Consensus in Diverse Corporate Boards*

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Abstract

Many directors are not simply insiders or outsiders. For example, an officer of a supplier firm is neither independent nor captive of management. We use a spatial model of board decision-making to analyze bargaining among multiple types of directors. Board decisions are modeled using a new bargaining solution concept called consensus. We use consensus to develop the idea that the information a new director brings to a large board is more important than the new director's impact on bargaining, provided existing preferences on the board are not too diverse. Our model suggests broadening the regulatory definition of independence of directors and requiring a supermajority of outsiders. The analysis also shows that strong penalties such as those imposed by Sarbanes-Oxley tend to erode incentives given that board performance is difficult to measure.

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1 Introduction

Discussions of corporate boards often focus on insiders and outsiders. However, this classification has been less than successful empirically, and many questions of interest consider diverse directors. For example, what is the impact of a "gray" director who has a business relationship with the firm but is neither an insider aligned with management nor a disinterested outsider? How effective are directors who are independent but still have their own agendas and disagree with each other? The purpose of this paper is to find a framework in which we can analyze decisions in diverse corporate boards. By diverse we mean that directors' preferences may differ over several dimensions, which we believe to be true to a greater or lesser extent for all boards. We use this framework, including our equilibrium notion called *consensus*, to analyze policy questions and to make testable predictions. Our analysis suggests that existing regulatory definitions of what is an outside director should be broadened and that firms should be required to have a super-majority of outside directors. Our model also predicts that boards will perform better if they include some gray directors who have some conflict of interest but also bring some information to the board, especially if the board has a super-majority of outsiders whose preferences are not too diverse.

To accommodate directors with diverse preferences, the board in our model makes a decision over multidimensional actions. Directors may disagree in different ways about different actions, which allows us to differentiate disagreements between insiders and outsiders, among outsiders, and between gray directors and either insiders or outsiders. We keep the model tractable by assuming that each director's utility is decreasing in distance from the director's most preferred action (ideal). This assumption allows us to use geometry to analyze the solution. The board choice is modeled by a new solution concept called *consensus*, discussed in more detail later. This concept is good for describing decisions made through discussion and bargaining subject to majority vote in the event of disagreement. We take a simple view on information, assuming that each director has an endowment of information that is shared with the rest of the board (as is the incentive in our examples). The simple form of information allows us to analyze trade-offs between information and preferences in spite of the richness of the set of possible actions.

Our model of board decision-making can be used to address a number of policy questions. For example, there has been a lot of attention recently to populating a board with a majority of independent directors. However, a conflicted director may be better than a strictly independent director if the conflicted director also brings information to the board. For example, an officer of a major supplier may bring useful information that is more important than the marginal impact on voting. Our model shows that a conflicted director is more valuable for the firm when the director's conflict differs from that of the insiders, and is most valuable when the conflict is opposing.

Our model also shows that a simple majority of outsiders may be unable to have a significant impact on the board's decisions. A minority with concentrated preferences may prevail over a majority with conflicting interests, since the directors in the majority might spend much of their persuasive effort fighting each other. And, if insiders can dictate the agenda or more generally have higher bargaining weights than the outsiders, they may be able to obtain their preferred outcome even if in the minority in numbers.

Another policy question relates to the stiffening of penalties for deviations from measured value-maximization as seems to be the intent of parts of Sarbanes-Oxley. The ability to improve matters using such a penalty depends a lot on being able to measure board performance *ex post*. We show that if only part of performance can be measured, directors facing the penalty will be driven to optimize the measurable part and neglect the rest as in Holmström and Milgrom (1991). Thus strict penalties are undesirable given that performance measures are very noisy and directors already have some commonality of interest with shareholders.

Our model offers a number of testable empirical implications. It suggests that adding new directors has a larger impact on performance when the board is small and diverse. Gray directors who have valuable information such as industry knowledge are beneficial for large boards, boards dominated by outsiders, and boards that are not too diverse, while adding a gray director to a small board may be costly. Our model also implies that defining a board as independent when it has a supermajority (as opposed to simple majority) of independent directors should improve the significance of the relationship between board independence and firm performance.

A major innovation in the paper is the introduction of a new equilibrium concept, *consensus*, which is used to model decisions of the board. The corporate board makes many decisions through discussion and bargaining rather than formal voting. The existing approaches to analyzing bargaining, however, seem inappropriate for modelling a corporate board. The most popular cooperative solution concept, the Nash bargaining solution, is predicated on the notion that each party to the negotiation has veto power. On a board, however, the threat of a vote limits the influence of individual directors, especially if their views are much different from those of the rest of the board. For example, a majority with identical preferences and control of the agenda should be able to dictate the outcome regardless of the preferences of the other directors. That is our motivation for introducing a new bargaining solution, consensus, rather than using the Nash bargaining solution.

Formally, the consensus sets to zero a weighted sum of the directors' normalized marginal utility vectors. We model directors' preferences using a spatial model in which a director's utility from an action (or decision on all issues) depends on the distance from the action to the director's possibly random *ideal* action choice. A similar approach to specifying preferences is used, for example, in the political model of Baron (1991) and in models of product differentiation such as Hotelling (1929). More specifically, we assume that utility is a negative expected squared distance from the director's ideal, and thus has concentric indifference curves. If the ideal is random, we refer to the director's expectation of the ideal as the director's *target*. With these preferences, the consensus minimizes a sum of bargaining weights times distances (not squared distances[1]) from the directors' targets.

The consensus defined in this paper has a number of attractive properties. It is always Pareto optimal (among the directors, who are not permitted to make side payments) because distance is a monotone decreasing function of utility, and thus the weighted sum of distances is a social welfare function for the directors. If a majority (by weights) of directors has the same target, the consensus is that target, which is sensible because a majority should be able to vote in their preferred action. If all directors' targets are collinear, then there is a natural ordering of the directors, and the consensus is the target of the (weighted) median director, a result similar to the median voter

[1] minimizing squared distances would not respect the majority rule

theorem.

Importantly, the consensus handles reasonably a director with extreme preferences, meaning a director with a target far from the cluster of all the other directors' targets. Such a director cannot enforce a large deviation from what the other directors want but does have some influence. This is consistent with vote-trading and Baron's (1991) argument that even a small political party with extreme preferences can obtain some influence by trading votes. Specifically, in this model, an extreme director has an influence that depends on the direction of the director's target from a candidate action but not on the distance from it. Intuitively, such a director will spend all available political influence on moving in the direction of the target, which has the same effect whether the target is nearer or farther away. A moderate environmentalist on the board may convince profit-oriented board members to go along with more recycling. However, an extreme environmentalist will be able to do no more and will not have the votes needed to shut down the firm's factories and convert the properties to parks.

This paper is complementary to existing theoretical work on boards. We have excluded features such as costly effort, delegation, and incentives for sharing information, all of which are studied in the interesting paper by Harris and Raviv (2004). Another interesting theoretical paper is Hermalin and Weisbach (1998), which looks at CEO retention, the effect of negotiations between the CEO and the board on CEO salary and percentage of insiders on the board,[2] and the incentive for costly information-gathering by directors. In both papers, director preferences are one-dimensional: Harris and Raviv have two types, and Hermalin and Weisbach have preferences determined by one parameter. We do not model the adverse selection and moral hazard features that are the focus of the other two papers, but we do allow a rich diversity of director preferences that allow us to answer questions not addressed in the other papers.[3] Our analysis is also significantly different

[2]Recent regulatory changes are intended to give CEO's less influence on the selection of directors but it is an open empirical question how effective these changes will be. If the changes are effective, the model of Hermalin and Weisbach may be less relevant in future than it was in the past.

[3]Our paper seems less related to Cai (2005) and the theoretical part of Aggarwal and Nanda (2004). These papers focus on costly effort, by a team of directors collecting information in the case of Cai (2005) and by the manager who is contracted separately by

from the analysis of Gomes and Novaes (2005), who compare monitoring by a large shareholder with adding the large shareholder to the board. We do have one theme in common with their paper, which is that the usefulness of adding a conflicted director depends on whether the new director's conflict reinforces or neutralizes the conflict of insiders.

The paper is organized as follows. Section 2 presents the model of board decision-making using consensus. Section 3 considers some examples that address policy questions such as the trade-off between independence and information and the desirability of imposing stronger penalties, such as those in Sarbanes-Oxley, for not acting to maximize firm value. Section 4 discusses empirical implications of our model. Section 5 shows a number of properties of consensus, and serves to justify this decision rule. Section 5 also provides tools that are used to construct the examples in Section 3. Section 6 closes the paper.

