_n \right\} \right\}.$$ 

We note that \( p_n = f_{G_1}(1) + o_p(1) \). Since \( p \mapsto (p + \varepsilon) - p^2 \) is decreasing in \( p \) for \( p \geq 1/2 \), it follows that (72) is optimized for the choice \( f_{G_1}(1) = p = p_n \). Furthermore, \( \varepsilon \mapsto (p + \varepsilon) - p^2 \) is increasing in \( \varepsilon \), and so it is maximized for the largest admissible value of \( \varepsilon \). By the constraint in (72), we see that

$$\varepsilon \leq (\delta/\sqrt{n}) \cdot \sqrt{f_n(1)(1 - f_n(1))}.$$ 

The above constraint may be replaced by an equality, since the RHS above is of order \( O_p(1/\sqrt{n}) \) and so, the distribution \( F \) with \( f(1) = p_n + \varepsilon \) would be included in both \( F \)-localizations (68) and (69) for \( n \) large enough. We conclude that

$$\omega_n(\delta) = p_n(1 - p_n) + \frac{\delta}{\sqrt{n}} \cdot \sqrt{f_n(1)(1 - f_n(1))},$$

which is differentiable in \( \delta \) with:

$$\omega_n'(\delta) = \frac{1}{\sqrt{n}} \cdot \sqrt{f_n(1)(1 - f_n(1))}.$$ 

Let us plug the above into (30) to find the optimal \( Q(\cdot) \). First, we consider the part of \( Q \) that is a function of \( z \). For \( z = 1 \) we get

$$\frac{n \cdot \omega_n'(\delta) \cdot f_{G_1}(1) - f_{G_{-1}}(1)}{f_n(1)} = \frac{\tilde{f}_n(1)(1 - \tilde{f}_n(1))}{f_n(1)} = 1 - \tilde{f}_n(1).$$

Similarly, for \( z = 0 \), we get \(-\tilde{f}_n(1)\). It remains to evaluate the additive component in (30) that does not depend on \( z \). Let \( G_0 = (G_1 + G_{-1})/2 \), where \( G_{-1}, G_1 \) are any priors that have marginal pmf at \( z = 1 \) equal to \( p_n \), respectively \( p_n + \varepsilon \). Then

$$L(G_0) = \frac{1}{2} \left\{ (p_n + \varepsilon) + \frac{\delta}{2} \right\} = \frac{p_n(1 + p_n)}{2} + \frac{\delta}{2\sqrt{n}} \cdot \sqrt{f_n(1)(1 - f_n(1))}.$$

$$f_{G_0}(1) = p_n + \frac{\varepsilon}{2} = p_n + \frac{\delta}{2\sqrt{n}} \cdot \sqrt{f_n(1)(1 - f_n(1))}.$$
Using the above two results, we find that the constant term is equal to:

\[- \frac{n \cdot \omega'_n(\delta)}{\delta} \cdot \sum_{z=0}^{1} \frac{\left(f_{G_1}(z) - f_{G_0}(z)\right) f_{G_0}(z)}{f_n(z)} + L(G_0)\]

\[= \tilde{f}_n(1) - f_{G_0}(1) + L(G_0)\]

\[= \tilde{f}_n(1) - p_n + \frac{p_n(1 + p_n)}{2}\]

\[= \tilde{f}_n(1) - \frac{p_n(1 - p_n)}{2}.

Hence we now have an explicit expression for \(Q(z)\) in (30) for \(z \in \{0, 1\}\):

\[Q(z) = 1(z = 1) - \frac{p_n(1 - p_n)}{2}.

This means that \(\hat{L} = \tilde{f}_n(1) - \frac{p_n(1 - p_n)}{2}\), where \(\tilde{f}_n(1) = \# \{Z_i = 1\} / n\). The worst case absolute bias of \(\hat{L}\) is given by:

\[\hat{B} = \frac{1}{2} (\omega_n(\delta) - \delta \omega'_n(\delta)) = \frac{p_n(1 - p_n)}{2}.

With \(\hat{V}\) as in (13), we finally get the confidence interval (15):

\[I_\alpha = \hat{L} \pm t_\alpha(\hat{B}, \hat{V}) = \tilde{f}_n(1) - \frac{p_n(1 - p_n)}{2} \pm t_\alpha(\hat{B}, \hat{V}).\]

Under the given asymptotics \(\hat{V} = o_P(1)\), \(\hat{B} = f_G(1)(1 - f_G(1))/2 + o_P(1)\) and so it follows that for \(\alpha \in (0, 1)\), \(I_\alpha(\hat{B}, \hat{V}) = f_G(1)(1 - f_G(1))/2 + o_P(1)\). We conclude that the left endpoint of the confidence interval converges in probability to:

\[f_G(1) - f_G(1)(1 - f_G(1))/2 - f_G(1)(1 - f_G(1))/2 = f_G(1)^2.

