

Learning how to cooperate when contingencies are ambiguous

Sylvain Chassang*, †

Massachusetts Institute of Technology

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Abstract

This paper studies how economic agents learn to cooperate when the details of what cooperation means are ambiguous. It considers a dynamic game in which one player's cost for the cooperative action is private information. From the perspective of the other player, this cost is an unknown but stationary function of observable states of the world. Initially, because of information asymmetries, full cooperation can be sustained only at the cost of inefficient punishment. As players gain common experience, however, the uninformed player may learn how to predict her partner's cost, thereby resolving informational asymmetries. Once learning has occurred, players can sustain cooperation more efficiently and reduce the partnership's sensitivity to adverse economic conditions. Nevertheless, because inducing information revelation has an efficiency cost, it may sometimes be optimal for the uninformed player to remain uninformed even though that limits the amount of cooperation that can be sustained in equilibrium.

KEYWORDS: cooperation, private information, learning, ambiguity, common understanding, indescribability.

JEL classification codes: C72, C73, D23

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†E-mail: chassang@mit.edu, Land-Mail: Department of Economics, Massachusetts Institute of Technology, 50 Memorial drive, Cambridge MA 02139, USA.

1 Introduction

Real-life partnerships are difficult to build and difficult to maintain. Breakdowns in cooperation and inefficient rigidities – which are unnecessary in full-information cooperation games – seem to be the rule rather than the exception. As a plausible explanation for these observations, private information has been recognized as an important hurdle to cooperation. In their seminal work on repeated games with private information, Green and Porter (1984) consider a model with unobservable actions (later extended by Abreu, Pearce and Stacchetti (1986, 1990) and Fudenberg, Levine, and Maskin (1994)) and show that inefficient breakdown must happen on any equilibrium path in order to induce cooperation. More recently, Athey and Bagwell (2001), Levin (2003) and Athey, Bagwell and Sanchirico (2004) have considered models in which it is the players' cost for the cooperative action that is the source of informational asymmetry. They find that when costs are private, cooperation does not necessarily require inefficient breakdown, but may involve inefficient rigidity.

The papers mentioned above are concerned with the difficulties of maintaining cooperation. This paper focuses on the particular hurdles involved in building cooperative relationships. Similarly to Athey and Bagwell (2001), Levin (2003) and Athey, Bagwell, and Sanchirico (2004), the players' cost for cooperation is initially private. The idea is to introduce the possibility of learning and to take into account the fact that as they gain common experience, players also become better judges of their partner's particular economic circumstances. For instance, knowing the history of disputes may help a workers' union determine when wage cuts implemented by management are necessary or not. The paper focuses on how learning dynamics interact with the patterns of cooperation.

The model considers two players engaged in an infinite horizon cooperation game which gives them asymmetric roles. Each period proceeds as follows: Player 1 first decides whether to terminate the partnership or not (stay or exit); when Player 1 stays, Player 2 gets a profit π while Player 1 incurs a cost of effort κ . Then, Player 2 has the option to take a

cooperative action (C) which has a variable cost c and gives Player 1 a benefit b . As in Athey and Bagwell (2001), Levin (2003) or Athey, Bagwell, and Sanchirico (2004), the distribution of c is known by both players, but only Player 2 observes its realization perfectly. The modeling innovation of this paper is that although only Player 2 observes c directly, Player 1 knows that it is a stationary function $c(w)$ of states of the world w which are observable by both players. Some states w may have clear payoff implications: then it will be obvious to Player 1 whether Player 2 should cooperate or not. Other states may be ambiguous, in the sense that: (i) these states cannot be distinguished in a language that is common to Player 1 and Player 2, (ii) Player 1 does not know which states correspond to a low cost of cooperation and which correspond to a high cost of cooperation. These ambiguous states represent a hurdle for cooperation because Player 2 can be tempted to exploit Player 1's confusion in order to avoid paying the cost c . However, because Player 2's cost structure is stationary, learning is possible and ambiguity can be resolved in equilibrium as Player 1 infers $c(w)$ from Player 2's actions.

Because Player 2 can exploit Player 1's confusion only temporarily, learning will be costless when players are patient enough. For intermediate degrees of patience, however, inducing information revelation will come at an efficiency cost. More precisely, on the path of a Pareto-efficient equilibrium, and while learning takes place, the partnership will be sensitive to adverse economic events: it will terminate with positive probability if an ambiguous state where cooperation is very costly occurs. Once ambiguity has been resolved, however, the partnership becomes resilient in the sense that it survives negative shocks that would have caused termination earlier on. Nevertheless, the cost of resolving ambiguity can be prohibitively high, and it may be optimal for Player 1 to remain uninformed.

Because the distribution of c is common knowledge, this model exhibits neither "good" nor "bad" types: players agree on the general principle of cooperation, but need to work through the details of implementation. As a consequence, the impact of ambiguity on cooperation is tightly related to indescribability: when ex ante communication about ambiguous

states is possible, the truthful revelation of Player 2's cost function $c(\cdot)$ via cheap-talk imposes no constraints on equilibrium continuation values. In other words, the game with asymmetric information and describable states has the same Pareto frontier as the game with full information. Hence, in this model it is both private information and indescribability – the lack of a precise enough common language – that make it delicate to build a cooperative arrangement.

As noted above, this paper is related to the literature on cooperation games with private costs. Athey and Bagwell (2001) and Athey, Bagwell, and Sanchirico (2004) consider a game of repeated Bertrand competition where the firms' production costs are private, and analyze efficient collusion schemes under symmetric and asymmetric strategies. Levin (2003) considers a dynamic principal-agent relationship where the agent's cost of effort is private information and highlights the tension between efficient flexibility and incentive compatibility. From the perspective of learning, the current paper is also related to the contributions of Watson (1999, 2002), which consider a cooperation game with “cooperative” and “uncooperative” types and show how delaying full cooperation emerges as a mechanism to efficiently sort out “uncooperative” types. In the current paper, there are no “cooperative” and “uncooperative” types: players agree on the broad features of cooperation. It is the specific details of implementation that are ambiguous and need to be elucidated.

The paper is also related to the literature on the relevance of indescribable contingencies initiated by Maskin and Tirole (1999) and developed in Hart and Moore (1999), Segal (1999) and Battigalli and Maggi (2002). Here, because indescribability implies that Player 1 does not observe the cost of Player 2 when the state is ambiguous, the results of Maskin and Tirole (1999) on the irrelevance of describability do not apply. In the dynamic game studied here, if costs were observable, indescribability would indeed have no impact. However, because indescribability creates confusion, it does affect the players' ability to cooperate. In the particular game considered in this paper, there are only two ambiguous states and indescribability implicitly results from a coarseness of language. The work of Al-Najjar,

Anderloni and Felli (2006) shows how indescribable events can be modelled even for rich languages. However, their construction relies on a large enough space of states, and analyzing the dynamics of learning for a such a rich state space is beyond the scope of this paper.

The paper is structured as follows: Section 2 describes the model. Section 3 studies the joint dynamics of cooperation and learning. Section 4 generalizes the notion of ambiguity and proposes a model of empathy building. Section 5 concludes. Appendix A examines the robustness of results to specific modeling assumptions and presents an extension to three-states ambiguity. Proofs are contained in Appendix B unless mentioned otherwise.

2 The Framework: ambiguity, indescribability, and cooperation

2.1 The game

Consider a game with two players $i \in \{1, 2\}$, infinite horizon $t \in \{1, \dots, \infty\}$, and discount rate β . Each period t consists of the following four subperiods:

1. Player 1 chooses to stay (S) at cost $\kappa > 0$ or exit (E) at zero cost. If Player 1 exits, the game ends and both players get zero continuation values.
2. If Player 1 has chosen to stay, an i.i.d. state of the world w is drawn from $\{\underline{w}, \bar{w}, w_a^1, w_a^2\}$ with respective probabilities $\{\underline{p}, \bar{p}, p_a, p_a\}$, where $p_a = \frac{1-\underline{p}-\bar{p}}{2}$.
3. Player 2 observes the state w and her private cost of cooperation $c(w)$. Player 1 observes only w . Both players observe some payoff-irrelevant random variable x_t , uniformly distributed over $[0, 1]$, that allows for public randomizations.
4. Player 2 decides whether to cooperate (C) or defect (D). Conditional on Player 1 having

stayed in stage 1, players' payoffs at the end of stage 4 are

	C	D
$Player\ 2$	$\pi - c(w)$	π
$Player\ 1$	$b - \kappa$	$-\kappa$

where π , b , and κ are common knowledge, strictly positive, and $b > \kappa$.