2 Board Decision-Making

The board's task is to choose an action $a \in \Re^M$. The board consists of N directors. Each director n's von Neumann-Morgenstern utility is a negative squared deviation from the director's possibly random ideal action y_n:

$$(1) \quad u_n(a) = -(y_n - a)'(y_n - a).$$

The ideal y_n may depend on stock and output market conditions, firm and individual characteristics, and may not be fully known to the director at the time the board chooses an action. Each director n possesses information I_n which our examples assume to be the pooled information of all directors on the board. There is no costly information gathering (which is interesting but not our focus, see Harris and Raviv (2004)). It is useful to restate the director's preferences in terms of what the director knows (which may not

each director in the case of Aggarwal and Nanda (2004). Another relatively unrelated paper is Song and Thakor (forthcoming), which analyses how career concerns affect information sharing between the CEO and the board and the CEO's preferences over board composition.

include y_n). We first decompose the ideal as $y_n = t_n + (y_n - t_n)$, where

(2) $\quad t_n = E[y_n|I_n]$

is the *target* and $(y_n - t_n)$ is the forecast error. Then, we can express the utility (1) as

(3) $\quad u_n(a) = -E[(y_n - t_n)'(y_n - t_n)|I_n] - (t_n - a)'(t_n - a),$

where the first term is the negative conditional variance of the forecast error and the second term is the squared deviation of the action from the target. Because the conditional variance of the forecast error does not depend on the action a, the director's preferences over actions depend only on the final term $-(t_n - a)'(t_n - a)$.

Given a collection of N directors, we need to have a model of what action the board will select. We wish the action choice to be consistent with recourse to majority voting for resolving any disagreements, and therefore a director with extreme preferences (a target far from the other directors' targets) should have a limited influence on the board's decisions because any proposal to accommodate significantly the extreme director would get voted down. For generality, we allow directors to have different bargaining weights b_n; usually, we assume that all directors have the same bargaining power: $b_n = 1$ for all n. We model the directors' decision using the following definition of *consensus* (which, as we show later, has the desired properties):

Definition 1. *Given targets t_n and bargaining weights b_n, action a is a* consensus *if and only if there exist z_n such that*

(4) $$\sum_{n=1}^{N} b_n z_n = 0,$$

where each $z_n \in e_n(a)$, and

(5) $$e_n(a) \equiv \begin{cases} \{(t_n - a)/\|t_n - a)\|\} & \text{if } t_n \neq a \\ \{\varepsilon | \|\varepsilon\| \leq 1\} & \text{otherwise} \end{cases},$$

for all n.[4]

[4]Throughout, $\|\ \|$ denotes the Euclidean distance: $\|x\| \equiv \sqrt{\sum_i x_i{}^2}$.

Intuitively, each director n pulls in the direction of the director's ideal (which is also the direction of steepest increase in the utility function) with an intensity of b_n. If the action coincides with the director's ideal, the director pulls with an intensity of up to b_n in whatever direction is needed to stay at the ideal. The bargaining power b_n describes the director's relative influence on the board's decision. For example, directors who control the agenda, represent large investors or have better negotiation skills may have more influence on board decisions. Notice that the definition implies that doubling the bargaining power of an agent has the same impact on the consensus as adding another director with the same preferences.

If the action space $\Re^M$ is the plane $\Re^2$, a physical interpretation may be useful. Think of the action space as the frictionless surface of a table, with weight b_n for director n suspended by a long weightless wire at the director's target. The wires extend around pulleys to a common knot on the surface of the table; the equilibrium location of the physical knot is the consensus of our model.

The following theorem offers a useful alternative characterization of the consensus.

Theorem 1. ***(characterization)*** *Suppose that directors have concentric preferences given by (1) with fixed information I_n, and are therefore characterized by their targets (2). Then, an action a^* is a consensus (according to Definition 1) if and only if the action solves*

$$(6) \quad \min_a \sum_{n=1}^{N} b_n \|t_n - a\|,$$

Proof. Given preferences (1), Definition 1 of consensus is the first-order condition for the convex but not-everywhere-differentiable minimization (6). The detailed proof is in the Appendix. □

In our spatial model, the characterization in Theorem 1 could be used as a definition of consensus. Definition 1, however, is easier to generalize to other utility functions (by replacing $t_n - a$ with the gradient of the utility function[5]). Theorem 1 also suggests an alternative definition that would minimize $\sum b_n \|t_n - a\|^2$ (equivalent to maximizing a weighted sum of utilities). Although analytically easier to solve, this definition gives unreasonably large power to directors with extreme preferences and is not consistent with a fallback to majority rule.

Consensus characterized in Theorem 1 is a simple reduced-form that is intended to have qualitative features that might be found in a more elaborate model with such complex features as agenda-setting and vote-trading. Section 5 quantifies and proves that the consensus in Definition 1 has the following desirable properties:

1. Consensus always exists.
2. Consensus is unique if the targets are not collinear. [6]
3. Consensus is Pareto optimal.[7]
4. When the targets are collinear, consensus is the target of the weighted median voter.
5. If there is a majority by weights with the same preferences, the consensus is the majority's target.
6. A director with extreme preferences has little or no influence on the consensus.

These properties are used in some of the results developed in the following sections.

[5]Consensus can be further extended to more general preference orderings. Let P_n's be preference orderings for agent n that are irreflexive (for all x and n, not xP_nx) with convex preferred sets (for all x and n, $\{y|yP_nx\}$ is convex), and lower-hemicontinuous. Then we can define consensus action a^* by $\sum_n b_n z_n = 0$ for some z_n's satisfying

$$z_n \in \text{HULL}\{z|\|z\| = 1 \text{ and for } yP_na^*,\ z'(y - a^*) > 0\},$$

where HULL denotes the convex hull.

[6]Even if the targets are collinear, consensus may still be unique.

[7]Given Pareto optimality of consensus, it may be natural to ask what is the relationship between the consensus and the core. When the board consists of only two directors, the set of consensus actions coincides with the core. In general, however, it is difficult to compare the two notions, because it is not clear how to define a winning coalition in our setting.

3 Board Composition and Action Choice

3.1 Directors with Extreme Preferences

Our first example illustrates two different ways adding a director can impact the board's decision. First, the new director brings information that can be used by all directors. Second, the director's preferences have an impact that depends on the director's bargaining power. Whether adding a director is an improvement from the perspective of the existing directors depends on the tradeoff of these two effects. Whether shareholders are better off is considered in the following subsection in a rich example with inside, independent, and grey directors.

This example also illustrates that when the new director's preferences are extreme, the director's bargaining power has little or no influence on the consensus. We have in mind the intuition that a director with extreme preferences has only one vote and any proposal to accommodate the extreme preferences would get voted down. This director, however, may have some influence by trading votes with more influential directors, in line with Baron's (1991) argument why a small political party may have some influence in a parliamentary system dominated by two large parties.

Example 1. ***(Information vs. Private Interests)*** *Consider a board with $N = N_0 > 2$ directors with ideals located on a circle with radius $r > 0$ in $\Re^3$:*

$$y_n = \lambda \bar{y} + (0, \sin(2\pi n/N_0), \cos(2\pi n/N_0))r,$$

where $\lambda \sim \mathcal{N}(1, \sigma_\lambda^2)$ and $\bar{y}$ is a constant vector with the first coordinate $\bar{y}_1 = 0$. We assume that these initial N_0 directors have no knowledge of λ and have the same bargaining power $b_n = 1$.

Suppose another director $N_0 + 1$ with bargaining power b_{N_0+1} joins the board. We assume this director knows λ, and the new director's preferences diverge, perhaps significantly, from the preferences of the other directors. The new director is assumed to have the ideal

$$y_{N_0+1} = \lambda \bar{y} + (h, 0, 0),$$

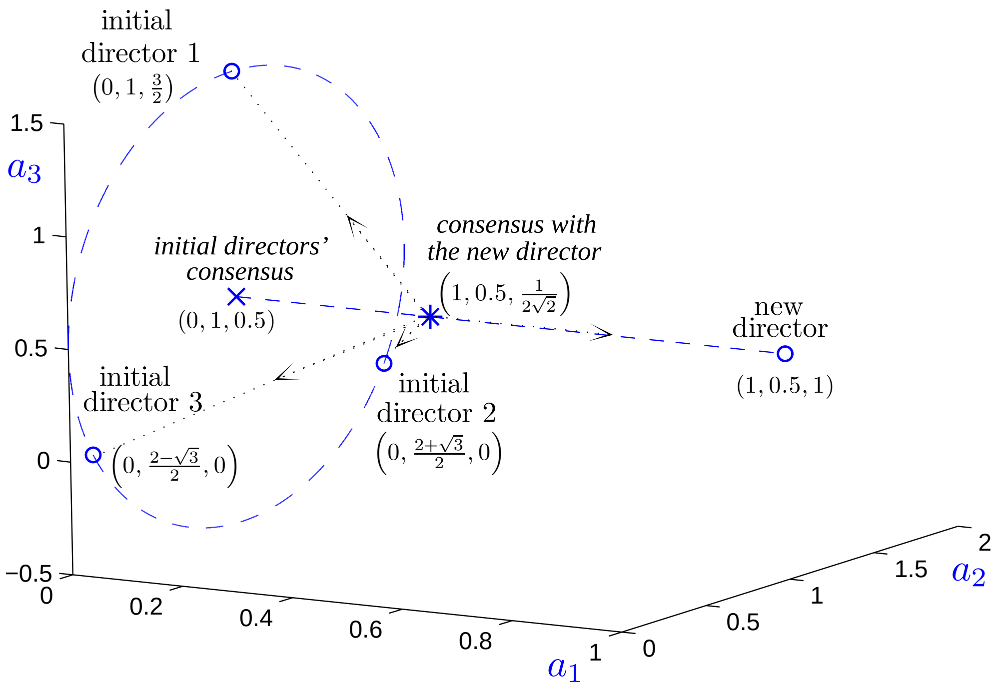

Figure 1: **Illustration for Example 1.** The graph assumes $N_0 = 3$, $\sigma_\lambda^2 = 0$, $r = 1$, $b_{N_0+1} = 1$, $h = 1$, and $\bar{y} = (0, 1, 0.5)$. The coordinates of all actions are (a_1, a_2, a_3). The arrows show the force with which each director "pulls" the action towards the director's ideal. The consensus balances these forces.

where the idiosyncracy $h > 0$. The new director is assumed to share the knowledge of λ with the other directors (indeed in this example the director will want to share the information because $\bar{y}_1 = 0$ implies that the new information is orthogonal to the disagreement between the new and the original directors). Figure 1 illustrates the example when $N_0 = 3$ and $\lambda \equiv 1$ ($\sigma_\lambda^2 = 0$).