The right point of the AMARI confidence interval converges in probability to:

\[f_G(1) - f_G(1)(1 - f_G(1))/2 + f_G(1)(1 - f_G(1))/2 = f_G(1).

Hence we conclude that the length of the AMARI confidence intervals converges to the length of the partial identification interval. \(\square\)

\section*{C.2.2 Proof of Proposition 8: Posterior mean}

\textbf{Proof.} We now turn to study,

\[\theta_G(1) = \mathbb{E}_G \left[ \mu \mid Z = 1 \right] = \int \frac{\mu^2 dG(\mu)}{f_G(1)}.

For a fixed value of the denominator \(f_G(\mu)(1)\), we derived partial identification intervals for the numerator in (70). It directly follows that the partial identification intervals for \(\theta_G(1)\) are equal to:

\[\theta_G(1) \in [f_G(1), 1].\]
**F-localization:** The argument now is very similar to that for the second moment. With the DKW-F-localization, in view of (68), we get the interval

\[ I^{DKW}(1) = \left[ \hat{f}_n(1) - \sqrt{\log(2/\alpha)/(2n)}, \ 1 \right] \cap [0, 1]. \]

Similarly, for the \( \chi^2 \)-F-localization (69) we get the interval (with \( \tau^2 = \chi^2_{1,1-\alpha} \)):

\[ I^{\chi^2}(1) = \left[ \frac{\hat{f}_n(1) + \tau^2/(2n)}{1 + \tau^2/n} - \frac{\sqrt{\tau^2/n}}{1 + \tau^2/n} \cdot \sqrt{\hat{f}_n(1)(1 - \hat{f}_n(1)) + \tau^2/(4n)}; \ 1 \right] \cap [0, 1]. \]

Since \( \hat{f}_n(1) = f_G(1) + o_p(1) \), as \( n \to \infty \), it follows for both \( I(1) = I^{DKW}(1) \) and \( I(1) = I^{\chi^2}(1) \), that:

\[ |I(1)| = 1 - f_G(1) + o_p(1). \]

**AMARI:** For fixed \( c \in [0, 1] \), we start by studying the modulus problem (71) for the linear functional

\[ L(G) = \theta^{lin}_G(z; c) = \int \mu^2 dG(\mu) - c \int \mu dG(\mu). \]

Following precisely the derivation in the proof for AMARI in Supplement C.2.1 and the notation used therein, we find that on an event with asymptotic probability 1, it holds that the optimal \( Q^c(\cdot) \) (30) for all \( c \in [0, 1] \) takes the form:

\[ Q^c(z) = (1 - c)1(z = 1) - \frac{p_n(1 - p_n)}{2}, \]

with worst-case bias \( \hat{B} = p_n(1 - p_n)/2 \) and \( \hat{V} = o_p(1) \). Write \( \tilde{I}_\alpha(z; c) \) for the confidence interval for \( \theta^{lin}_G(z; c) \) and \( \tilde{\theta}^{lin}_c(z; c) \), resp. \( \tilde{\theta}^{lin}_+(z; c) \) for its left and right endpoints. All \( o_p(1) \) terms above are uniform with respect to \( c \), and so arguing again as in Supplement C.2.1, it follows that:

\[ \sup_{c \in [0, 1]} |\tilde{\theta}^{lin}_c(z; c) - f_G(1)(f_G(1) - c)| = o_p(1), \quad \sup_{c \in [0, 1]} |\tilde{\theta}^{lin}_+(z; c) - f_G(1)(1 - c)| = o_p(1). \]

Recall that in Algorithm 2 we seek to find all \( c \in [c^l, c^u] \) such that \( 0 \in \tilde{I}_\alpha(z; c) \), where \( [c^l, c^u] \subset [0, 1] \) is the pilot interval. Fix \( \zeta > 0 \) small, then by the above uniform convergence, we have that:

\[ \mathbb{P} \left[ 0 \in \tilde{I}_\alpha(z; c) \text{ for any } 0 \leq c \leq f_G(1) - \zeta \right] \to 0 \text{ as } n \to \infty, \]

and that:

\[ \mathbb{P} \left[ 0 \in \tilde{I}_\alpha(z; c) \text{ for all } 1 \geq c \geq f_G(1) + \zeta \right] \to 1 \text{ as } n \to \infty. \]

Since \( \zeta > 0 \) was arbitrary, we thus find that the left-most endpoint of \( \tilde{I}_\alpha(z) \) converges in probability to \( f_G(1) \) and the right-most endpoint converges to 1, i.e., the asymptotic confidence interval length is equal to \( 1 - f_G(1) \).