Ambiguity. The players' differing knowledge of the cost structure $c(\cdot)$ is the key element of this environment: the observability of a state is distinguished from the observability of the cost of cooperation that this state implies. More precisely, when the state w belongs to $\{\underline{w}, \bar{w}\}$, the cost $c(w)$ is common knowledge for the players. However, when w belongs to $\{w_a^1, w_a^2\}$, the state w is observable by both players, but the associated cost $c(w)$ is perfectly known only to Player 2. Player 1 merely knows that $(c(w_a^1), c(w_a^2))$ is equal to (c_L, c_H) with probability 1/2 and equal to (c_H, c_L) with probability 1/2, where $c_H > c_L > 0$. The states \underline{w} and \bar{w} , which have clear implications for the cost of cooperation, are contrasted with the ambiguous states w_a^1 and w_a^2 , which are harder to interpret. The game with ambiguous information is denoted by Γ_A . The game with full information, in which c is perfectly observable by both players in all states, is denoted by Γ_{FI} . The object of this paper is to study perfect Bayesian equilibria of both Γ_{FI} and Γ_A . Note that although they act at different moments, both players observe the same histories from period $t = 1$ on.

Definition 1 (histories) Denote by d_t Player 1's decision to stay or not and a_t Player 2's decision to cooperate or not at time t . We respectively define by

$$h_t^1 \equiv \{d_1, w_1, x_1, a_1, \dots, d_{t-1}, w_{t-1}, x_{t-1}, a_{t-1}\}$$

and $h_t^2 \equiv \{d_1, w_1, x_1, a_1, \dots, d_{t-1}, w_{t-1}, x_{t-1}, a_{t-1}, d_t, w_t, x_t\}$

the histories respectively observed by Player 1 and Player 2 when they make their decisions

in period t .

The set of all possible histories is denoted by \mathcal{H} . Pure strategies of Player 1 are mappings $s_{P_1} : \mathcal{H} \rightarrow \{S, E\}$ and pure strategies of Player 2 are mappings $s_{P_2} : \mathcal{H} \rightarrow \{C, D\}$.

Note that there is no aggregate uncertainty about the amount of cooperation that could be sustained under full information. This game represents a situation where players agree on the possibility of cooperation but are trying to work out the details of when cooperation should occur or not.

Because the cost function is stationary, the observability of states allows learning: if at some ambiguous state, the equilibrium behavior of Player 2 is different when her cost is c_L than when her cost is c_H , Player 1 will learn the mapping $c(\cdot)$ perfectly and the continuation game will involve no informational asymmetry.

2.2 Interpretation

The game presented in Section 2.1, although very stylized, can be interpreted as a rough model for a variety of economic settings where the detailed contingencies in which cooperation should occur are ambiguous. For example, the game Γ_A can be used to model the problem of a union (Player 1) having to decide when wage cuts implemented by management (Player 2) are acceptable or not. It could also be a model of shareholders (Player 1) trying to decide when it is tolerable that the firm's CEO (Player 2) does not give out dividends. It could also be a model of a firm (Player 1) deciding to relocate its production if the local government (Player 2) raises unjustified taxes too frequently. One might also view the game Γ_A as a description of a public good provision problem in which a public authority (Player 1) is trying to find out whether a citizen (Player 2) is not contributing because she is short of money, or because she is being selfish. Finally this game can also be understood as a model of a lender/borrower relationship in which the lender (Player 1) decides whether or not she should tolerate delayed repayment by the borrower (Player 2), or demand liquidation.

In this paper “ambiguity” refers to a setting in which players agree on the principle of cooperation but do not know the details of when and how cooperation should be implemented. The main result of this paper is that although there is no uncertainty about the sustainability of cooperation under full-information, the inability to communicate about states of the world may limit the partnership to inefficient cooperative agreements. In particular while learning occurs, the partnership will be sensitive to negative economic shocks and inefficient termination may happen when an ambiguous state with cost c_H occurs. As ambiguity is resolved, the partnership becomes resilient to adverse circumstances and players can sustain efficient cooperation. However, because inducing information revelation has an efficiency cost of its own, under some conditions it will be optimal not to resolve informational asymmetries.

2.3 The role of indescribability

The game presented in Section 2.1 does not allow for communication. The assumption is that ambiguous states are recognizable but cannot be described using words from the players’ common language. This indescribability is a key element of the setup. Indeed, Proposition 1 shows that if players have the ability to communicate ex ante about ambiguous states, then the Pareto frontier under ambiguity is the same as the Pareto frontier under full information.

More precisely, assume that at time $t = 0$, before the game described in Section 2.1 begins, Player 2 can send Player 1 a message that is either “ w_a^1 is the low cost state” or “ w_a^1 is the high cost state”. In that setting, the following proposition holds.

Proposition 1 (communication and ambiguity) *If Player 2 can engage in ex ante communication, then ambiguity is innocuous. More formally, the Pareto frontier of the game with ambiguity is the same as the Pareto frontier of the game with full information: Player 2 can be induced to reveal her type without constraining continuation values.*

The intuition for this result is straightforward. Pick an equilibrium under full information, and let Player 1 play according to that equilibrium, taking Player 2’s declaration at face value.

Given Player 1's behavior, ex ante, Player 2 does not benefit from Player 1's confusion and information revelation is incentive compatible.

Section 3 studies the joint dynamics of cooperation and learning when communication is impossible and Player 1 must learn from Player 2's actions. Incentive compatibility constraints will now need to hold ex post rather than ex ante. In particular, when she is at a state where she is supposed to cooperate, Player 2 may be tempted to delay cooperation and temporarily exploit Player 1's confusion. As Section 3 will show, Player 1 may need to use inefficient exit in order to induce Player 2 to cooperate at ambiguous states.

3 Learning how to cooperate

This section explores the joint dynamics of learning and cooperation under ambiguity. Section 3.1 studies the problem of optimal information revelation. It shows that when learning occurs, efficient cooperative equilibria will be non-stationary: early on, while ambiguity has not been resolved, negative economic shocks will cause the partnership to breakdown with positive probability; once players have built common understanding, however, the partnership will be resilient to such shocks and inefficient breakdowns become unnecessary. Because information revelation comes at an efficiency cost, Section 3.2 explores whether it may be optimal for Player 1 to remain uninformed. Depending on parameter values, both under-cooperation and over-cooperation on the part of Player 2 may be optimal.

3.1 The joint dynamics of learning and cooperation

This section studies game Γ_A under the assumption that $c(\underline{w}) = c(w_a^1) = c_L > 0$, and $c(\bar{w}) = c(w_a^2) = c_H = +\infty$ (where the fact that w_a^1 is the ambiguous state with cost c_L is of course unknown to Player 1). States w fall into four categories, depending on whether cooperation is possible or not (c equal to c_L or $c_H = +\infty$), and whether that information is private or not (w in $\{\underline{w}, \bar{w}\}$ or $\{w_a^1, w_a^2\}$). Section 3.1.1 characterizes the Pareto frontier

under full information. Section 3.1.2 gives sufficient and necessary conditions for information revelation to be costly and characterizes optimal equilibria among the class of ambiguity-resolving equilibria.

3.1.1 The Pareto frontier under full information.

This section characterizes the Pareto frontier of the full-information game Γ_{FI} . In particular, it shows that on the full-information Pareto-efficient frontier, Player 1 should never exit in equilibrium.