Because the initial set of N_0 directors has no knowledge of λ, by (2) director

n's target is

$$(7) \quad \begin{aligned} t_n &= E[y_n] \\ &= \bar{y}E[\lambda] + (0, \sin(2\pi n/N_0), \cos(2\pi n/N_0))r \\ &= \bar{y} + (0, \sin(2\pi n/N_0), \cos(2\pi n/N_0))r \end{aligned}$$

After the new director joins the board and shares the information about λ, each original director ($n \leq N_0$) will have the target

$$(8) \quad \begin{aligned} t_n &= E[y_n|\lambda] \\ &= \bar{y}\lambda + (0, \sin(2\pi n/N_0), \cos(2\pi n/N_0))r, \end{aligned}$$

and the new director's target is

$$(9) \quad \begin{aligned} t_{N_0+1} &= E[y_{N_0+1}|\lambda] \\ &= \bar{y}\lambda + (h, 0, 0). \end{aligned}$$

Lemma 1. ***(Extreme Director)*** *In Example 1, for the initial set of N_0 directors, the consensus is located at the center of the circle of their targets:*

$$a_0^* = \bar{y},$$

while the consensus with the new director is

$$a^* = \bar{y}\lambda + (h_0, 0, 0),$$

where

$$(10) \quad h_0 = \begin{cases} h, & \text{if} \quad r^2 b_{N_0+1}^2 > h^2(N_0^2 - b_{N_0+1}^2); \\ \dfrac{r b_{N_0+1}}{\sqrt{N_0^2 - b_{N_0+1}^2}}, & \text{otherwise.} \end{cases}$$

Proof. See the Appendix. □

The change in the consensus when the new director joins comes from two sources: the influence of the new director's information and the influence of the new director's preferences. From Lemma 1, adding the new director rescales the consensus by λ (from $a_0^* = \bar{y}$ to $a_1^* \equiv \bar{y}\lambda$ due to the new director's information) and translates the consensus by h_0 in the direction of the new director's target (from $a_1^* = \bar{y}\lambda$ to $a^* = \bar{y}\lambda + (h_0, 0, 0)$) due to the new director's bargaining power.

When the new director's preferences are extreme (h is large), the impact of the new director's bargaining power h_0 is independent of h (because this implies the second case in (10)). Having more extreme preferences does not change the direction or the intensity of the new director's pull in the bargaining process. In this case, h_0 is small when the new director's bargaining power b_{N_0+1} is small, the number of original directors N_0 is large, and if the dispersion r of the original directors is small. If the original directors are relatively disperse they are less effective because they tend to fight among themselves instead of resisting the new director.

Lemma 2. ***(Information vs. Private Interests)*** *In Example 1, after the new director joins, the original directors' utilities change by*

$$u_n(a^*) - u_n(a_0^*) = \|\bar{y}\|^2\sigma_\lambda^2 - h_0^2.$$

Proof. From (1), the utility of an original director $n \leq N_0$ in the initial consensus is

$$u_n(a_0^*) = -\|\bar{y}\|^2\sigma_\lambda^2 - r^2.$$

and the utility of the original director n after the new director $N_0 + 1$ joins is

$$u_n(a^*) = -(r^2 + h_0^2).$$

□

The change in original directors' utilities when the new director joins comes from two sources: the influence of the new director's information and the influence of the new director's preferences. The term $\|\bar{y}\|^2\sigma_\lambda^2$ (the expected squared distance from a_0^* to a_1^*) is the increase in the original director's utility due to the new information; and h_0^2 (the squared distance from a_1^* to a^*) is the reduction of the original director's utility due to the new director's bargaining power.

3.2 Director Independence

Conflicts of interest may arise when directors have ties (beyond board membership) to managers they are meant to supervise. This seems to imply that director independence is crucial for board effectiveness and suggests that regulations encouraging board independence may be beneficial. In this subsection, we use our model to explore issues of director independence. We find that the current definition of director independence may be too restrictive. Perhaps the definition of independence should encompass conflicted directors who bring information, provided that their conflict is different from that of other insiders. We also find that having a strict majority of independent directors may not be good enough given that insiders' interests are more focused. Perhaps firms should be encouraged to have a supermajority of independent directors.

In the United States, regulation of the composition of corporate boards has been largely delegated to Self-Regulatory Organizations (SROs)[8] such as the NYSE and NASD. Both of these organizations have recently adopted rules requiring a majority of the board to be composed of independent directors. According to NASD rule 4350(c),[9]

> A majority of the board of directors must be comprised of independent directors as defined in Rule 4200.

[8]Self-regulation of industry, operating under the threat that Congress and governmental agencies will take over with more severe and less efficient regulation, is intended to be an effective, efficient, and cost-effective alternative to direct regulation by governmental agencies such as the SEC.

[9]The NASD rules are quoted from the "Rules of the Association" in the *NASD Manual*, which is available online at *http://www.nasd.com*. The quotes are current as of July 11, 2005.

The definition of independence excludes employees, high-paid consultants, close relatives of executives, and significantly for our purposes executives of a significant customer or supplier. Specifically, NASD rule 4200-1(a)(14)(C) says a person satisfying the following description should not be considered independent:

> a director who is a partner in, or a controlling shareholder or an executive officer of, any for-profit business organization to which the corporation made, or from which the corporation received, payments (other than those arising solely from investments in the corporation's securities) that exceed 5% of the corporation's or business organization's consolidated gross revenues for that year, or $200,000, whichever is more, in any of the past three years.

The NYSE has a similar rule. According to NYSE rule 303A.01,[10]

> Listed companies must have a majority of independent directors,

and according to NYSE rule 303A.02(b)(v), the director is not considered independent if

> the director is a current employee, or an immediate family member is a current executive officer, of a company that has made payments to, or received payments from, the listed company for property or services in an amount which, in any of the last three fiscal years, exceeds the greater of $1 million, or 2% of such other company's consolidated gross revenues.

The following example illustrates why it might be preferable to add a new director who would fail the current test of independence. It also illustrates that having a majority of independent directors may not be good enough.

[10] The NYSE rules are quoted from the *NYSE Listed Company Manual*, which is available online at *http://www.nyse.com*. The quotes are current as of July 11, 2005.

Example 2. *There is a vacancy on a board, and we want to consider the choice between two candidates. The first candidate C_1 is a strictly independent director who wants to maximize firm value but brings no information. The second candidate C_2 is somewhat conflicted but also brings some information to the board. For example, the second candidate may be part owner and an executive of a major supplier.*

The action space is $\Re^5$; having five actions allows us to separate the dispersion of the original directors, the insiders' conflict, the conflict with the new candidates, and the impact of the new information. The firm value maximizing action is

$$a_{fv} = (0, 0, 0, 0, \lambda),$$

where $\lambda \sim \mathcal{N}(1, \sigma_\lambda^2)$ is a random variable known by candidate C_2 but none of the other directors.

Besides the vacancy, there are $N_o \geq 2$ slots on the board held by existing outsiders and $N_i > 0$ slots held by existing insiders. Existing outside directors $n = 1, ..., N_o$ have ideals

$$y_n = (0, 0, r \sin(2\pi n/N_o), r \cos(2\pi n/N_o), \lambda). \tag{11}$$

The sine and cosine functions have been chosen to space the existing outsiders evenly along a circle with radius $r > 0$ centered at the firm value maximizing action $(0, 0, 0, 0, \lambda)$.

Existing inside directors, $n = N_o + 1, ..., N_o + N_i$, all fear retribution from the CEO and act exactly as the CEO would like, with identical ideals

$$y_n = (\delta, 0, 0, 0, \lambda), \tag{12}$$

for some constant $\delta > 0$.

The first candidate new director C_1 is strictly independent, which we interpret as wanting to maximize profits. Therefore, C_1 has ideal

$$y^1_{N_o+N_i+1} = (0, 0, 0, 0, \lambda). \tag{13}$$

The second candidate director C_2 is somewhat conflicted, but also brings knowledge of λ to the board. The candidate C_2 has ideal

$$y^2_{N_o+N_i+1} = (g_1, g_2, 0, 0, \lambda), \tag{14}$$

with both $g_1 \neq 0$ and $g_2 \neq 0$. All directors (including both candidates) have the same bargaining power $b_n = 1$.