\[ \square \]
D Computational aspects for \( F \)-localization

D.1 Parametric convex programming for \( F \)-localization intervals

We explain how to compute \( \hat{\theta}_\alpha(z) \) in (7) (the steps for \( \hat{\theta}_\alpha(z) \) being analogous) when \( G \) and \( F_n \) are convex, but not necessarily representable through linear constraints. Recall that \( \theta_G(z) = a_G(z)/f_G(z) \). We first compute confidence intervals for \( f_G(z) \) using the same \( F \)-localization, i.e.,

\[
\hat{f}_n^-(z) = \inf \{ f_G(z) : G \in G(\mathcal{F}_n(\alpha)) \}, \quad \hat{f}_n^+(z) = \sup \{ f_G(z) : G \in G(\mathcal{F}_n(\alpha)) \}.
\]

The objective here is linear and the constraints are convex, and so the above is a convex programming problem. We then observe that

\[
\hat{\theta}_n^+(z) = \sup \{ \theta_G(z) : G \in G(\mathcal{F}_n(\alpha)) \} = \sup \left[ \sup \{ a_G(z)/t : G \in G(\mathcal{F}_n(\alpha)) \} : f_G(z) = t \right] \in [\hat{f}_n^-(z), \hat{f}_n^+(z)].
\]

Hence, we can proceed as follows. Let \( T \) be a fine discretization of \([\hat{f}_n^-(z), \hat{f}_n^+(z)]\), say with 100 equidistant points. Then, for each \( t \in T \), compute:

\[
\hat{\theta}_n^+(z; t) = \sup \{ a_G(z)/t : G \in G(\mathcal{F}_n(\alpha)) \}.
\]

Note that this problem also has an objective that is linear in \( G \), and specifies convex constraints on \( G \), i.e., it is a convex programming problem. Finally, we report \( \hat{\theta}_n^+(z) = \max_{t \in T} \hat{\theta}_n^+(z; t) \).

D.2 Considerations for discretization of \( G \)

If an infinite-dimensional \( G \) is specified, then it is important to guarantee that the error incurred when solving (7) with a discretized class \( \tilde{G} \) instead of \( G \) is negligible compared to e.g., the confidence interval width. This typically requires \( G \) to be tight, and in our applications we used the prior classes (4), (34) and (35) with \( K \) chosen as a compact set.

The following proposition can be used to verify the accuracy of a discretization \( \tilde{G} \).

**Proposition 17.** Consider the linear functional \( L(G) = \int \psi(\mu)dG(\mu) \) for a function \( \psi(\cdot) \).

\begin{enumerate}
\item If \( |\psi(\mu)| \leq C_\psi \), then, \( \inf_{\tilde{G} \in \tilde{G}} |L(G) - L(\tilde{G})| \leq C_\psi/2 \inf_{\tilde{G} \in \tilde{G}} \{TV(G, \tilde{G})\} \), where \( TV(G, \tilde{G}) = \sup_{A} |G(A) - \tilde{G}(A)| \) is the total variation distance between \( G \) and \( \tilde{G} \).
\item If \( \psi(\mu) \) is \( C_\psi \)-Lipschitz continuous, then, \( \inf_{\tilde{G} \in \tilde{G}} |L(G) - L(\tilde{G})| \leq C_\psi \inf_{\tilde{G} \in \tilde{G}} \{W_1(G, \tilde{G})\} \), with \( W_1(G, \tilde{G}) = \inf \{ \mathbb{E}[|\mu - \tilde{\mu}|] : (\mu, \tilde{\mu}) \text{ random variables s.t. } \mu \sim G, \tilde{\mu} \sim \tilde{G} \} \) the Wasserstein distance between \( G \) and \( \tilde{G} \) (cf. Panaretos and Zemel [2019] and references therein).
\end{enumerate}

**Proof.** a) Recall that \( TV(G, \tilde{G}) = \frac{1}{2} \int |dG(\mu) - d\tilde{G}(\mu)| \). Thus,

\[
|L(G) - L(\tilde{G})| = \left| \int \psi(\mu) \left( dG(\mu) - d\tilde{G}(\mu) \right) \right| \leq C_\psi \int |dG(\mu) - d\tilde{G}(\mu)| \leq 2C_\psi TV(G, \tilde{G}).
\]

b) Letting \( \mu \sim G, \tilde{\mu} \sim \tilde{G} \), the optimal Wasserstein coupling, we get

\[
|L(G) - L(\tilde{G})| = |\mathbb{E}[\psi(\mu)] - \mathbb{E}[\psi(\tilde{\mu})]| \leq \mathbb{E}[|\psi(\mu) - \psi(\tilde{\mu})|] \leq C_\psi \mathbb{E}[|\mu - \tilde{\mu}|] = C_\psi W_1(G, \tilde{G}).
\]
For example, when part b) of the Proposition is applicable, then it suffices for \( \tilde{G} \) to be a cover of \( G \) in terms of the Wasserstein distance. In some cases, part b) is not applicable. For example, when constructing intervals for the local false sign rate in the standard Gaussian empirical Bayes problem, then the numerator \( a_G(z) \) in (10) takes the form \( a_G(z) = \int \psi(\mu) dG(\mu) \) with \( \psi(\mu) = 1(\mu \geq 0) \varphi(z - \mu) \), and so \( \psi(\cdot) \) is not Lipschitz continuous. Instead, part a) of the Proposition is applicable, and so a cover in total variation suffices.