Proposition 2 (no exit) *Under full information, the Pareto frontier is such that:*

1. *either it is reduced to a unique equilibrium for which Player 1 exits with certainty in period $t = 1$*
2. *or, Player 1 always chooses to stay following an action of Player 2 that is possible on the equilibrium path.*

Lemma 1 *Conditional on the existence of equilibria that are not reduced to immediate exit, any pair of values on the full-information Pareto frontier can be attained by equilibria such that*

1. *On the equilibrium path, Player 1 always stays;*
2. *Player 2 never cooperates when her cost is $c_H = +\infty$;*
3. *If she has never cooperated when her cost was c_L before, Player 2 cooperates with probability r_1^L at a state with cost c_L . If she has cooperated when her cost was c_L before, Player 2 cooperates with probability $r_2^L \leq r_1^L$ when her cost is c_L .*

Consider one of the equilibria described in Lemma 1. For a given rate of cooperation r_2^L , after the first time Player 2's cost is c_L and she cooperated, the long run continuation values

are

$$V_{P_2}^{r_2^L} = \frac{1}{1-\beta} (\pi - (\underline{p} + p_a)r_2^L c_L) \quad (1)$$

$$V_{P_1}^{r_2^L} = \frac{1}{1-\beta} (-\kappa + (\underline{p} + p_a)r_2^L b) . \quad (2)$$

For $r_2^L > 0$, this cooperation scheme will be sustainable in equilibrium if and only if

$$V_{P_1}^{r_2^L} \geq 0 , \quad \text{and} \quad \beta V_{P_2}^{r_2^L} \geq c_L \quad (3)$$

These two conditions ensure that staying and cooperating are respectively incentive compatible for Player 1 and Player 2. Denote by $r_{min} \equiv \frac{\kappa}{b(\underline{p}+p_a)}$ the minimum constant cooperation rate required from Player 2 to induce Player 1 to stay. There are three cases:

1. If $r_{min} \in \left[0, \frac{\underline{p}}{\underline{p}+p_a}\right]$, then if Player 2 cooperates frequently enough at the unambiguous state \underline{w} , Player 1 can be induced to stay.
2. If $r_{min} \in \left[\frac{\underline{p}}{\underline{p}+p_a}, 1\right]$, then Player 2 must cooperate at both the unambiguous state \underline{w} and the ambiguous state with low cost (state w_a^1 by convention) to induce Player 1 to stay.
3. If $r_{min} > 1$, then Player 2 cannot get Player 1 to stay, independently of how much she cooperates when her cost is c_L .

It is incentive compatible for Player 2 to cooperate at the minimum required rate r_{min} if and only if

$$\frac{1}{1-\beta} (\pi - (\underline{p} + p_a)r_{min}c_L) \geq \frac{c_L}{\beta} . \quad (4)$$

It is assumed that condition (4) holds and $r_{min} < 1$, so that there exists a cooperative equilibrium under full information. The maximum value attainable by Player 2 is

$$V_{P_2}^{max} = V_{P_2}^{r_{min}} = \frac{1}{1-\beta} (\pi - (\underline{p} + p_a)r_{min}c_L) . \quad (5)$$

Define r_{max} as the maximum $r \in [0, 1]$ such that,

$$\frac{1}{1 - \beta} (\pi - (\underline{p} + p_a)rc_L) \geq \frac{c_L}{\beta}. \quad (6)$$

Ambiguous states are relevant only if $r_{max} > \frac{\underline{p}}{\underline{p} + p_a}$. Otherwise, the Pareto frontier under full information can be spanned by equilibria such that Player 2 cooperates only at the unambiguous state \underline{w} . The maximum continuation value Player 1 can expect after Player 2 has cooperated at a state with cost c_L is

$$V_{P_1}^{max} = V_{P_1}^{r_{max}} = \frac{1}{1 - \beta} (-\kappa + (\underline{p} + p_a) r_{max} b). \quad (7)$$

3.1.2 Optimal ambiguity-resolving strategies

This section considers strategies that induce revelation of Player 2's type in equilibrium. Since $c_H = +\infty$, revelation will occur whenever Player 2 cooperates at an ambiguous state. Because misbehavior (i.e. failing to cooperate in the ambiguous state when $c = c_L$) can be detected in finite time, inducing revelation need not always be costly. Proposition 3 and Lemma 5 provide conditions under which resolving ambiguity will be costly and characterize the extent of the efficiency loss. Proposition 4 characterizes optimal revelation inducing strategies.

Definition 2 Consider a pair of equilibrium strategies (s_{P_1}, s_{P_2}) and some history h_t at which Player 2 can take action. A history h_t is a **revelation stage** if it is the first history at which the two types of Player 2 – with respective cost structures $(c(w_a^1), c(w_a^2))$ equal to (c_H, c_L) and (c_L, c_H) – take different actions. A revelation stage h_t is said to be **conclusive** (resp. **inconclusive**) when Player 2's type is such that she chooses to cooperate (resp. Player 2 chooses not to cooperate).

A history h_{t+s} weakly following a revelation stage h_t is called a **confirmation stage** if it is the first history subsequent to h_t at which Player 2 cooperates at an ambiguous state.

In equilibrium, Player 1 will know Player 2's type after a revelation stage. Still, conclusive and inconclusive revelation stages differ in the sense that after an inconclusive revelation, Player 1 may worry that Player 2 simply misrepresented her cost to avoid cooperating in the short run. For this reason, inconclusive revelations may have to be followed by inefficient exit.

Proposition 3 (costly revelation) *Assume that some cooperation is feasible under full information and define*

$$V_{P_2}^{Liar} \equiv \max \left\{ \frac{1}{1 - (\bar{p} + p_a)\beta} \pi ; \frac{1}{1 - (\underline{p} + \bar{p} + p_a)\beta} (\pi - \underline{p}c_L) \right\}. \quad (8)$$

If

$$\beta(V_{P_2}^{max} - V_{P_2}^{Liar}) < c_L \quad (9)$$

then, the revelation of Player 2's type is necessarily costly: no equilibrium of the game with ambiguity that involves revelation on the equilibrium path can be on the full-information Pareto frontier.

The value $V_{P_2}^{Liar}$ corresponds to the min-max value that Player 2 can attain when she chooses not to cooperate although her cost is low, and Player 1 never exits following actions that are consistent with Player 2 being truthful¹. Under this constraint, the only way Player 1 can detect misbehavior is when Player 2 does not cooperate at both ambiguous states. The time multiplier comes from the fact that when there is no inefficient exit, Player 2 knows that she will keep getting benefits from the partnership at least until the other ambiguous state comes up.

As Proposition 3 suggests and Lemma 5 confirms, when Player 2's surplus is sufficiently high, revelation is incentive compatible at no cost: because it is possible to discover past misbehavior in finite time, when Player 2 gets enough surplus, she will reveal her type rather

¹In other terms, $V_{P_2}^{Liar}$ is the min-max value of Player 2 when she misbehaves and Player 1 uses a strategy that would coincide with a Pareto-efficient equilibrium under full information.

take the chance of causing the partnership breakdown in the near future. This result is consistent with the usual folk theorem intuition². When Player 2 gets only moderate surplus however, she can be tempted to misbehave and exploit Player 1's confusion for a while. In such a case, revelation will be induced only by having Player 1 exit on the equilibrium path, which is inefficient.

The rest of this section focuses on the case where the revelation of Player 2's type is costly, and characterizes the dynamics of optimal, ambiguity resolving, strategies.

Lemma 2 (revelation stages) *On the Pareto frontier of game Γ_A , revelation will occur only at a history for which the current state w_t is ambiguous, i.e. $w_t \in \{w_a^1, w_a^2\}$.*

Lemma 3 (forgiveness) *For any Pareto-efficient equilibrium of game Γ_A , once Player 2 has cooperated at an ambiguous state, then players behave according to some Pareto-efficient equilibrium of the game with full information Γ_{FI} .*

In other terms, if inefficient behavior must occur to induce revelation, it should not happen any more once Player 2 has cooperated at an ambiguous state. Indeed, the sole purpose of inefficient exit is to diminish the expected continuation value of Player 2 upon misbehavior. Once confirmation occurs, Player 1 knows for sure that Player 2 cannot have misbehaved, and hence there is no need for further punishment.

The next lemma concerns optimal behavior following an inconclusive revelation. It shows that optimally, Player 2 should cooperate as much as is incentive compatible in order to reduce Player 2's incentive to misrepresent her cost structure.

Lemma 4 (maximum cooperation) *Consider a Pareto efficient equilibrium (s_{P_1}, s_{P_2}) of game Γ_A that involves costly revelation. If revelation occurs at state w_a^i with $i \in \{1, 2\}$, then*

1. *Player 2 must cooperate with probability 1 the first time state w_a^{-i} or \underline{w} comes up;*

²See for instance Fudenberg, Levine, and Maskin (1994).

2. *There exists an equilibrium of Γ_A that gives players the same ex-ante values as (s_{P_1}, s_{P_2}) and in which following 1) inconclusive revelation and 2) cooperation at a state $w \in \{w_a^{-i}, \underline{w}\}$, Player 2 cooperates at a rate r_{max} at all further states $w \in \{w_a^{-i}, \underline{w}\}$.*

The rationale for maximum cooperation is that it minimizes the time that Player 1 needs to identify for sure whether or not Player 2 has misbehaved. After an inconclusive revelation, maximum cooperation will tend to bias surplus division in favor of Player 1, but this can be undone by delaying the revelation stage itself. Finally, note that when revelation is not costly, equilibria on the Pareto frontier need not exhibit maximum cooperation. However, such equilibria are sufficient to describe the Pareto frontier. Using Lemmas 2, 3, and 4, we can now characterize optimal revelation-inducing equilibria.