In this example, because insiders collude (have the same ideal) while independent directors have dispersed ideals, the consensus action may coincide with the insiders' target even if there is a majority by weights of independent directors. Notice that the third and fourth consensus actions are zero with or without any new candidate. Calculations similar to those in Lemma 1 imply that if the majority of the board is independent: if $N_i < N_o$, then the consensus action without any new candidate is

$$a_0^* = (h_0, 0, 0, 0, 0)\,,$$

where $h_0 = \min(\delta, rN_i/\sqrt{N_o^2 - N_i^2})$, which coincides with the insiders' target when $h_0 = \delta$.

Lemma 3. ***(Ineffective Majority of Independent Directors)*** *Adding any new director, including the strictly independent candidate C_1 or a conflicted candidate C_2 with any g_1 and g_2, preserves the consensus at the insiders' target if*

$$\text{(15)}\quad \frac{(N_i^2 - 1)\sqrt{1 + r^2/\delta^2}}{2N_o} - \frac{N_o}{2\sqrt{1 + r^2/\delta^2}} \geq 1.$$

Proof. See the Appendix. □

For any given board composition N_o and N_i, the above inequality is satisfied when the disagreement among independent directors r is large enough compared to their disagreement with the insiders δ.

The intuition from Example 1 suggests that adding the conflicted but informed candidate C_2 results in a higher firm value than adding a strictly independent candidate C_1 if the reduction in the uncertainty from the new information compensates for the bias in the action choice. Because it is difficult to find a closed-form expression for the consensus with the new candidate C_2, we illustrate this intuition using numerical solution with the following parameter values: $N_o = 4$, $N_i = 3$, $\delta = 1$, and $r = 0.3$.

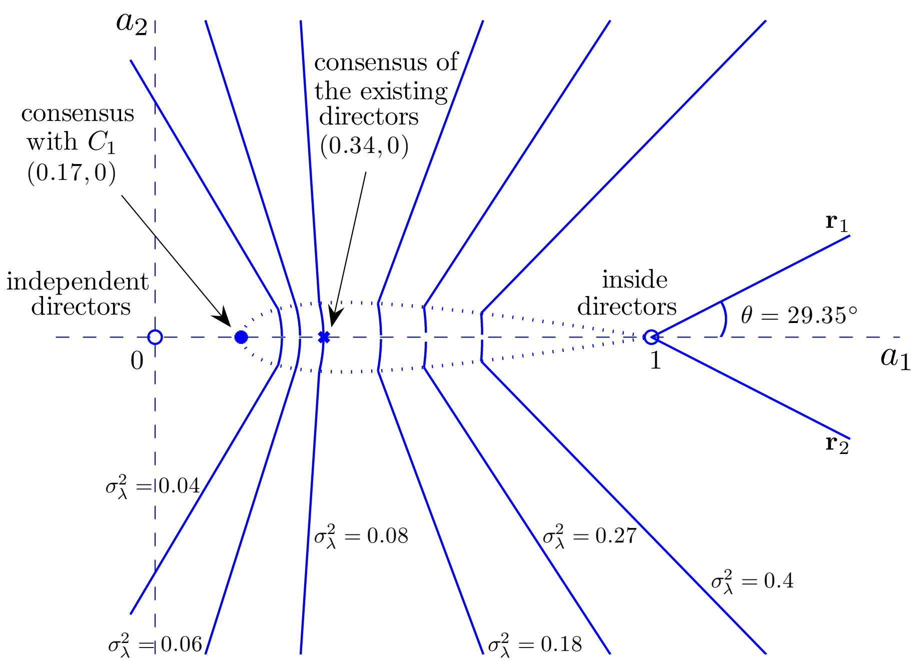

Figure 2: **Illustration for Example 2.** A candidate informed director whose conflict is closely aligned with insiders (as it is on the right in this figure) requires a larger amount of information ${\sigma_\lambda}^2$ to be a good addition to the board than a director who is unconflicted or has a conflict opposing the insiders (on the left).

For any new director of type C_2, the third and the fourth coordinates of the consensus action are zero and the fifth coordinate is λ. Figure 2 shows the projection of our solution on the first two coordinates. The projection of all possible consensus actions with the new candidate is the area bounded by the dotted line. Each solid line shows the locus of the new director's targets that result in the consensus actions with the same distance from the value maximizing action $(0, 0, 0, 0, \lambda)$. If the new director's target is located in the cone formed by rays $\mathbf{r}_1$ and $\mathbf{r}_2$, then the consensus coincides with the insider's target.

For each locus, there is a threshold value of σ_λ^2 (indicated on the graph), above which adding the new informed director with a target on the locus results in a higher firm value than adding the strictly independent candidate C_1. The figure shows that the firm owners may prefer to add a director with a significant conflict of interest (large $\sqrt{g_1^2 + g_2^2}$) and little new information (small σ_λ^2) if the director's conflict of interest differs from that of the existing insiders (g_2 may be large, but g_1 is positive and small or negative).

We have seen that insiders acting on behalf of the CEO may get what they want in spite of a majority of outsiders. In the model, that can happen because the outsiders have diverse interests centered around but far from the optimum (when r is larger) or when insiders have larger bargaining weights (perhaps as a result of agenda-setting power). In practice, a minority of insiders could maintain control for other reasons not in the model. For example, insiders seem likely to have better attendance at meetings and could have a majority of a quorum at most meetings but a minority of the whole board. Or, an outside director may be a CEO or ex-CEO who thinks of board meddling as intrusive and counterproductive. Such an outsider would side with management and would also tend to select like-minded candidates when serving on a nominating committee.

3.3 Fiduciary Duties

Incentives or penalties for deviations from the value-maximizing actions can be used to mitigate the conflicts of interests that influence board's actions. Stronger incentives seem desirable when directors face more conflicts of interests. Strict penalties, however, may backfire because it is difficult to measure performance ex-post. Our results illustrate how penalties, such as those imposed by Sarbanes-Oxley, may be counter-productive.

The current legal system imposes fiduciary duties (duties of diligence, loyalty, and obedience) on directors. Directors are generally protected by the business judgment rule. The business judgment rule, as stated by the Delaware courts, is a presumption that in making a business decision the directors of a corporation acted on an informed basis, in good faith and in the honest belief that the action taken was in the best interests of the company. Stricter standards are applied when apparent conflicts of interest are present. According to Knepper and Bailey (2004),

> When a challenger of the business judgment rule has shown that directors have a self-interest in the transaction at issue – that is, that the corporate fiduciaries, because of a conflict are disabled from safeguarding the interests of the shareholders to whom they owe a duty – the burden of proof shifts to the directors.

For similar reasons, inside directors may also be subject to more scrutiny:

> Under numerous statutory theories of liability, an "inside" director faces greater liability exposure than a disinterested "outside" director".

The extent to which the courts impose fiduciary duties is limited; courts generally do not "second-guess the directors' choice of procedure absent gross negligence" (Knepper and Bailey (2004)). In the recent years, however, the penalties for corporate wrongdoing have increased, primarily due to Sarbanes-Oxley Act of 2002.[11] Sarbanes-Oxley also prohibits certain actions that are deemed to create conflicts of interest such as loans to directors and officers.

We want to model penalties for deviating from the value-maximizing action. The penalty can arise from various sources such as compensation, requirements imposed by SROs, governmental regulation and law enforcement. It would be unreasonable to assume a forcing contract that penalizes deviations from the optimum because the optimum may depend on information available to the directors but unavailable to the party imposing the penalty, even ex-post. Instead, suppose that directors' preferences have two parts: deviation from the ideal action y_n and a penalty for deviation from some possibly random action a_{0n},

$$u_n = -E[(y_n - a)'(y_n - a) + \beta_n(a_{0n} - a)'(a_{0n} - a)|I_n], \tag{16}$$

[11] According to Bloomenthal (2003), while Sarbanes-Oxley did not change the existing scheme of penalties substantially, it has probably increased the likelihood of going to prison for violations of securities laws .

where $\beta_n > -1$, or else there is not a unique a that maximizes (16). Action a_{0n} may depend only on the information available to the party that imposes the penalty. Because the penalty is applied after the board decision-making is over, a_{0n} may in particular depend on some information unavailable to directors during the decision making process. In that case, director n views a_{0n} as random when determining the target action. From (16),

$$\begin{aligned}
(17)\quad u_n &= -E[(y_n - a)'(y_n - a)|I_n] - \beta_n E[(a_{0n} - a)'(a_{0n} - a)|I_n] \\
&= -E[(y_n - a)'(y_n - a)|I_n] - \beta_n (a'a - 2a'E[a_{0n}|I_n] + E[a_{0n}'a_{0n}|I_n]) \\
&= -E[(y_n - E[y_n|I_n])'(y_n - E[y_n|I_n])|I_n] \\
&\quad - \beta_n E[(a_{0n} - E[a_{0n}|I_n])'(a_{0n} - E[a_{0n}|I_n])|I_n] \\
&\quad - \frac{\beta_n}{1+\beta_n}(E[y_n|I_n] + E[a_{0n}|I_n])'(E[y_n|I_n] + E[a_{0n}|I_n]) \\
&\quad - (1+\beta_n)\left(\frac{E[y_n + \beta_n a_{0n}|I_n]}{1+\beta_n} - a\right)'\left(\frac{E[y_n + \beta_n a_{0n}|I_n]}{1+\beta_n} - a\right)
\end{aligned}$$

Note that only the last term depends on the chosen action a. Therefore, given preferences with the penalty in (16), the target is

$$(18)\quad t_n = \frac{E[y_n|I_n] + \beta_n E[a_{0n}|I_n]}{1+\beta_n}.$$

and cardinal preferences are an affine transform of what they were in (3) so that ordinal preferences[12] are the same as they were in (3), so we use the same definition of consensus as before (using t_n's in (18)).