Below we provide details for the discretization of \( P(K) \) (4) and \( LN(\tau, K) \) (34), when \( K \) is a compact interval.

D.2.1 Discretization of compactly supported distributions
Consider \( P([L, U]) \) (4), the class of all distributions supported on the compact interval \( K = [L, U] \). We first discretize \( K = [L, U] \) as the finite grid \( K(p, L, U) = \{ L, L + \frac{U - L}{p}, L + \frac{2(U - L)}{p}, \ldots, U \} \), \( p \in \mathbb{N} \). (73)

Then \( P([L, U]) \) may be discretized by considering \( P(K(p, L, U)) \), the class of all distributions supported on the grid \( K(p, L, U) \). This class is amenable to our optimization tasks. By enumerating the grid elements as \( K(p, L, U) = \{ \mu_1, \ldots, \mu_{p+1} \} \), we may represent every \( G \in P(K(p, L, U)) \) by the probabilities \( \pi_j = P_G[\mu = \mu_j] \) assigned to \( \mu_j \), and so we may identify \( P(K(p, L, U)) \) with the probability simplex:

\[
S_{p+1} = \left\{ (\pi_1, \ldots, \pi_{p+1}) \in [0,1]^{p+1} : \sum_{j=1}^{p+1} \pi_j = 1 \right\}.
\] (74)

\( S_{p+1} \) is a linear polytope. See (17) for an explicit example of how it is used for computations. In Section 5.1 we discretized \( P([0,1]) \) as above with \( p + 1 = 300 \).

Finally, we note that the discretization \( P(K(p, L, U)) \) provides an \( O((U - L)/p) \) covering of \( P([L, U]) \) in Wasserstein distance, but not in total variation distance. For this discretization scheme, Proposition 17 justifies inference for functionals satisfying b) of its statement. The implication for inference on empirical Bayes estimands is that we may use the above discretization to conduct inference for the posterior mean, e.g., in the Gaussian empirical Bayes problem.

D.2.2 Discretization of Gaussian location mixtures
Consider the class of distributions \( LN(\tau, K) \) from (34) in the special case where \( K = [L, U] \) is a compact interval. Then letting \( K(p, L, U) \) (73) the equidistant discretization of \([L, U]\), we use \( LN(\tau, K(p, L, U)) \) as a discretization of \( LN(\tau, K) \) that is amenable to efficient computation. We have the following numerical representation of \( LN(\tau, K(p, L, U)) \):

\[
G \in LN(\tau, K(p, L, U)) \iff G = \sum_{j=1}^{p+1} \pi_j N(\mu_j, \tau^2), \quad (\pi_1, \ldots, \pi_{p+1}) \in S_{p+1}.
\] (75)

Here (75) refers to the probability simplex (74) and so \( LN(\tau, K(p, L, U)) \) can also be represented in term of \( S_{p+1} \). Furthermore in the Gaussian empirical Bayes problem (1) with
$Z \mid \mu \sim \mathcal{N}(\mu, \sigma^2)$, and $G$ discretized as in (75) we typically do not need to resort to numerical quadrature. To see this, note that for $G \in \mathcal{LN}(\tau, \mathcal{K}(p, L, U))$:

$$L(G) = \sum_j \pi_j L(\mathcal{N}(\mu_j, \tau^2)),$$

and for many functionals of interest there exist explicit expressions for $L(\mathcal{N}(\mu_j, \tau^2))$. A special case of the above result is marginalization:

$$Z \sim \sum_j \pi_j \mathcal{N}(\mu_j, \tau^2 + \sigma^2).$$

Finally, we note that $\mathcal{LN}(\tau, \mathcal{K}(p, L, U))$ provides a $O((U - L)/p)$ covering of $\mathcal{LN}(\tau, [L, U])$ in both total variation and Wasserstein distance. Inference based on the discretized class will thus be valid for linear functionals satisfying a) or b) of Proposition 17 as long as $p$ is large enough.

### E Computational aspects for AMARI

#### E.1 Discretization

Discretization of $G$: Similar considerations apply as in Supplement D.2.