Proposition 4 (early dispute, fast forgiveness) *Consider any equilibrium of Γ_A that involves revelation. There exists a weakly Pareto-dominating equilibrium in which exit – if it happens at all – needs to happen only in the period immediately following an inconclusive revelation of Player 2’s type.*

In other term, there is no option value in delaying confrontation. Delaying exit may allow Player 2 to confirm its type in the meanwhile, but also makes the crisis more violent if Player 2 isn’t able to dispel doubts. Proposition 4 states that these two forces exactly offset each other.

Proposition 4 shows that optimal ambiguity-resolving equilibria take a simple form. Some histories are designated as revelation stages. At a revelation stage, Player 2 either cooperates or defects. If Player 2 cooperates then players play some Pareto-efficient equilibrium of the full-information continuation game. If Player 2 defects then Player 1 exits with some probability in the next period. When exit does not happen, players follow some Pareto-efficient equilibrium of the full-information continuation game. The probability of exit following an inconclusive revelation is given by Lemma 5.

Lemma 5 *Let (s_{P_1}, s_{P_2}) be a Pareto-efficient equilibrium with early dispute as defined in Proposition 4. Consider a revelation stage h at which Player 2 would get a continuation value $V_{P_2}^{Coop}$ upon cooperation. Following an inconclusive revelation, Player 1 must exit with probability*

$$1 - q(V_{P_2}^{Coop}) = 1 - \min \left\{ \frac{\beta V_{P_2}^{Coop} - c_L}{\beta V_{P_2}^{Liar}}, 1 \right\}. \quad (10)$$

Lemma 5 implies that the likelihood of exit following an inconclusive revelation is decreasing in Player 2's continuation value upon cooperation, $V_{P_2}^{Coop}$. This means that transferring surplus from the informed player to the uninformed player increases inefficiency: because it increases Player 2's incentives to misrepresent its cost, Player 1 must exit with greater probability following an inconclusive revelation in order to keep truthful revelation incentive compatible. In fact, Section 3.2 will show that in some cases, the inefficiency can be so large that Player 1 would rather have Player 2 cooperate only in the unambiguous state \underline{w} , rather than provide the costly incentives needed to induce revelation.

This section has shown that when states are indescribable, ambiguity is at the source of a monitoring problem which generates inefficiencies. When ambiguous states occur, indescribability creates an information asymmetry that Player 2 is tempted to exploit. Because misbehavior on the part of Player 2 can be detected in finite time, ambiguity can be resolved at no efficiency cost when the surplus is large enough. However, because Player 2 can benefit from the partnership for multiple periods after she has misbehaved, Player 1 must resort to inefficient exit following an inconclusive revelation when the surplus is moderate.

Optimal revealing equilibria take a simple form that is reminiscent of the bang-bang property of Abreu, Pearce, and Stacchetti (1990): some histories are selected as revelation stages. If revelation is conclusive, the continuation game is played according to some Pareto-efficient full-information equilibrium. If revelation is inconclusive, Player 1 exits with some probability. If exit does not happen, then players fall back to playing some Pareto-efficient equilibrium of the full-information game. While learning occurs, the partnership will be

sensitive to adverse economic shocks and termination will happen with positive probability if an ambiguous state with high cost occurs. Once learning has happened, however, players behave according to a Pareto-efficient full information equilibrium which does not involve inefficient exit. In other words, once players have reached a common understanding, the partnership becomes resilient to adverse shocks.

This section was restricted to revealing equilibria. However some cooperation can be sustained in game Γ_A without revealing Player 2's type. There are two ways to achieve this: Player 2 can cooperate only in the unambiguous state w — this corresponds to under-cooperation; or Player 2 can cooperate in both states w_a^1, w_a^2 with the same probability — this corresponds to over-cooperation. The levels of cooperation achieved in revealing and non-revealing equilibria follow different dynamic patterns: revealing equilibria will initially exhibit some inefficient termination, later followed by greater cooperation, while non-revealing equilibria will sustain constant intermediate levels of cooperation. In that sense, resolving ambiguity is an investment in relational capital which comes at the immediate cost of inefficient exit but yields the future benefit of efficient cooperation. Section 3.2 studies the relative benefits of revelation and non-revelation and establishes that non-revelation, in the form of either under- and over-cooperation, may sometimes be optimal.

3.2 Optimal learning

Section 3.1.2 showed that reducing informational asymmetries comes at an efficiency cost. Given these costs, this section considers whether there should be any information revelation on the Pareto frontier. It is shown that depending on the costs and benefits of revelation, it can be optimal for Player 2 to under-cooperate or over-cooperate.

3.2.1 Under-cooperation

This section maintains the assumption that $c(\underline{w}) = c(w_a^1) = c_L > 0$, and $c(w_a^2) = c(\bar{w}) = c_H = +\infty$. Consider the region of the parameter space such that,

$$r_{min} \in \left[0, \frac{\underline{p}}{\underline{p} + p_a}\right] \quad \text{and} \quad r_{max} \in \left(\frac{\underline{p}}{\underline{p} + p_a}, 1\right). \quad (11)$$

When condition (11) is satisfied, Player 2 needs to cooperate only in the unambiguous state \underline{w} to induce Player 1 to stay. Furthermore, under full information, it is incentive compatible for Player 2 to cooperate with probability 1 in the unambiguous state \underline{w} and with some probability strictly less than 1 in the ambiguous state with low cost w_a^1 . This section shows that under ambiguity, the efficiency cost of revelation can be so prohibitive that for all equilibria on the Pareto frontier of Γ_A , Player 2 should cooperate only in the unambiguous state \underline{w} . In this case, we say that *under-cooperation* is optimal.

Proposition 5 (sufficient condition for optimal under-cooperation) *Whenever*

$$b + \frac{\beta}{1-\beta} p_a b \leq \frac{\beta}{1-\beta} (\underline{p}b - \kappa), \quad (12)$$

all equilibria on the Pareto frontier of the game with ambiguity involve under-cooperation.

Under this condition, on the Pareto frontier of the game Γ_A , Player 2 will never cooperate at an ambiguous state: even from the perspective of Player 1, the cost of inducing revelation is greater than the potential benefit of full cooperation.

According to condition (12), under-cooperation will be optimal whenever the greatest possible increase in benefit Player 1 could get from full cooperation, $b + \frac{\beta}{1-\beta} p_a b$, is smaller than the value, $\frac{\beta}{1-\beta} (\underline{p}b - \kappa)$, it expects from having Player 2 cooperate whenever the state is \underline{w} . This will be the case when the unambiguous state \underline{w} occurs with greater frequency than the ambiguous state w_a^1 , and players' discount rate β is close enough to 1.

3.2.2 Over-cooperation

Section 3.2.1 showed that the efficiency cost of revelation can be so prohibitive that Pareto optimal equilibria of game Γ_A simply do not involve cooperation at an ambiguous state. Over-cooperation – meaning that Player 2 cooperates at both ambiguous states with equal probability – is another possible non-revealing strategy. So far, over-cooperation was ruled out by the assumption that $c(w_a^2) = c_H = \infty$. In order to study potential over-cooperation, this section considers the case where $c(w_a^1) = c(\underline{w}) = c_L$, $c(w_a^2) = c_H < +\infty$, and $c(\bar{w}) = \infty$. It may now be possible for Player 2 to cooperate at an ambiguous state with cost c_H . The section begins by extending Propositions 2 and 3: there must be no exit on the Pareto frontier of game Γ_{FI} , while under ambiguity revelation may require inefficient exit on the equilibrium path .

Lemma 6 (no exit) *The Pareto frontier under full information is such that:*

1. *either it is reduced to a unique equilibrium for which Player 1 exits with certainty in period $t = 1$*
2. *or Player 1 always chooses to stay following an action of Player 2 that is possible on the equilibrium path.*

Lemma 7 (sufficient condition for costly revelation) *Define $\underline{V}_{P_2}^{Liar} \equiv \frac{1}{1-\beta}\pi$. Whenever*

$$\beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar}) < c_L \tag{13}$$

revelation is necessarily costly. In other terms, an equilibrium of the game with ambiguity that involves revelation cannot lie on the Pareto frontier of the game with perfect information.

When the inefficiency cost of revelation is high, non-revealing equilibria may be optimal. An equilibrium will be non-revealing whenever Player 2 cooperates at the same rate in both ambiguous states. Given a value V_{P_1} that Player 1 can obtain in a full information

equilibrium, there may or may not be a non-revealing equilibrium that grants Player 1 the same utility. Proposition 6 provides conditions under which, if there exists a non-revealing equilibrium that grants Player 1 value V_{P_1} , then Pareto-efficient equilibria of Γ_A that give value V_{P_1} to Player 1 are non-revealing.