Lemma 4. *Consider a board in which directors $1, 2, ..., N_0$ have the same ideal $y_n = y$ and the same information $I_n = I$, and consequently the same target before penalty $t = E[y|I]$. Suppose these directors face the same penalty $\beta(a_0 - a)'(a_0 - a)$ and have more bargaining power than the rest of the board: $\sum_{n=1}^{N_0} b_n > \sum_{n=N_0+1}^{N} b_n$. Then, $\beta \geq -1$ and a_0 maximize the expected firm value, assumed to be $-E[(a - a_{fv})^2]$, if and only if they shift the majority directors' expected target (18) to $E[a_{fv}]$:*

$$(19)\quad E[a_{fv}] = \frac{E[t] + \beta E[a_0]}{1+\beta}$$

[12] Ordinal preferences concern the ordering of outcomes, in contrast to cardinal preferences that concern the actual utility numbers. A monotone transformation of the utility function changes cardinal preferences but not ordinal preferences.

and minimize the variance

$$(20)\quad V\left(a_{fv} - \frac{t+\beta E[a_0|I]}{1+\beta}\right)$$

of the deviation of the majority directors' target from the full value action.

Proof. Because directors $n \leq N_0$ have the majority of the bargaining power and identical targets, the consensus is their common target from (18), i.e. $a^* = (t+\beta E[a_0|I])/(1+\beta)$. Therefore, the expected squared deviation of a^* from a_{fv} can be rewritten as

$$\begin{aligned} E[(a_{fv}-a^*)^2] &= E[(a_{fv} - E[a_{fv}] + E[a_{fv}] - E[a^*] + E[a^*] - a^*))^2] \\ &= E[(E[a_{fv}] - E[a^*])^2] + E[(a_{fv} - a^* - (E[a_{fv}] - E[a^*]))^2] \\ &= E\left[\left(E[a_{fv}] - \frac{E[t]+\beta E[a_0]}{1+\beta}\right)^2\right] + V\left(a_{fv} - \frac{t+\beta E[a_0|I]}{1+\beta}\right). \end{aligned}$$

Consider the last expression. The first term is minimized at 0 when (19) is satisfied and the second term is (20). If β and a_0 satisfy (19) and minimize (20), both terms are minimized and therefore the sum is minimized. Conversely, we want to show that if β and a_0 do not satisfy (19) or do not minimize (20), then the sum of both terms is not minimized.

Suppose β and a_0 do not satisfy (19). If $\beta \neq 0$, then adding a constant to a_0 to satisfy (19) reduces the first term and does not affect the second term, showing that β and a_0 did not minimize the sum. If $\beta = 0$, then changing β to any nonzero value and adding a constant to a_0 to satisfy (19) reduces the first term to zero. The resulting change in the second term can be made arbitrarily small by setting β to a small enough value. Therefore, small enough β and corresponding a_0 to satisfy (19) must reduce the overall sum, showing that the original values did not minimize the sum.

If β and a_0 do not minimize (20), then changing β to some nonzero[13] value that reduces the second term and adding a constant to a_0 to satisfy (19)

[13] If moving β to 0 reduces the second term, continuity of the second term implies that there is a neighborhood of 0 such that moving β anywhere in the neighborhood also reduces the second term, so without loss of generality we can choose a nonzero value.

reduces the second term and does not increase the first term, showing that β and a_0 did not minimize the sum. □

Notice that, for any $\beta \neq 0$ (which is without loss of generality as noted in the proof), $E[a_0]$ can be set to $(E[a_{fv}](1+\beta)-E[t])/\beta$ to satisfy (19). Therefore, the strength of penalty β and the random part of the target $E[a_0|I] - E[a_0]$ can be chosen to minimize (20). Minimizing the variance (20) is like a linear regression. Given $\beta > -1$, $\beta \neq 0$, the choice of a_0 gets rid of the volatility in the a_{fv} and t terms that can be spanned by the information available in $E[a_0|I]$ for different choices of a_0. The determination of β therefore is related to the parts of a_{fv} and t that cannot be spanned. If, for example, the parts of a_{fv} and t that cannot be spanned are negatively correlated, minimization of the variance would want to pick $1/(1+\beta)$ to be negative, which is infeasible. In this case, there is a closure problem and the optimal value would be approached in the limit as $\beta \uparrow \infty$: since it is infeasible to use t to hedge the risk in a_{fv}, the best that can be done is to "kill off" the term in t by making the denominator large.

Lemma 4 assumes that the penalty can be implemented at no cost. In reality, imposing small penalties and fines can be considered costless or even beneficial for the rest of the society, but more serious punishments such as imprisonment are fairly costly. The penalty may also impose an indirect cost on the firm, because it reduces directors' utility and, in the absence of rents, would lead directors to require higher compensation for their services.

Implementing the penalty suggested in Lemma 4 may be difficult, since it requires an ability to identify what actions would increase the firm value. Indeed, the empirical studies that analyze the link between various board characteristics and firm value typically rely on such obviously flawed measures of board performance as Tobin's Q, the "universal proxy" (see, for example, Yermack (1996)). As shown in Holmström and Milgrom (1991), when there is a lot of uncertainty about what maximizes the firm value, any incentive scheme is likely to cause undesirable biases in the agents' choices.

4 Empirical Implications

Our model's most interesting implications are normative. However, the model also offers testable hypotheses.

Hypothesis 1. Adding a new director has a larger impact when the board is diverse.

This implication arises from Example 1 as a direct consequence of Lemma 1, which shows that the new director has a larger effect on consensus when the existing directors have diverse targets.

Hypothesis 2. Adding a gray director who brings valuable information such as industry knowledge is more beneficial for larger boards, and may be costly for small boards.

Hypothesis 3. The benefit of adding a gray director who brings valuable information is greatest in boards with large numbers of outsiders who are not too diverse.

Hypothesis 4. Boards with gray directors who possess valuable information are more effective than boards with completely independent directors.

Hypothesis 2 follows from Lemma 2, which shows that the value of new information is more important than the new director's impact on voting when the board is large. Hypotheses 3 and 4 are illustrated in Example 2. Hypothesis 3 is a consequence of Lemma 3, which shows that any new director, even a strictly independent one, would preserve the consensus at the insider's target when there are few independent directors with diverse targets. Hypothesis 4 is illustrated in Figure 2 that shows a tradeoff between the new director's information and the new director's conflict of interest. When the new information is important, adding a conflicted but informed gray director is more valuable than adding an independent director who is uninformed.

We obtain the following hypotheses from Lemma 3.

Hypothesis 5. Defining board as independent when it has a supermajority of independent directors (as opposed to simple majority) should improve the significance of the relationship between board independence and firm performance.

Hypothesis 6. Given the mix of insiders and independent directors, diverse independent directors are less effective.

Empirical studies often either classify boards as independent if the majority of the directors are independent (for example, Byrd and Hickman (1992), Bhagat and Black (2000)), or characterize boards by the percentage of independent directors (Mehran (1995), Hermalin and Weisbach (1991), Yermack (1996)). Lemma 3, however, shows that a supermajority of independent directors may be required to affect the board decisions when the insiders' conflicts are large and when the independent directors have diverse targets. If true, hypothesis 6 may also help explain why large boards have poor performance on average (Yermack (1996)). As boards grow, there may be more tendency to accept directors representing special interests and thus add diversity. Hypothesis 6 implies that including measures of diversity as controls may eliminate the negative relationship between board size and performance. The amount of dispersion may depend on the issue under consideration. This may explain the more significant relationship between board independence and performance on specific issues such as CEO turnover (Weisbach (1988), Huson, Parrino, and Starks (2000)), merger and acquisition strategies (Shivdasani (1993), Byrd and Hickman (1992)), and executive compensation (Core, Holthausen, and Larcker (1999)).

5 Properties of Consensus

Decision-making on a corporate board is a complex dynamic process that includes such institutions as formal votes, agenda-setting, coalition formation, and vote-trading. Like the Nash bargaining solution, our analysis does not attempt to build a detailed structural model and instead provides a simple and robust mechanism that captures the economics of the situation but is more tractable than a detailed structural model would be. Our model of consensus is more appropriate for a corporate board or other committee than the Nash bargaining solution would be, since the Nash bargaining solution is predicated on the assumption that unanimity is required so that the solution can only improve on a given disagreement point. In a corporate board or other committee with majority voting, the outcome must do well by majorities, even if the outcome is very undesirable for some minority members.