Discretization of $Z$: Computing (28) for a continuous likelihood, such as $\mathcal{N}(\mu, \sigma^2)$, requires numerical integration, e.g., to compute $\int (f_G(z) - f_{G_n}(z))^2 / \bar{f}_n(z) d\lambda(z)$. In this section we explain how to conduct the discretization rigorously. Our goal is to allow an arbitrary discretization (that may be coarse) by accounting for the discretization in the calculation of the worst-case bias. In particular, even if the discretization is too coarse, our intervals will have correct coverage, although they may be overly wide.

Fix $M > 0$ as in (27), and consider the grid,$^{29}$

$$\mathcal{I} = \mathcal{I}_n = \{-M = t_{1,n} < t_{2,n} < \ldots < t_{K_n-1,n} = M\},$$

where the grid may depend on $n$. Also let us define $t_{0,n} = -\infty$, $t_{K_n,n} = +\infty$ and $I_{k,n} = [t_{k-1,n}, t_{k,n})$ for $k \in \{1, \ldots, K_n\}$ and $K_n \in \mathbb{N}$. In analogy to (27), we define:

$$Z_i^T := \sum_{k=1}^{K_n} k \mathbf{1}(Z_i \in I_{k,n}) \in \{1, \ldots, K_n\}.$$  

(77)

Also let $\lambda^T$ the counting measure on $\{1, \ldots, K_n\}$ and analogously to the development after (27), define $f_G^T$ to be the marginal density of $Z_i^T$ with respect to $\lambda^T$, i.e., $f_G^T(k) = \int f_G(z) \mathbf{1}(z \in I_{k,n}) d\lambda(z)$ for $k \in \{1, \ldots, K_n\}$ and also define $\bar{f}^T(k)$ analogously.

---

$^{29}$In practice, to guarantee shorter confidence intervals, the grid should be made as dense as possible, subject to computational constraints, and should also become denser as $n$ increases. In the Gaussian problem for example, we discretize $[-M, M]$ as a dense equidistant grid, with step size $\ll \sigma$. While we do not pursue this further here, existing theory for discretization in statistical inverse problems [Johnstone and Silverman, 1991] suggests that even relatively coarse gridding suffices to maintain the minimax risk. The result of Theorem 2 allows for an arbitrary discretization of $[-M, M]$ (by applying the results of that theorem to the ‘discretized’ likelihood) that can also change with $n$. 

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In view of the above considerations, the modulus problem (28) takes on the following discrete form:

\[
\sup \left\{ L(G_1) - L(G_{-1}) : G_1, G_{-1} \in \mathcal{G}_n, \ n \cdot \sum_{k=1}^{K_n} \frac{(f_{G_1}^T(k) - f_{G_{-1}}^T(k))^2}{f^T(k)} \leq \delta^2 \right\}.
\] (78)

Below we discuss the solution of this discretized form of the modulus.

### E.2 Computing the affine minimax estimator

#### E.2.1 Direct form of the modulus problem

To solve (78) with modern convex optimization solvers, it is convenient to represent it as follows

\[
\sup_{G_1, G_{-1}} L(G_1) - L(G_{-1}) \quad \text{subject to} \quad \left( \sum_{k=1}^{K_n} \frac{(f_{G_1}^T(k) - f_{G_{-1}}^T(k))^2}{f^T(k)} \leq \delta \sqrt{n} \right) \quad \text{and} \quad G_1, G_{-1} \in \mathcal{G}\]

(79b)

\[
G_1, G_{-1} \text{ are } F\text{-localized, i.e., } F_{G_1}, F_{G_{-1}} \in \mathcal{F}_n
\]

(79d)

We make the following observations:

- The optimization variables are \( G_1, G_2 \in \mathcal{G} \). With \( \mathcal{G} \) suitably discretized, as in Supplement D.2, these have finite-dimensional representations. The choices of (discretized) \( \mathcal{G} \) considered in this work may be represented using a finite number of linear constraints.
- The objective (79a) is linear in the optimization variables.
- The maps \( G_{\ell} \mapsto f_{G_{\ell}}^T(k) \) are linear in \( G_{\ell} \) and so (79b) corresponds to a second order cone constraint.
- The localization constraints in (79d) may be implemented as a finite number of constraints on \( G_1 \), resp. \( G_{-1} \). These can be either linear (DKW and Gauss-F-localizations) or quadratic (\( \chi^2 \)-F-localization) in the optimization variables. To see these two claims, first note that the maps \( G_{\ell} \mapsto f_{G_{\ell}}^T(k) \) are linear. Furthermore, inspecting the proof of Theorem 2, we see that (32) only needs to hold for the discretized distributions, \( F_{G_1}, F_{G_{-1}}, F^T_{G_1}, F^T_{G_{-1}}, F^T_G \) (with triangular array asymptotics accounting for \( I \) changing with \( n \)).