Proposition 6 (sufficient condition for optimal over-cooperation) *Assume that Player 1 can be granted value V_{P_1} in a non-revealing equilibrium of game Γ_A . Whenever*

$$\frac{\beta}{1-\beta} p_a (c_H - c_L) < c_L - \beta (V_{P_2}^{max} - \underline{V}_{P_2}^{Liar}), \quad (14)$$

the most efficient equilibrium of Γ_A that gives value V_{P_1} to Player 1 is non-revealing.

Note that condition (14) will be satisfied whenever c_L is high enough for condition (13) to hold and $c_H - c_L$ is small enough. Proposition 6 is particularly interesting when $r_{min} \in \left(\frac{p}{p+p_a}, 1\right)$, since in that case Player 2 must cooperate in an ambiguous state in order to induce Player 1 to stay. In that case, non-revelation means that Player 2 would rather over-cooperate than go through the potentially inefficient process of information revelation. Intuitively, revelation generates inefficiencies because it creates a temptation for Player 2 to misrepresent her cost and avoid cooperating in the short term. Because such misbehavior cannot be detected immediately, Player 1 must resort to inefficient exit in order to provide adequate incentives. Propositions 5 and 6 show that these efficiency costs can be greater than the gains of using a precise cooperative arrangement that distinguishes between ambiguous states.

4 A model of empathy building

The simple model of ambiguity presented in Section 2, the fact that there are only two states of the world results in a very stark learning process: whenever Player 2 cooperates at an ambiguous state, Player 1 learns the cost structure $c(w)$ perfectly. This section suggests a framework in which to model richer information structures. The model presented in

what follows is perhaps more about what might be called empathy-building than ambiguity-resolution: Player 1 can observe a number of signals that may or may not be good predictors of Player 2’s cost for cooperation. As players gain common experience, Player 1 learns about the relative predictive power of the various signals she observes, and can start using them to provide flexible contingent incentives for cooperation. Section 4.1 describes the model. Section 4.2 gives some examples of information structures that fit in this framework and shows that it is a generalization of the ambiguity framework. Section 4.3 shows that the essential properties of the ambiguity framework still hold.

4.1 The framework

Consider a game with two players, infinite horizon, $t \in \{1, \dots, +\infty\}$, and discount rate β . The timing of each period is similar to that of Section 2:

1. Player 1 decides to either stay (S) at a cost κ or exit (E) at zero cost. If Player 1 exits, the game ends and both players get zero continuation values.
2. If Player 1 has stayed, an i.i.d. random variable \tilde{c} is drawn. It takes values in $\{c_L, c_H\}$ with respective probabilities $(p, 1 - p)$.
3. Player 2 observes her cost of cooperation \tilde{c} . Player 1 observes a signal s of \tilde{c} . Both players observe a payoff-irrelevant random variable that allows public randomizations. Players can also exchange messages belonging to $\{\text{“cooperation”}, \text{“no cooperation”}\}$.
4. Player 2 decides to cooperate (C) or defect (D). Conditional on Player 1 having stayed in the first stage, final payoffs are given by

	C	D
<i>Player 1</i>	$\pi - \tilde{c}$	π
<i>Player 2</i>	$b - \kappa$	$-\kappa$

where π , b , and κ are common knowledge and strictly positive, and it is common knowledge that the cost of cooperation \tilde{c} is either c_L or c_H , with $0 < c_L < c_H = +\infty$.

Empathy. At the beginning of each period t , Player 1 can choose to watch one signal s from a family \mathcal{S} of possible signals. Each of these signals is a real-valued i.i.d. stochastic process $(s_t)_{t \in \mathbb{N}}$, normalized to share an identical distribution f , but that may or may not be good predictors of \tilde{c} . The precision of a signal is given by the conditional distribution $D_s(x) = \text{Prob}(\tilde{c} = c_H \mid s = x)$. When Player 1 has watched signals $\{s_1, \dots, s_n\}$ and chooses to draw a new signal, the precision of the new signal is drawn according to a probability distribution $f_{\mathcal{S} \setminus \{s_1, \dots, s_n\}}$ on $[0, 1]^{\mathbb{R}}$ (the set of functions mapping \mathbb{R} into $[0, 1]$). Which signal she observes is private information to Player 1. This game will be denoted by $\Gamma_{\mathcal{S}}$.

In this framework, Player 1 may not be able to learn how to perfectly predict the cost structure $c(w)$ immediately. In particular, when \mathcal{S} is large, finding a good signal of Player 2's behavior will take time. As players gain common experience, Player 1 will be able to better understand Player 2's cost of cooperation. This greater empathy will allow for the provision of more efficient contingent incentives.

4.2 Some examples

This section gives a few examples of the empathy framework presented above. In particular it shows that this framework generalizes the game with ambiguity Γ_A defined in Section 2.

4.2.1 Ambiguity

This section shows that the game with ambiguity Γ_A is a specific case of the framework presented in Section 4.1. Let \tilde{c} take values c_L and c_H with respective probabilities $p = \underline{p} + p_a$ and $1 - p = \bar{p} + p_a$. The set of possible signals contains only two signals, $\mathcal{S} = \{s_-, s_+\}$ taking

values in $\{-2, -1, 1, 2\}$ with respective probabilities $\{\underline{p}, p_a, p_a, \bar{p}\}$ such that

$$Prob(\tilde{c} = c_H | s_-) = \begin{cases} 1 & \text{if } s_- = 2 \\ 0 & \text{if } s_- = 1 \\ 1 & \text{if } s_- = -1 \\ 0 & \text{if } s_- = -2 \end{cases} \quad \text{and} \quad Prob(\tilde{c} = c_H | s_+) = \begin{cases} 1 & \text{if } s_+ = 2 \\ 1 & \text{if } s_+ = 1 \\ 0 & \text{if } s_+ = -1 \\ 0 & \text{if } s_+ = -2 \end{cases}$$

States \underline{w} and \bar{w} can respectively be defined as the events $\{s_+ = 2 \cup s_- = 2\}$ and $\{s_+ = -2 \cup s_- = -2\}$, while states w_a^1 and w_a^2 respectively correspond to events $\{s_+ = 1 \cup s_- = -1\}$, and $\{s_+ = -1 \cup s_- = 1\}$.

4.2.2 Finding a reliable signal

The set of signals \mathcal{S} can also be infinite. This would naturally be the case if states of the world were infinite dimensional vectors, and Player 1 could pay attention only to a particular dimension of the state of the world. This can be modeled as follows: \mathcal{S} is an infinite, countable, set of i.i.d. stochastic processes s such that for all $t \geq 1$, s_t takes values in $\{-1, 1\}$, with respective probabilities $(p, 1 - p)$. The precision of a signal s is entirely characterized by the value $d_s = Prob(\tilde{c} = c_H | s = 1)$. Each time Player 1 draws a new signal from \mathcal{S} , its precision d_s is drawn from some constant distribution f over $[0, 1]$.

4.3 Indescribability and costly revelation

This section shows that the class of games described in Section 4.1 satisfies the basic properties of the ambiguity framework.

Lemma 8 (no exit) *Under full information the Pareto frontier is such that:*

1. *either it is reduced to a unique equilibrium for which Player 1 exits with certainty in period $t = 1$*

2. or, Player 1 always chooses to stay following an action of Player 2 that is possible on the equilibrium path.

Lemma 9 (ex ante truthful communication) *Assume that there exists a signal s^* that is a perfect predictor of \tilde{c} — that is, for all $x \in \mathbb{R}$, $D_{s^*}(x) \in \{0, 1\}$ — and that Player 2 can inform Player 1 of this signal at a time $t = 0$, before the game begins. Then the Pareto frontier of the game with private information and ex ante communication is the same as the Pareto frontier of the game with full information.*

In other terms, if Player 2 can costlessly inform Player 1 of which signal is a perfect predictor of c , Player 2 can be induced to reveal her type without constraining continuation values. Lemma 9 assumes that signals are describable. When this is not the case and the precision of signals has to be learned from play, limited empathy will generate inefficiencies. As in the game with ambiguity, the pervasiveness of these inefficiencies depends on how fast misbehavior from Player 2 can be detected.