First, we show that consensus always exists and the consensus is unique except in a degenerate case. (It is also easy to modify the definition of consensus—using a measurable selection—to be unique all the time, for example, by selecting the midpoint of any interval of solutions.)

Theorem 2. ***(Existence and Uniqueness)*** *Given any t_1, ... t_N, there exists a consensus defined in Definition 1. Provided not all the t_1, ... t_N are collinear, consensus is unique.*

Proof. By Theorem 1, consensus minimizes the objective (6). Consensus exists because the objective in (6) is continuous and dominated outside a compact set. If the t_n's are not collinear, the objective in (6) is strictly convex, which implies uniqueness. The detailed proof is in the Appendix. □

Theorem 3. ***(Pareto Optimality)*** *If utilities are concave, differentiable, and have a unique maximum (ideal) or no maximum, then a consensus defined in Definition 1 is Pareto optimal. In particular, with a target t_n, the ideal is unique and consensus is Pareto optimal.*

Proof. Let a^* be the consensus. Note that, given the quadratic utility functions (1), we have $(t_n - a) = u_n'(a)$, where $u_n'(a)$ is the vector of marginal utilities. Thus, $z_n = u_n'(a)/\|u_n'(a)\|$ if $a \neq t_n$ and $z_n = u_n'(a) = 0$ if $a = t_n$.

If for some n, $u_n'(a^*) = z_n = 0$, then a^* is the unique maximum of u_n: any $a \neq a^*$, $u_n(a) < u_n(a^*)$. Therefore, a^* is Pareto optimal. If, on the other hand, $u_n'(a^*) \neq 0$ for all n, consider the following social welfare function,

$$W(a) = \sum_{n=1}^{N} \frac{b_n}{\|u_n'(a^*)\|} u_n(a).$$

Because utilities $u_n(a)$ are concave, $W(a)$ is also concave. Therefore, actions that satisfy the first-order condition $W'(a) = 0$ are Pareto optimal. According to Definition 1, a^* satisfies (4), which is equivalent to $W'(a^*) = 0$. □

If a group of directors has the same target and more total bargaining power than the rest of the board, then their target is the consensus.

Theorem 4. ***(Majority Rule)*** *Suppose directors $n = 1, ..., N_0$ have the same target t.*
1. If directors $1, ..., N_0$ have more bargaining power than the rest of the board: $\sum_{n=1}^{N_0} b_n > \sum_{n=N_0+1}^{N} b_n$, then t is the unique consensus.
2. If directors $1, ..., N_0$ have the same bargaining power as the rest of the board: $\sum_{n=1}^{N_0} b_n = \sum_{n=N_0+1}^{N} b_n$, then t is a consensus. It is unique if not all the t_1, ... t_N are collinear.

Proof. The majority at t can have z_n's offsetting all the other z_n's in the characterization in Theorem 1. The detailed proof is in the Appendix. □

The following result shows that when targets are collinear, a version of the median voter result holds.

Theorem 5. ***(Median Voter)*** *Suppose all directors have collinear targets: $t_n = a_0 + \rho_n(a_1 - a_0)$, where $\rho_1 \leq \rho_2, ..., \leq \rho_n$. Let N_0 be such that*

$$\sum_{n=1}^{N_0-1} b_n \leq \frac{1}{2} \sum_n b_n \tag{21}$$

and

$$\sum_{n=1}^{N_0} b_n > \frac{1}{2} \sum_n b_n.$$

1. If (21) holds with inequality, then t_{N_0} is the unique consensus.
2. If (21) holds with equality, then $[t_{N_0-1}, t_{N_0}]$ is the set of consensus actions, which is a single point if and only if $t_{N_0-1} = t_{N_0}$.

Proof. See the Appendix. □

In the theorem, the weighted median voter's target is not unique if it is possible to strictly separate two groups with equal total weight that prefer to move in opposite directions. For example, consensus is never unique if there is an even number of directors with equal weights and distinct collinear targets. In these cases, every action in the closed interval between the two groups is a consensus.

The following theorem generalizes the result of Lemma 1, which offers a bound on an extreme director's impact on the consensus.

Theorem 6. ***(Extreme Director)*** *Consider a board with $N > 2$ directors with targets $t_1, ..., t_N$ and same bargaining power $b_n = 1$ for all n. Suppose that for all $i, j < N-1$, $\|t_i - t_j\| \leq K$ and let a^* be the consensus. Then,*

$$d(H, a^*) \leq \frac{K}{N-2}, \tag{22}$$

where $d(H, a^*) \equiv \min_{h \in H}(\|h - a^*\|)$ is the distance from a^* to a compact set H, $H = \text{HULL}(t_1, ..., t_{N-1})$, and HULL *denotes the convex hull.*[14]

Proof. See the Appendix. □

We devote the rest of the section to the result that while directors have utility functions that are separable across actions,

$$\begin{aligned} U_n(a) &= \|a - t_n\|^2 \\ &= (a_1 - t_{n1})^2 + (a_2 - t_{n2})^2 + ... + (a_M - t_{nM})^2, \end{aligned}$$

where $a = (a_1, ..., a_M)$ and $t_n = (t_{n1}, ..., t_{nM})$, consensus is not separable on the actions. This result is illustrated in the following example.

Example 3. ***(Action Separability)*** *Consider a board with 4 directors with no uncertainly about their ideals. Let their ideals (same as targets) be: $t_1 =$*

[14] Recall that the convex hull of a finite set X is a set of all convex combinations of elements of X.

$(-1,0)$, $t_2=(2,0)$, $t_3=(0,-1)$, and $t_4=(1,h)$. All directors have the same bargaining power $b_n=1$. The consensus is given by

$$a^* = \begin{cases} (\frac{1}{1+h}, 0) & \text{for } h \geq 0; \\ (1, h) & \text{for } 0 \geq h \geq -\frac{1}{2}; \\ \left(\frac{2+h}{1-h}, \frac{3h}{2(1-h)}\right) & \text{for } -\frac{1}{2} \geq h \geq -2; \\ (0, -1) & \text{for } -2 \geq h, \end{cases}$$

as can be verified by checking the first-order conditions given by Theorem 1.

Although directors' preferences are separable across actions and changing h affects only one director's preferences for action a_2, the consensus action a_1^* changes, and not even in a monotone way. However, this is reasonable, and there is a simple explanation: when either $h \gg 0$ or $h \ll 0$, director 4's strong preferences for a_2 imply that the director focuses political capital on a_2. Therefore, director 4's preference to move a_1^* to the right has little impact when $|h|$ is large and large impact when $|h|$ is small.

6 Conclusion

We have developed a simple model of board decision-making in the presence of diverse directors. Our new solution concept, consensus, has two particularly appealing characteristics: first, it is consistent with a recourse to a majority voting in case of disagreement, and second, it allows directors with extreme preferences to have only a limited influence on the board's action.

We apply our model to analyze the influence of independent and grey directors on the board's decisions. Most importantly, we show that a grey director who has important information may be more valuable to the firm than an independent and uninformed director, especially if the board is large and not too diverse. Our model can also be applied to study the impact on boards' decisions of penalties such as those imposed by Sarbanes-Oxley. We show that strict penalties may be undesirable because it is difficult to access the quality of the board's decisions ex-post.

We have touched on only a few of the issues involved in board composition. For example, we have not considered the incentives for information sharing, costly effort, and delegation. While these issues are investigated in Harris and Raviv (2004) in a model with a one-dimensional action space, it may be interesting to combine our approaches to analyze how the results change when directors' goals differ on different dimensions (for different actions). Another interesting extension would allow for side payments between directors. It may be particularly applicable for analyzing to what extent side payments disguised as business-related transfers can induce directors who have a business relationship with the firm to vote with the firm's management. This problem is probably most severe when the board member also consults for the firm. We conjecture that this would be a bigger problem for a consultant who is paid directly than for a supplier who benefits indirectly.

In this paper, we have also omitted such important issues as directors' compensation and the need for having various types of expertise on the board. While there is little theoretical work on these issues, empirical analysis of some of these aspects can be found for example in Adams and Ferreira (2005), Agrawal and Knoeber (2001), and Fich and Shivdasani (2006), and it would be nice to have a better theoretical understanding as well.

Appendix

The order of proofs in the Appendix differs from the order in which the results are presented in the body of the paper. Because some results in Sections 2 and 3 rely on the properties of consensus properties in Theorems 2 – 6, the Appendix offers proofs of these theorems first and then turns to the results of Sections 2 and 3.