As a consequence of the above observations, the discretized modulus problem may be represented as a finite-dimensional second order conic program (SOCP) [Boyd and Vandenberghe, 2004], which in turn is efficiently solvable by modern convex optimization solvers such as Mosek [ApS, 2020] or Hypatia [Coey et al., 2020]. In our numerical examples we use Mosek; our implementation can also use Hypatia.

#### E.2.2 Superdifferential of the modulus problem and duality

Evaluation of the estimator (30) requires access to \( \omega_1'(\delta) \), an element of the superdifferential of the modulus of continuity \( \omega_n(\delta) \) at \( \delta = \delta_n \). An element \( \omega_n' \) of \( \sqrt{n} \) may be directly extracted upon solving (79) as the dual variable associated to the constraint (79b), provided
that strong duality holds, and the primal and dual optima are attained. Many convex solvers,
including Mosek and Hypatia, return the dual variables.

**Argument sketch.** Define the Lagrangian of (79) for \( \lambda \geq 0 \):

\[
\mathcal{L}(G_1, G_{-1}, \lambda; \delta) = L(G_1) - L(G_{-1}) - \lambda \left[ \sum_{k=1}^{K_n} \frac{(f_{G_1}^T(k) - f_{G_{-1}}^T(k))^2}{f^T(k)} \right]^{1/2} - \frac{\delta}{\sqrt{n}}.
\]

Note that we parametrize the optimization problem and also the Lagrangian by \( \delta \). For any feasible \( G_1, G_{-1} \) and \( \lambda \geq 0 \)

\[
\mathcal{L}(G_1, G_{-1}, \lambda; \delta) \geq L(G_1) - L(G_{-1}).
\]  

(80)

Let \( G_1^\delta, G_{-1}^\delta \) be primal optimal solutions to (79) and let \( \lambda^\delta \) be the optimal dual variable, then [Boyd and Vandenberghe, 2004, Chapter 5.5.2]:

\[
\omega_n(\delta) = L(G_1^\delta) - L(G_{-1}^\delta) = \mathcal{L}(G_1^\delta, G_{-1}^\delta, \lambda^\delta; \delta) = \sup_{G_1, G_{-1}} \mathcal{L}(G_1, G_{-1}, \lambda^\delta; \delta)
\]  

(81)

Now fix \( \delta > 0 \) and take any \( \Delta \delta \) such that \( \Delta \delta > -\delta \). Also let \( G_1^{\delta+\Delta \delta}, G_{-1}^{\delta+\Delta \delta} \) be solutions to (79) at \( \delta + \Delta \delta \). Putting all results together

\[
\omega_n(\delta + \Delta \delta) = L(G_1^{\delta+\Delta \delta}) - L(G_{-1}^{\delta+\Delta \delta}) \leq \mathcal{L}(G_1^{\delta+\Delta \delta}, G_{-1}^{\delta+\Delta \delta}, \lambda^\delta; \delta + \Delta \delta)
\]

\[
= L(G_1^{\delta+\Delta \delta}) - L(G_{-1}^{\delta+\Delta \delta}) - \lambda^\delta \left[ \sum_{k=1}^{K_n} \frac{(f_{G_1}^T(\delta + \Delta \delta)) - f_{G_{-1}}^T(\delta + \Delta \delta))^2}{f^T(k)} \right]^{1/2} - \frac{\delta + \Delta \delta}{\sqrt{n}}
\]

\[
= L(G_1^{\delta+\Delta \delta}) - L(G_{-1}^{\delta+\Delta \delta}) - \lambda^\delta \left[ \sum_{k=1}^{K_n} \frac{(f_{G_1}^T(\delta + \Delta \delta)) - f_{G_{-1}}^T(\delta + \Delta \delta))^2}{f^T(k)} \right]^{1/2} - \frac{\delta}{\sqrt{n}} \right] + \frac{\lambda^\delta \Delta \delta}{\sqrt{n}}
\]

(81)

\[
\leq \omega_n(\delta) + (\lambda^\delta/\sqrt{n})\Delta \delta
\]

Thus \( \lambda^\delta/\sqrt{n} \in \partial \omega_n(\delta) \), that is, \( \lambda^\delta/\sqrt{n} \) is an element of the superdifferential of \( \omega_n(\cdot) \) at \( \delta \). 