Definition 3 (slow learning) *For all $n \in \mathbb{N}$, consider $\mathbf{s}^n = (s_1, \dots, s_n)$ a vector of signals of \mathcal{S} drawn using the joint distribution $f_{\mathcal{S}} \times f_{\mathcal{S} \setminus \{s_1\}} \times \dots \times f_{\mathcal{S} \setminus \{s_1, \dots, s_{n-1}\}}$. The family of signals \mathcal{S} exhibits slow learning whenever,*

$$\begin{aligned} \forall n \in \mathbb{N}, \text{ with proba } 1, \quad & \text{Prob}(\tilde{c}_1 = c_H \mid \mathbf{s}^n) > 0 \quad \text{and} \\ & \text{Prob}(\tilde{c}_1 = c_L \mid \mathbf{s}^n) > 0. \end{aligned} \tag{15}$$

When slow learning holds, no matter how much information Player 1 has access to, both values of \tilde{c} are always possible given the signals she observes.

Definition 4 (fast learning) *The family of signals \mathcal{S} exhibits fast learning if and only if*

1. when $\tilde{c}_1 = c_H$, there exists $n \in \mathbb{N}$ such that with strictly positive probability, $\text{Prob}(\tilde{c}_1 = c_L \mid \mathbf{s}^n) = 0$

2. when $\tilde{c}_1 = c_L$, there exists $n \in \mathbb{N}$ such that with strictly positive probability, $\text{Prob}(\tilde{c}_1 = c_H \mid \mathbf{s}^n) = 0$.

Lemma 10 *Whenever \mathcal{S} exhibits fast learning and Player 1 can observe the past history of the signals she pays attention to, there exists $\bar{\beta} < 1$ such that for all $\beta \geq \bar{\beta}$, Player 2 can be induced to cooperate fully at no efficiency cost.*

Lemma 11 (necessary exit) *Whenever \mathcal{S} exhibits slow learning, then, on any equilibrium of game $\Gamma_{\mathcal{S}}$, exit will happen on the equilibrium path with strictly positive probability.*

In view of Lemma 8, this implies that in games with slow learning, limited empathy *always* generates inefficiencies. The intuition for this result is straightforward: when learning is slow, it is possible – though unlikely – that Player 2’s true cost \tilde{c} is equal to c_H for arbitrarily long periods of time, independently of what the signal observed by Player 1 indicates. If Player 1 did not exit no matter the length of these sequences, Player 2 would be tempted to misrepresent her cost and claim her cost is c_H independently of what the true cost is. Hence exit must happen on the equilibrium path with some positive probability.

5 Conclusion.

This paper explored settings in which players agree on the principle of cooperation, but the details of how cooperation should be implemented are ambiguous. It shows that although there is no uncertainty about the sustainability of cooperation under full information, ambiguity about the contingencies in which cooperation should happen can cause inefficient termination on the equilibrium path. These inefficiencies are closely linked to the indescribability of ambiguous states. In particular, if ambiguous states were describable in the players’ common language, Pareto-efficient equilibria would never involve termination on the equilibrium path.

Because ambiguous states are observable, the framework naturally authorizes learning from the other player's actions. When learning takes place, cooperation exhibits interesting dynamics: while learning occurs, the partnership is sensitive to adverse economic shocks, and terminates with positive probability if an ambiguous state with a high cost of cooperation occurs. Once learning is over, however, the players can sustain greater cooperation and the partnership becomes resilient to negative economic shocks. Because inducing information revelation has an efficiency cost, resolving ambiguity can be regarded as an investment in relational capital that comes at the short run cost of potential early termination but yields the future benefit of improved cooperation. When revelation cost are high, it may be optimal for the uninformed player to remain uninformed and have the informed player either under- or over-cooperate.

Finally, the paper shows that resolving ambiguity can be seen as a particular case of an empathy building problem in which players learn how to predict each others' cost. In this richer setup, the basic result that indescribability and lack of empathy hinder cooperation and generate inefficient exit on the equilibrium path still holds. A full-fledged analysis of optimal learning dynamics in this richer setup is beyond the scope of this paper but is an interesting topic for future research.

Appendix A: Extensions

A.1 Repeated game

This section examines the simplifying assumption made in Section 2 that Player 2's decision is an exit decision. The setup used in this section is a variation on that of Section 2: now Player 1's decision to stay or exit is not irreversible any longer: each period, Player 1 decides to participate (P) or not (NP). Player 1's cost of participation is $e > 0$. This section shows that Proposition 3 extends in this repeated game setup. The assumption that $c_H = +\infty$ is maintained. The timing of each period is the following:

1. Player 1 decides to participate or not.
2. An i.i.d. state of the world $w \in \{\underline{w}, \bar{w}, w_a^1, w_a^2\}$ is drawn with probabilities $(\underline{p}, \bar{p}, p_a, p_a)$.
3. Player 2 observes her cost of cooperation $c(w)$. Player 1 observes only w .
4. Player 2 decides whether to cooperate or defect. Final payoffs, are given by

	C	D
P	$(b - e, \pi - c(w))$	$(-e, \pi)$
NP	$(b, -c(w))$	$(0, 0)$.

As in Section 2, states w_a^1 and w_a^2 are ambiguous from the perspective of Player 1.

Consider the following conditions,

$$\underline{p}b < e; \quad \frac{\beta}{1-\beta}(\pi - \underline{p}c_L) > c_L \quad \text{and} \quad \frac{\beta}{1-\beta}(\pi - (p_a + \underline{p})c_L) < c_L \quad (16)$$

Under Assumption 16, it is necessary for Player 1 to cooperate at some frequency at an ambiguous state for Player 1 to be induced to stay. Moreover, Player 2 can be induced to cooperate at an ambiguous state associated with cost c_L in addition to cooperating in state \underline{w} , but not with probability 1.

Lemma 12 (full participation) *Under Assumption 16 equilibria on the Pareto frontier of the repeated game with full information are such that once the Player 2 has cooperated at any state, Player 1 always participates following an action that is possible on the equilibrium path.*

Proposition 7 (sufficient conditions for costly revelation) *Define $\underline{V}_{P_2}^{Liar} = \frac{1}{1-(\bar{p}+p_a)\beta}\pi$. Under Assumption 16, whenever*

$$\beta(V_{P_2}^{Max} - \underline{V}_{P_2}^{Liar}) < c_L \quad (17)$$

then the Pareto frontier of the game with ambiguity lies strictly below the Pareto frontier of the game with full information.

Note that there is a non-empty set of parameter values such that Assumption 16 and inequality (17) are satisfied together.

Proposition 7 shows that ambiguity can generate inefficiencies even though Player 1's action is not an irreversible exit decision. Inefficiencies will occur whenever under efficient strategies, Player 2 can hope to ride the partnership for a long enough time before her deviation is detected.

A.2 Three-states ambiguity

Section 3 considered forms of ambiguity in which only two states were ambiguous. By studying an example with three states ambiguity, this section makes the point that in all generality, there can be different degrees of ambiguity revelation. As a consequence, it is shown that on the Pareto frontier, the players may end up using cooperative agreements of various precisions depending on the particular path of play.

Consider a model with the same payoff structure as that of section 2, but such that there are no unambiguous states and three ambiguous states $\{w_a^1, w_a^2, w_a^3\}$. These three ambiguous states have the same likelihood p_a . Player 1 believes that $(c(w_a^1), c(w_a^2), c(w_a^3))$ is uniformly distributed over $\{(c_L, c_L, c_H), (c_L, c_H, c_L), (c_H, c_L, c_L)\}$. We assume that $c_H = +\infty$.

Since $c_H = +\infty$ and all states are ambiguous, some initial revelation must happen on any equilibrium path for which there is some cooperation. As information is being revealed, there are two levels of ambiguity that can be reached: if the state with cost c_H is first revealed, then there is no more ambiguity about the game; however, if a state with cost c_L is revealed, the two remaining states are ambiguous, and we are exactly in the case studied in Section 3.

In Section 3, we showed that it may be that under ambiguity, the entire Pareto frontier involves under-cooperation on the part of Player 2. In such a case, on the Pareto frontier, depending on the amount of ambiguity that is resolved initially, players will end up using different long run cooperative agreements.

Appendix B: Proofs

Lemma 13 (No Exit) *For any specification of the costs $c(w_a^1)$ and $c(w_a^2)$, under full information, the Pareto frontier of Γ_{FI} is such that:*

- (i) *either it is reduced to a unique equilibrium for which Player 1 exits with certainty in period $t = 1$*
- (ii) *or, Player 1 always chooses to stay following an action of Player 2 that is possible on the equilibrium path.*

Proof: First, if there exist an equilibrium with some cooperation, it must Pareto dominate immediate exit since both players can always guarantee the payoffs of exit.