Given $t_1, ..., t_N$, define function $f(a)$ on the action space $a \in \Re^M$ as

$$(23) \quad f(a) = \sum_{n=1}^{N} b_n \|t_n - a\|.$$

Lemma 5. *Function $f(a)$ given by (23) is convex. Moreover, if not all $t_1, ..., t_N$ are collinear, it is strictly convex.*

Proof. For any two actions a_1 and a_2,

$$\begin{aligned}
f(\beta a_1 + (1-\beta)a_2) &= \sum_{n=1}^{N} b_n \|t_n - \beta a_1 - (1-\beta)a_2\| \\
&= \sum_{n=1}^{N} b_n \|\beta(t_n - a_1) + (1-\beta)(t_n - a_2)\| \\
&\leq \sum_{n=1}^{N} b_n (\|\beta(t_n - a_1)\| + \|(1-\beta)(t_n - a_2)\|) \qquad (24)\\
&= \beta \sum_{n=1}^{N} b_n \|(t_n - a_1)\| + \sum_{n=1}^{N} b_n (1-\beta)\|(t_n - a_2)\| \\
&= \beta f(a_1) + (1-\beta) f(a_2).
\end{aligned}$$

Therefore, $f(a)$ is convex.

Function $f(a)$ is strictly convex if and only if (24) is a strict inequality for every a_1 and a_2. Notice that (24) holds with equality if and only if $(t_n - a_1)$ and $(t_n - a_2)$ are collinear for all n. Collinearity of $(t_n - a_1)$ and $(t_n - a_2)$ implies that

$$t_n - a_1 = \beta_n (t_n - a_2).$$

Rearranging, find that t_n is a weighted average of a_1 and a_2:

$$t_n = \frac{1}{1-\beta_n} a_1 + \frac{\beta_n}{1-\beta_n} a_2$$

for all n. Hence, collinearity of $(t_n - a_1)$ and $(t_n - a_2)$ implies that $t_1, ..., t_N$ are collinear. Therefore, (24) is a strict inequality if $t_1, ..., t_N$ are not collinear. □

Proof of Theorem 1: Characterization

Proof. According to Lemma 5, $f(a)$ is convex. Therefore, the action a^* minimizes $f(a)$ if and only if it satisfies the first-order condition. Because $\sum b_n e_n(a)$ is the subgradient correspondence for $f(a)$ (as we show below), the first-order condition is given by (4).

The subgradient correspondence for $f(a)$ can be derived as follows. Let $g_n(a) = \|t_n - a\|$. If $a \neq t_n$, then the gradient of $g_n(a)$ at a is

$$\frac{d}{da} g_n(a) = \frac{d}{da}\|t_n - a\| = -\frac{t_n - a}{\|t_n - a\|}.$$

If $a = t_n$, then $\varepsilon \in dg(a)$ if and only if

$$\varepsilon' \delta \leq g(t_n + \delta) - g(t_n) = \|t_n - t_n - \delta\| - \|t_n - t_n\| = \|\delta\|.$$

The above holds if and only if $\|\varepsilon\| \leq 1$, or in other words, if and only if $\varepsilon \in e(a)$. Hence, for any a, $e(a)$ is the subgradient correspondence for $g(a) = \|t_n - a\|$. Therefore, $\sum b_n e_n(a)$ is the subgradient correspondence for $f(a)$. □

Proof of Theorem 2: Existence and Uniqueness

Proof. Let $\mathbf{S}$ denote the ball in the action space that is centered at zero and has a radius of $(\frac{2}{B}\sum_n b_n \|t_n\|)$, where $B = \sum_n b_n$; therefore $\mathbf{S} \equiv \{a \in \Re^M : \|a\| \leq \frac{2}{B}\sum_n b_n\|t_n\|\}$. Because $f(a)$ is continuous, there exists an action a^* that minimizes $f(a)$ on the set $\mathbf{S}$, $a^* = \text{argmin}_{a \in B} f(a)$. To prove existence, it remains to show that a^* is a global minimum of $f(a)$.

Because a^* minimizes $f(a)$ on set $\mathbf{S}$, it in particular implies that

$$f(a^*) \leq f(0) = \sum_{n=1}^{N} b_n \|t_n - 0\| = \sum_{n=1}^{N} b_n \|t_n\| \tag{25}$$

For the actions a outside ball $\mathbf{S}$, $a \notin \mathbf{S}$,

(26)

$$\begin{aligned} f(a) =& \sum_{n=1}^{N} b_n \|t_n - a\| \geq \sum_{n=1}^{N} b_n(\|a\| - \|t_n\|) > \sum_{n=1}^{N} b_n \left(\frac{2}{B}\sum_{k=1}^{N} b_k\|t_k\| - \|t_n\|\right) \\ =& 2\sum_{k=1}^{N} b_k \|t_k\| - \sum_{n=1}^{N} b_n\|t_n\| = \sum_{n=1}^{N} b_n\|t_n\|, \end{aligned}$$

where the first inequality follows from the triangle inequality, and the second follows because $a \notin \mathbf{S}$ implies $\|a\| > \frac{2}{B}\sum_{k=1}^{N}\|t_k\|$. Combining (25) and (26) obtains that $f(a^*) < f(a)$ for all a. In other words, a^* is a global minimum.

From Lemma 5, the function $f(a)$ is strictly convex when not all $t_1, ..., t_N$ are collinear, and thus has a unique minimum. Therefore, the solution to (6) is unique if not all t_1,...,t_n are collinear. □

Proof of Theorem 4: Majority Rule

Proof. Let $B_0 = \sum_{n=1}^{N_0} b_n$ and $B_1 = \sum_{n=N_0+1}^{N} b_n$. Define vectors z_n for $n = 1, ..., N$ as follows:

$$z_n = \begin{cases} \frac{t_n - a}{\|t_n - a\|} & \text{if } N_0 + 1 \leq n \leq N \\ \frac{1}{B_0}\sum_{k=N_0+1}^{N} b_k z_k & \text{if } n \leq N_0. \end{cases}$$

These vectors z_n satisfy (4). Therefore, from Theorem 1, t solves problem (6). From Theorem 2, the consensus is unique when not all $t_1, ..., t_N$ are collinear.

To prove the theorem, it remains to show that when all $t_1, ..., t_N$ are collinear, the consensus is unique when $B_0 > B_1$. Indeed, for any action $a \neq t$, let $\hat{z}_n \in e_n(a)$ and observe that

$$\|\sum_{n=1}^{N} \hat{b}_n z_n\| \geq \|B_0\| - \|\sum_{n=N_a+1}^{N} b_n \hat{z}_n\| \geq B_0 - B_1 > 0.$$

Therefore, from Theorem 1, a does not solve problem (6). □

Proof of Theorem 5: Median Voter

Proof. We show first that any consensus a is collinear with $t_1, ..., t_N$: $a = a_0 + \beta(a_1 - a_0)$. Suppose that, to the contrary, $a = a_0 + \beta(a_1 - a_0) + x$ is a consensus, where x is a non-zero vector, orthogonal to $(a_1 - a_0)$: $x'(a_1 - a_0) = 0$. Let $\hat{a} = a_0 + \beta(a_1 - a_0)$. Then, for any n,

$$\begin{aligned}\|t_n - a\| &= \|a_0 + \rho_n(a_1 - a_0) - a_0 - \beta(a_1 - a_0) - x\| \\ &= \|(\rho_n - \beta)(a_1 - a_0) - x\| > \|(\rho_n - \beta)(a_1 - a_0)\| = \|t_n - \hat{a}\|,\end{aligned}$$

Therefore, moving from a to $\hat{a}$ reduces every term in the objective of (6), contradicting the assumption that a is a consensus.

Consider $a = a_0 + \beta(a_1 - a_0)$. Let N_l be be the number of ρ_n's that are strictly less than β, let N_g be the number of ρ_n's that are strictly greater than β, and let N_e be the number of ρ_n's that equal β. Also, let $b_l = \sum_{n=1}^{N_l} b_n$, let $b_e = \sum_{n=N_l+1}^{N_l+N_e} b_n$, and let $b_g = \sum_{n=N_l+N_e+1}^{N} b_n$. Then,

$$\begin{aligned}\sum_{n:a\neq t_n} b_n z_n &= \sum_{n:a\neq t_n} b_n \frac{t_n - a}{\|t_n - a\|} = \sum_{n:a\neq t_n} b_n \frac{(\rho_n - \beta)(a_1 - a_0)}{\|(\rho_n - \beta)(a_1 - a_0)\|} \\ &= \frac{a_1 - a_0}{\|a_1 - a_0\|} \sum_{n:a\neq t_n} b_n \frac{\rho_n - \beta}{|\rho_n - \beta|} = \frac{a_1 - a_0}{\|a_1 - a_0\|}(b_g N_g - b_l N_l).\end{aligned}$$

Action $a = a_0 + \beta(a_1 - a_0)$ is a consensus if and only if $|b_g N_g - b_l N_l| \leq b_e N_e$, which is equivalent to

$$\sum_{n=1}^{N} b_n N = b_l N_l + b_g N_g + b_e N_e \geq b_l N_l + b_g N_g + |b_g N_g - b_l N_l| = 2\max\{b_l N_l, b_g N_g\}.$$

Hence, a is a consensus if and only if $\max\{N_l, N_g\} \leq N_0$. When (21) holds with inequality, this implies that $\beta = \rho_{N_0}$. Otherwise, $\beta \in [\rho_{N_0-1}, \rho_{N_0}]$. $\square$