**E.3 Bias-aware Normal confidence interval**

Recall that for constructing the confidence intervals from (15), we need to calculate (with \( W \sim \mathcal{N}(0, 1) \)):

\[
t_\alpha(B, V) = \inf \left\{ t : \mathbb{P} \left[ \left| B + V^{1/2}W \right| \leq t \right] \geq 1 - \alpha \text{ for all } |b| \leq B \right\}
\]

This is the same as:

\[
t_\alpha(B, V) = V^{1/2} \inf \left\{ t : \mathbb{P} \left[ \left| B/V^{1/2} + W \right| \leq t \right] \geq 1 - \alpha \text{ for all } |b| \leq B \right\}
\]

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We will argue that the calculation reduces to calculating the quantile of the absolute value of a Normal distribution (and hence can be efficiently computed); this expression is also given in Armstrong and Kolesár [2018]:

**Proposition 18.** Under the above setting it holds that:

\[ t_\alpha(B,V) = V^{1/2} \text{cv}_\alpha(B/V^{1/2}), \]

where \( \text{cv}_\alpha(u) \) is the \( 1 - \alpha \) quantile of the absolute value of a \( \mathcal{N}(u,1) \) distributed random variable.

**Proof.** For convenience of notation and without any loss of generality, let us assume \( V = 1 \).

First let us note that \( |b + W| \overset{D}{=} |−b + W| \) for any \( b \), hence:

\[ t_\alpha(B,V) = \inf \{ t : P[|b + W| \leq t] \geq 1 - \alpha \text{ for all } 0 \leq b \leq B \} \]

Next, observe that for \( b = B \), \( B + W \sim \mathcal{N}(B,1) \), and thus by definition:

\[ \inf \{ t : P[|B + W| \leq t] \geq 1 - \alpha \} = \text{cv}_\alpha(B). \]

We now just need to check what happens for \( 0 \leq b \leq B \), and indeed we will need some stochastic dominance argument. It suffices to argue that for any fixed \( t > 0 \) and \( 0 \leq b \leq B \):

\[ P[|B + W| \leq t] \leq P[|b + W| \leq t]. \]

Thus, if we let \( h(b) = P[|b + W| \leq t] \) it suffices to show \( h'(b) \leq 0 \) for all \( b \geq 0 \), so that it is decreasing. A direct calculation yields (with \( \Phi, \varphi \) the standard Normal CDF and pdf respectively):

\[ h(b) = \Phi(t - b) - \Phi(-t - b). \]

So:

\[ h'(b) = -\varphi(t - b) + \varphi(-t - b) \leq 0. \]

The last inequality holds since \( |t - b| \leq |-t - b| \) for \( t, b \geq 0 \). \( \square \)

**F  Exponential family (logspline) G-modeling**

In this section we summarize the empirical Bayes approach introduced by Efron [2016] and Narasimhan and Efron [2020]. The main idea is to specify \( \mathcal{H} \) as a flexible exponential family of effect size distributions with natural parameters \( \alpha = (\alpha_1, \ldots, \alpha_p) \), sufficient statistic \( Q(\mu) : \mathbb{R} \to \mathbb{R}^p \) and base measure \( H \). Concretely, distributions \( G \in \mathcal{H} \) are parametrized by \( \alpha \) and have Radon-Nikodym derivative \( g_\alpha(\mu) = dG/dH(\mu) \) defined as

\[ g_\alpha(\mu) = \exp(Q(\mu)\top \alpha - A(\alpha)). \tag{82} \]

\( A(\alpha) \) is such that \( \int g_\alpha(\mu)dH(\mu) = 1 \). It is worth pointing out, that in contrast to our setting, \( \mathcal{H} \) is not a convex class. \( \alpha \) is estimated by \( \hat{\alpha} \), the maximizer of the log (marginal) likelihood \( \ell(\alpha) \) in model (1):

\[ \ell(\alpha) = \sum_{i=1}^n \log \left( \int p(Z_i | \mu)g_\alpha(\mu)dH(\mu) \right). \tag{83} \]

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Efron [2016] further recommends maximizing the penalized likelihood $\ell(\alpha) - s(\alpha)$ instead, where $s(\alpha) = c_0 \| \alpha \|_2$ for some $c_0 > 0$. The empirical Bayes quantity $\theta_G(z) = \mathbb{E}_G [ h(\mu_i) \mid Z_i = z]$ can then be estimated by the plug-in estimator $\hat{\theta}(z) = \mathbb{E}_{\hat{G}} [ h(\mu_i) \mid Z_i = z]$, where $\hat{G}$ is the prior with $dH$-density $g_\alpha(\cdot)$. Standard delta method calculations and maximum-likelihood asymptotics can then be used to estimate standard errors and correct bias due to the penalization (but not due to misspecification). Efron [2016] demonstrates that even under asymptotics can then be used to estimate standard errors and correct bias due to the penalization (but not due to misspecification). In this Supplement, we explore some of the issues raised in Section 8.1 by revisiting the prostate data analysis of Section 5.2. There we posited that $G$ in the class $\mathcal{G}$ with $\tau = 0.25$. The first question we ask, is whether a goodness-of-fit test can guide the choice of $\tau$ in a data-driven way. To test $H_0(\tau) : G \in \mathcal{LN}(\tau^2, [-3,3])$, we use the Split Likelihood-Ratio (SLR) test of Wasserman, Ramdas, and Balakrishnan [2020], of which we provide a brief explanation.