Now, consider a subgame perfect equilibrium (s_{P_1}, s_{P_2}) , involving some cooperation, and such that there exists a history h , attainable on the equilibrium path, such that $s_{P_1}(h) = E$.

Denote \sqcup the concatenation operator for histories, where $h \sqcup h'$ means history h followed by history h' . We say that h includes h' – denoted $h' \subset h$ – if and only if there exists h'' such that $h = h' \sqcup h''$.

Since $s_W(h) = E$, no history of the form $h' = h \sqcup h''$ where $h'' \neq \{E\}$ is reached under (s_{P_1}, s_{P_2}) . Hence, without breaking incentive compatibility conditions or changing payoffs on the equilibrium path, we can assume that at any history of the form $h' = h \sqcup h''$ where Player 1 makes a decision $s_{P_1}(h') = E$. From here, the proof is by construction: define the pair of strategies \tilde{s}_{P_1} and \tilde{s}_{P_2} as follows:

1. whenever $h' \in \mathcal{H}$ is of the form $h' = h \sqcup h''$, then $\tilde{s}_{P_2}(h') \equiv s_{P_2}(h'')$ and $\tilde{s}_{P_1}(h') \equiv s_W(h'')$.
2. otherwise, when h' isn't of the form $h \sqcup h''$, then $\tilde{s}_{P_2}(h') \equiv s_{P_2}(h')$ and $\tilde{s}_{P_1}(h') \equiv s_{P_1}(h')$.

Let us first show that when they play according to $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ then at any history h' the players have greater value than if they used (s_{P_2}, s_{P_1}) . This is obvious at any history of the form $h' = h \sqcup h''$, since under (s_{P_1}, s_{P_2}) both the worker and the firm would get 0 utility, while under $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$, by construction the firm and the worker are playing a Nash equilibrium which must give them values weakly greater than 0. For any history h' such that $h' \not\subseteq h$ and $h \not\subseteq h'$, the two pairs of strategies (s_{P_1}, s_{P_2}) and $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ prescribe the same behavior at all future histories, and hence they imply the same continuation values. Finally, for any history h' for which there exists $h'' \in \mathcal{H}$ such that $h' \sqcup h'' = h$, the players play the same actions for both pairs of strategies until h is reached and, we have just shown, obtain greater continuation values with $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ once h is reached. It follows that at h' the players must have greater continuation values under $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$.

We now show that the pair $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ forms a subgame perfect equilibrium. At any history of the form $h'' = h \sqcup h'$, by construction, the firm and the worker are playing a Nash equilibrium. The same holds at any history h'' such that $h'' \not\subseteq h$ and $h \not\subseteq h''$. Now consider a history h'' such that $h'' \sqcup h' = h$, with $h' \neq \emptyset$. Note that at h'' , (s_{P_1}, s_{P_2}) and $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ prescribe the same actions. Should the players deviate from their prescribed move, then the resulting history cannot be connected to h and at any history following the deviation, s and \tilde{s} prescribe the same behavior. Hence, should they deviate when playing according to $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$, the two players would get the same values they would have had under (s_{P_1}, s_{P_2}) . Should they follow their prescribed move, we know that the players get greater continuation value under $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ than under (s_{P_2}, s_{P_1}) . Since following their prescribed action was incentive compatible under (s_{P_2}, s_{P_1}) , it must be incentive compatible under $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$. It follows that $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ is indeed a subgame perfect equilibrium.

Since at any history where Player 1 stays, the expected value of Player 2 is weakly greater than $\pi > 0$, equilibrium $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ strictly dominates (s_{P_2}, s_{P_1}) . It follows that on the Pareto frontier, Player 1 must always stay following actions on the equilibrium path, otherwise it is possible to construct a strictly dominating equilibrium. ■

Proof of Proposition 1: Consider the game with full information Γ_{FI} where by convention $c(w_a^1) = c_L$. The Pareto frontier of Γ_{FI} is entirely spanned by equilibria in which Player 1

exits with probability 1 following any action of player 2 that is not on the equilibrium path. Consider such an equilibrium (s_{P_1}, s_{P_2}) of Γ_{FI} .

Consider now the game with ambiguity and ex ante communication at $t = 0$. Player 2 can be induced to send a message of the form “State w_a^1 [is/isn’t] the low cost state” by having Player 1 exit if Player 2 does not send a message at time $t = 0$. A message sent by Player 1 defines a permutation σ over $\{w_a^1, w_a^2\}$ such that $\sigma(w_a^1)$ is the state with cost c_L according to Player 2’s claim. Consider the strategy \tilde{s}_{P_1} of Player 1 defined³ by $\tilde{s}_{P_2} = s_{P_2} \circ \sigma$. When Player 2 tells the truth in period $t = 0$, then $s_{P_1} \circ \sigma$ is a best-reply to $s_{P_2} \circ \sigma$ since (s_{P_1}, s_{P_2}) is an equilibrium of the full-information game Γ_{FI} . Let us now show that when Player 2 uses strategy \tilde{s}_{P_2} in the subgame starting at time $t = 1$, it is incentive compatible for Player 1 to tell the truth at time $t = 0$.

We use some intermediary results. Let us define $z_t = \{w_1, x_1, \dots, w_t, x_t\}$. For any pair of strategies (s_i, s_{-i}) , we will denote by $s_i(z_t)$ the action prescribed by s_i when the state of the world is z_t and players have played according to their strategies in all previous periods. Because (s_{P_1}, s_{P_2}) is on the Pareto frontier of Γ_{FI} , it must be that for any realization of z_t , on the equilibrium path,

$$Prob(a_{t+1} = coop. \ \& \ c(w_{t+1}) = c_H \mid z_t) \leq Prob(a_{t+1} = coop. \ \& \ c(w_{t+1}) = c_L \mid z_t). \quad (18)$$

Indeed, otherwise, it is possible to improve players’ values by shifting cooperation from the high cost state to the low cost state.

Denote by σ^* the permutation defined by truth-telling, σ^\neg the permutation defined by lying and $\sigma^{\leftrightarrow} = \sigma^* \circ \sigma^\neg$. We know that $s_{P_2} \circ \sigma^*$ is a best reply to $s_{P_1} \circ \sigma^*$. Let us now study Player 2’s best reply s^\neg to $s_{P_1} \circ \sigma^\neg$. Under s_{P_1} Player 1 exits whenever Player 2 does not play according to s_{P_2} . This implies that under s^\neg , for a sequence z_t of states of the world, either s^\neg coincides with $s_{P_2} \circ \sigma^\neg$ at all preceding periods or the game has ended because Player 1 has exited. Hence whenever s^\neg deviates from $s_{P_2} \circ \sigma^\neg$, Player 2 gets a value equal to 0. This is weakly less than any value Player 2 obtains when players use $(s_{P_1} \circ \sigma^*, s_{P_2} \circ \sigma^*)$. Let us now compare the utility obtained by Player 2 in states where s^\neg coincides with $s_{P_2} \circ \sigma^\neg$. Define $Z^\neg \equiv \{z_t \mid \forall t' \leq t, s^\neg(z_{t'}) = s_{P_2} \circ \sigma^\neg(z_{t'})\}$. Let us show that there exists a set Z^* of states of the world z_t such that there is an injective mapping $m : Z^\neg \rightarrow Z^*$ and for all $z_t \in Z^\neg$,

$$u_{P_2}(s^\neg, s_{P_1} \circ \sigma^\neg, z_t) \leq u_{P_2}(s_{P_2} \circ \sigma^*, s_{P_1} \circ \sigma^*, m(z_t)) \quad \text{and} \quad Prob(z_t) = Prob \circ m(z_t)$$

where u_{P_2} denotes Player 2’s flow payoffs. The existence of such a mapping m implies that Player 2 obtains greater utility under $(s_{P_1} \circ \sigma^*, s_{P_2} \circ \sigma^*)$ than under $(s_{P_1} \circ \sigma^\neg, s_{P_2} \circ \sigma^\neg)$: at states $z_t \in Z^\neg$ states of equal mass can be found that provide greater flow payoffs; at states $z_t \notin Z^\neg$, Player 2 gets value zero under $(s_{P_1} \circ \sigma^\neg, s_{P_2} \circ \sigma^\neg)$ and value weakly greater than zero under $(s_{P_1} \circ \sigma^*, s_{P_2} \circ \sigma^*)$.

We now show that m does exist. At all these states, Pareto efficiency of (s_{P_1}, s_{P_2}) and Lemma13 imply that Player 1 will stay. The only difference is then the cost of cooperation.

³Where σ also permutes ambiguous states within histories.