Proof of Theorem 6: Extreme Director

Proof. The bound is satisfied trivially if $a^* \in H$. Suppose that $a^* \notin H$. Let

t^* be the projection of a^* onto H. Because a^* solves (6),[15]

$$\left.\frac{\partial}{\partial\beta}\right|_{\beta=0}\left(\sum_{n=1}^{N}\left\|t_n-\left(a^*+\beta\frac{t^*-a^*}{\|t^*-a^*\|}\right)\right\|\right) = \sum_{n=1}^{N}\left(\frac{(t_n-a^*)'}{\|t_n-a^*\|}\frac{(t-a^*)}{\|t^*-a^*\|}\right) = 0.$$

Note that $(\partial/\partial\beta)_{\beta=0}\|t_N-(a^*+\beta(t^*-a^*)/\|t^*-a^*\|)\|\geq -1$. Thus,

$$\left.\frac{\partial}{\partial\beta}\right|_{\beta=0}\left(\sum_{n=1}^{N-1}\left\|t_n-\left(a^*+\beta\frac{t^*-a^*}{\|t^*-a^*\|}\right)\right\|\right)\leq 1. \tag{27}$$

For $n<N$,

$$\left.\frac{\partial}{\partial\beta}\right|_{\beta=0}\left\|t_n-\left(a^*+\beta\frac{t^*-a^*}{\|t^*-a^*\|}\right)\right\| = \frac{(t_n-a^*)'(t^*-a^*)}{\|t^*-a^*\|\|t_n-a^*\|}$$
$$= \frac{(t_n-t^*+t^*-a^*)'(t^*-a^*)}{\|t^*-a^*\|\|t_n-a^*\|} = \frac{\|t^*-a^*\|}{\|t_n-a^*\|}+\frac{(t_n-t^*)'(t^*-a^*)}{\|t^*-a^*\|\|t_n-a^*\|}$$

Since t^* is the projection of a^* on H, t^*-a^* separates t^* from H so $t_n\in H$ implies $(t_n-t^*)'(t^*-a^*)\geq 0$. From a triangle inequality,

$$\begin{aligned}\|t_n-a^*\| &\leq \|t_n-t^*\|+\|t^*-a^*\|\\ &= \|t_n-\sum_{j=1}^{N}w_jt_j\|+\|t^*-a^*\|\\ &= \|\sum_{j=1}^{N}w_j(t_n-t_j)\|+\|t^*-a^*\|\\ &\leq \sum_{j=1}^{N}w_j\|t_n-t_j\|+\|t^*-a^*\|\\ &\leq K+d(H,a^*),\end{aligned}$$

[15] The formula in the text assumes that the derivative exists. We know that the distance measure is not differentiable when $a^*=t_n$ for some n. For $n<N$, $a^*\neq t_n$ because $a^*\notin H$. In the event that $a^*=t_N$, substitute the relevant member of the derivative correspondence and the rest of the proof goes through.

Therefore,

$$\frac{\partial}{\partial \beta}\bigg|_{\beta=0} \left\| t_n - \left(a^* + \beta \frac{t^* - a^*}{\|t^* - a^*\|} \right) \right\| \geq \frac{d(H, a^*)}{K + d(H, a^*)}$$

Substituting the above into (27), obtain

$$(N-1)\frac{d(H, a^*)}{K + d(H, a^*)} \leq 1$$

Rearranging the above, obtain that the claimed bound (22) on the distance holds. □

Proof of Lemma 1: (Extreme Director)

We first look at the original directors' consensus (before the new director joins). Then, their targets t_n are given by (7) because λ is unknown. By Theorem 1, $a_0^* = \bar{y}$ is the consensus for the original $N = N_0$ directors because, taking $z_n = (t_n - a_0^*)/\|t_n - a_0^*\|$,

$$\text{(28)} \quad \begin{aligned} \sum_{n=1}^{N_0} z_n &= \sum_{n=1}^{N_0} \frac{t_n - a_0^*}{\|t_n - a_0^*\|} \\ &= \sum_{n=1}^{N_0} \frac{r(0, \sin(2\pi n/N_0), \cos(2\pi n/N_0))}{r} \\ &= 0, \end{aligned}$$

where the last equality holds because the points are equally spaced on a circle. Formally, recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and observe that

$$\begin{aligned}
\sum_{n=1}^{N_0} e^{i2\pi n/N_0} =& \sum_{n=1}^{N_0} \left(e^{i2\pi/N_0}\right)^n \\
=& e^{i2\pi/N_0} \frac{\left(e^{i2\pi/N_0}\right)^{N_0} - 1}{e^{i2\pi/N_0} - 1} \\
=& e^{i2\pi/N_0} \frac{e^{i2\pi} - 1}{e^{i2\pi/N_0} - 1} \\
=& e^{i2\pi/N_0} \frac{\cos(2\pi) + i\sin(2\pi) - 1}{e^{i2\pi/N_0} - 1} \\
=& 0.
\end{aligned}$$

Note that, by construction, $z_n \in e_n(a_0^*)$, where $e_n(a)$ is defined in (5).

We next look at the board of directors after the new director joins. Then, the targets t_n of the original directors are given by (8) because they know λ. Let $a^* = \bar{y}\lambda + (h_0, 0, 0)$, where h_0 is defined in the statement of the lemma. We want to apply Theorem 1 to show that a^* is the consensus for the $N = N_0 + 1$ directors. We take

$$z_n = \begin{cases} \dfrac{t_n - a^*}{\|t_n - a^*\|}, & \text{for } n \leq N_0; \\ \left(\dfrac{N_0 h_0}{b_{N_0+1}\sqrt{r^2 + h_0^2}}, 0, 0\right), & \text{for } n = N_0 + 1. \end{cases}$$

Then, noting that, for $n = 1, .., N_0$, $\|t_n - a^*\| = \sqrt{r^2 + h_0^2}$ and $\|t_n - \bar{y}\lambda\| = r$, we obtain

$$\begin{aligned}
(29) \quad \sum_{n=1}^{N_0+1} b_n z_n \quad &= \quad b_{N_0+1} z_{N_0+1} + \sum_{n=1}^{N_0} \frac{t_n - a^*}{\|t_n - a^*\|} \\
&= \quad b_{N_0+1} z_{N_0+1} - \frac{N_0(a^* - \bar{y}\lambda)}{\sqrt{r^2 + h_0^2}} + \frac{r}{\sqrt{r^2 + h_0^2}} \sum_{n=1}^{N_0} \frac{t_n - \bar{y}\lambda}{\|t_n - \bar{y}\lambda\|}.
\end{aligned}$$

which is zero because the second term in the above expression is zero from a derivation similar to (28) and the first and the third term cancel from the definition of z_{N_0+1}. By construction, $z_n \in e_n(a^*)$ for $n \leq N_0$, as we now prove. If $r^2 b_{N_0+1}^2 \leq h^2(N_0^2 - b_{N_0+1}^2)$, then $h_0 = r b_{N_0+1}/\sqrt{N_0^2 - b_{N_0+1}^2}$ and $z_{N_0+1} = (1,0,0) \in e_{N_0+1}(a^*)$. If $r^2 b_{N_0+1}^2 > h^2(N_0^2 - b_{N_0+1}^2)$, then $h_0 = h$, and

$$\begin{aligned} \|z_{N_0+1}\|^2 &= \frac{N_0^2 h^2}{b_{N_0+1}(r^2+h^2)} = \frac{N_0^2}{b_{N_0+1}^2}\left(1 - \frac{r^2}{r^2+h^2}\right) \\ &\leq \frac{N_0^2}{b_{N_0+1}^2}\left(1 - \frac{r^2}{r^2 + r^2 b_{N_0+1}^2/(N_0^2 - b_{N_0+1}^2)}\right) = 1, \end{aligned}$$

and again, $z_{N_0+1} \in e_{N_0+1}(a^*)$

Proof of Lemma 3: Ineffective Majority of Independent Directors

Without loss of generality, let the new director's ideal be

$$t_{N_o+N_i+1} = (\delta + \gamma\cos(\theta), \gamma\sin(\theta), 0, 0, 0).$$

If $\gamma \neq 0$, from Theorem 1, the consensus with the new director is at $a^* = (\delta, 0, 0, 0, 0)$ if

$$\sum_{n=1}^{n=N_o} \frac{t_n - a^*}{\|t_n - a^*\|} + \frac{t_{N_o+N_i+1} - a^*}{\|t_{N_o+N_i+1} - a^*\|} = N_i \varepsilon,$$

where $\|\varepsilon\| \leq 1$, or equivalently, if

$$\left(-\frac{N_0\delta}{\sqrt{\delta^2+r^2}} + \cos(\theta)\right)^2 + (\sin(\theta))^2 \leq N_i^2.$$

Noting that $\cos(\theta)^2 + (\sin(\theta))^2 = 1$ and rearranging, obtain

$$\frac{N_o\delta}{2\sqrt{\delta^2+r^2}} - \frac{(N_i^2-1)\sqrt{\delta^2+r^2}}{2N_o\delta} \leq cos(\theta).$$

The sufficient condition (15) then follows from $\cos(\theta) \leq 1$

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