First, we randomly split our observations into two folds, $I_0$ and $I_1$. Then, let $\hat{G}_1$ be the nonparametric maximum likelihood estimator of $G$ in the class $\mathcal{P}(\mathbb{R})$ using $Z_i, i \in I_1$ and $\hat{G}_0$ the nonparametric maximum likelihood estimator of $G$ in the class $\mathcal{LN}(\tau^2, [-3,3])$ using $Z_i, i \in I_0$. The Split Likelihood-Ratio (SLR) is defined as:

$$\text{SLR} = \prod_{i \in I_0} \frac{f_{\hat{G}_1}(Z_i)}{f_{\hat{G}_0}(Z_i)}.$$

Wasserman, Ramdas, and Balakrishnan [2020] prove that the test $\{\text{SLR} > 1/\alpha\}$ is a finite-sample valid level $\alpha$ test for the null hypothesis $H_0(\tau)$. The second column of Table 3 shows the SLR for $\tau \in \{0.02, 0.1, 0.25, 0.5, 0.55\}$. The SLR test at level $\alpha = 0.05$ only rejects the model with $\tau = 0.55$, and the SLR statistic becomes smaller as $\tau$ decreases.

Next, we consider inference for the local false sign rate $\mathbb{P}_G[\mu \geq 0 \mid Z = 2]$ and posterior mean $\mathbb{E}_G[\mu \mid Z = 2]$ at $Z = 2$ using the Gauss-F-Localization approach. The last two columns of Table 3 show the confidence intervals for each choice of $\tau$ (that was not rejected by the SLR test). We observe that as $\tau$ becomes smaller, the confidence intervals for $\mathbb{P}_G[\mu \geq 0 \mid Z = 2]$ become substantially wider, while the confidence intervals for the posterior mean are less sensitive. One way of determining how pessimistic a given choice of $\tau$ may be, is to inspect the worst-case priors in (7). Figure 9 shows these for the local false sign rate $\mathbb{P}_G[\mu \geq 0 \mid Z = 2]$.

G Sensitivity analysis for the prostate dataset

In this Supplement, we explore some of the issues raised in Section 8.1 by revisiting the prostate data analysis of Section 5.2. There we posited that $G \in \mathcal{G} = \mathcal{LN}(\tau^2, [-3,3])$ with $\tau = 0.25$. The first question we ask, is whether a goodness-of-fit test can guide the choice of $\tau$ in a data-driven way. To test $H_0(\tau) : G \in \mathcal{LN}(\tau^2, [-3,3])$, we use the Split Likelihood-Ratio (SLR) test of Wasserman, Ramdas, and Balakrishnan [2020], of which we provide a brief explanation.

First, we randomly split our observations into two folds, $I_0$ and $I_1$. Then, let $\hat{G}_1$ be the nonparametric maximum likelihood estimator of $G$ in the class $\mathcal{P}(\mathbb{R})$ using $Z_i, i \in I_1$ and $\hat{G}_0$ the nonparametric maximum likelihood estimator of $G$ in the class $\mathcal{LN}(\tau^2, [-3,3])$ using $Z_i, i \in I_0$. The Split Likelihood-Ratio (SLR) is defined as:

$$\text{SLR} = \prod_{i \in I_0} \frac{f_{\hat{G}_1}(Z_i)}{f_{\hat{G}_0}(Z_i)}.$$
Figure 9: **Lebesgue density of worst-case priors** in the Gauss-$F$-Localization approach (7) for the local false sign rate $\Pr_{G} [\mu \geq 0 \mid Z = 2]$ in the Prostate dataset. Each panel corresponds to a different specification of the prior class $\mathcal{G}$, namely $\mathcal{G} = \mathcal{L}N(\tau^2, [-3.3])$ where $\tau$ varies across panels.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>SLR</th>
<th>Goodness of fit rejected</th>
<th>CI for $\Pr_{G} [\mu \geq 0 \mid Z = 2]$</th>
<th>CI for $\mathbb{E}_{G} [\mu \mid Z = 2]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.0030</td>
<td>X</td>
<td>0.1859 – 0.9996</td>
<td>0.0196 – 0.7156</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0032</td>
<td>X</td>
<td>0.4491 – 0.9746</td>
<td>0.0302 – 0.7127</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0042</td>
<td>X</td>
<td>0.6396 – 0.8905</td>
<td>0.0922 – 0.6960</td>
</tr>
<tr>
<td>0.50</td>
<td>1.8847</td>
<td>X</td>
<td>0.8043 – 0.8325</td>
<td>0.3834 – 0.5291</td>
</tr>
<tr>
<td>0.55</td>
<td>74.106</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Goodness-of-fit testing and sensitivity analysis for the Prostate data.