For all states $z_t \in Z^\neg$ such that $s^\neg(z_t) = D$, then m is defined by $m(z_t) = \sigma^{\leftrightarrow}(z_t)$. Indeed, if $s_{P_2} \circ \sigma^\neg(z_t) = D$ then by construction $s_{P_2} \circ \sigma^* \circ \sigma^{\leftrightarrow}(z_t) = D$. When $s^\neg(z_t) = C$ and $c(w_t) = c_L$ then equation 18 implies that there exists an injective mapping $(x_t, w_t) \mapsto (x'_t, w'_t)$ such that $c(w'_t) = c_L$ and $s_{P_2}(\{z_{t-1}, x'_t, w'_t\}) = C$. At such a z_t , m is defined by $m(z_t) = \{z_{t-1}, x'_t, w'_t\}$. Finally for any state z_t such that $s^\neg(z_t) = C$ and $c(w_t) = c_H$, any completion of m that makes it a bijection between the remaining states that maintains probability works. ■

Proof of Proposition 2: This is a direct application of Lemma 13. ■

Proof of Lemma 1: We know from Lemma 2 that on the Pareto frontier Player 1 always stays following equilibrium actions. Furthermore, it is clear that Player 2 never cooperates at states with cost $c_H = +\infty$. Hence, transfer of utility on the Pareto frontier is done entirely by having the Player 2 be cooperating more or less frequently in states with cost c_L . Because players have the same discount rate the timing of these transfers does not affect the efficiency of the equilibrium. The timing of transfers however will affect incentive compatibility conditions and we look for the timing that minimizes the worst case temptation to defect.

Consider a history h such that Player 2 has never cooperated when her cost was c_L before. Behavior at h does not affect any preceding or future incentive constraints. Therefore, as long as cooperation at h is incentive compatible, the rate of cooperation at such a history can be freely chosen. Note that at any history following a history at which Player 2 has cooperated when her cost was c_L , Player 2's behavior will affect past incentive compatibility constraints. In this sense, there is something special about the first time Player 2 cooperates with cost c_L : no past promises need to be upheld.

For any Pareto efficient equilibrium (s_{P_1}, s_{P_2}) , denote $V_{P_2}^*$ the continuation value of the Player 2 at any history where she has to cooperate at a state with cost c_L for the first time. Note that in all generality, $V_{P_2}^*$ is a random variable, however, it must satisfy $V_F^* \geq c_L/\beta$. We denote $\mathbf{E}V_F^*$ its expectation, it must satisfy $\mathbf{E}V_F^* \geq c_L/\beta$. We consider stationary strategies in which Player 2 cooperates with probability⁴ r_L^2 every time her cost is c_L and Player 1 enforces this by exiting when Player 2 does not comply. Denote $V_{P_2}^{r_L^2}$ the continuation value of the Player 2 at any of her decision point when she plays such a strategy. Since such strategies continuously span behavior going from “cooperating always” to “cooperating never”, there exists r_L^2 such that $V_{P_2}^{r_L^2} = \mathbf{E}V_F^* \geq c_L/\beta$. By construction, this r_L^2 makes cooperation incentive compatible and the stationary strategy of with cooperation rate r_L^2 is sustainable in equilibrium.

Given this r_L^2 , for all possible $r_L^1 \in [0, 1]$, consider strategies in which Player 2 cooperates with rate r_L^1 at states of cost c_L while she has never cooperated and starts cooperating at rate r_L^2 once she has cooperated. Denote $V_{P_2}^{r_L^1, r_L^2}$ the value Player 2 expects at time $t = 0$ under such a strategy. Let $V_{P_2}(0)$ denote the value expected by Player 2 at $t = 0$ under (s_{P_1}, s_{P_2}) . Clearly there exists r_L^1 such that $V_{P_2}^{r_L^1, r_L^2} = V_{P_2}(0)$. By construction, such a strategy $s_{P_2}^{r_L^1, r_L^2}$

⁴Players make use of the public randomization.

can be sustained in equilibrium. To conclude the proof, note that if $r_L^1 \leq r_L^2$ then one can increase r_L^1 and decrease r_L^2 while keeping the initial value of Player 2 constant and weakly relaxing all incentive compatibility constraints. ■

Proof of Proposition 3: Consider an equilibrium (s_{P_1}, s_{P_2}) on the Pareto frontier of Γ_A . Assume that this equilibrium yields values that are on the Pareto efficient frontier of Γ_{FI} . Then, Lemma 13 implies that under (s_{P_1}, s_{P_2}) , Player 1 never exits following behavior that is consistent with equilibrium behavior under full information for some cost structure c . At a revelation stage, not-cooperating is consistent with both possible cost structures. Hence Player 1 should not exit if Player 2 does not cooperate at the state w_a^i of a revelation stage h_t . For (s_{P_1}, s_{P_2}) to yield values that are one the Pareto frontier of Γ_{FI} the only moments when Player 2 may exit following a revelation stage h_t are at a state w_a^{-i} or \underline{w} where Player 2 does not cooperate. Hence, the value Player 2 can obtain when she does not cooperate at w_a^i even though $c(w_a^i) = c_L$ is minimized when Player 1 demands cooperation the first time w_a^{-i} occurs after h_t and whenever $w_t = \underline{w}$ until w_a^{-i} occurs. Hence, the minimum value $V_{P_2}^{Liar}$ that Player 2 can guarantee when she misrepresents her cost is

$$V_{P_2}^{Liar} \equiv \max \left\{ \frac{1}{1 - (\bar{p} + p_a)\beta} \pi ; \frac{1}{1 - (\underline{p} + \bar{p} + p_a)\beta} (\pi - \underline{p}c_L) \right\}$$

where the value depends on whether it is optimal for Player 2 to cooperate or not when $w_t = \underline{w}$.

The fact that revelation is incentive compatible under (s_{P_1}, s_{P_2}) implies that $\beta(V_{P_2}^{max} - V_{P_2}^{Liar}) \geq c_L$. Hence, if $\beta(V_{P_2}^{max} - V_{P_2}^{Liar}) < c_L$, then (s_{P_1}, s_{P_2}) cannot yield values that are on the Pareto frontier of the game Γ_{FI} . ■

Proof of Lemma 2: The only other circumstance where revelation can occur in equilibrium is when $w_t = \underline{w}$. By cooperating or not cooperating at this state Player 2 can potentially reveal information about ambiguous states. The incentive problem however is exactly the same as at an ambiguous state. Any continuation equilibrium that induces revelation at $w_t = \underline{w}$ would induce revelation at $w_t \in \{w_a^1, w_a^2\}$ and vice-versa. Player 2's temptation is the same in both cases: by misrepresenting her cost structure, Player 2 can save a cost c_L . If revelation is potentially costly, it is weakly Pareto dominant to have revelation occur as late as possible. Hence revelation may as well be delayed to ambiguous states. ■

Proof of Lemma 3: The role of exit is to reduce the continuation value of Player 2 when she misrepresents her cost. If she has misrepresented her cost, Player 2 cannot cooperate at an ambiguous state without revealing she misbehaved. Since it occurs out-of-equilibrium, it is clearly efficient for Player 1 to exit whenever past misbehavior is revealed: it reduces the continuation value of Player 2 when she misbehaves without changing behavior on the equilibrium path.

In these conditions, if she has misrepresented her cost, Player 2 will never choose to cooperate at an ambiguous state. Therefore, behavior that occurs after the Player 2 cooperates

at an ambiguous has no impact on the continuation value of a misbehaving Player 2, and hence behavior after Player 2 cooperates at an ambiguous state should be on the Pareto frontier of the game with full information Γ_{FI} . ■

Proof of Lemma 4: Consider an equilibrium (s_{P_1}, s_{P_2}) of game Γ_A that exhibits exit on the equilibrium path following some inconclusive revelation stage with state w_a^i . The point of Lemma 4 is that forcing Player 2 to cooperate is a more efficient way to reduce Player 2's incentive to misrepresent her cost structure than exit. This will bias values following an inconclusive revelation stage in favor of Player 1 but does not restrict ex-ante values since revelation itself can be delayed in order to transfer utility from Player 1 to Player 2.

Let us first consider the case where $r_{max} < 1$. Denote by h^0 the revelation stage under (s_{P_1}, s_{P_2}) and h^1 the first history following h^0 such that the state w is either \underline{w} or w_a^{-i} . Consider the strategies $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ such that at h^1 , Player 2 cooperates with probability 1 and at all following states $w \in \{\underline{w}, w_a^{-i}\}$ Player 2 cooperates at a rate r_{max} . At h^1 , the continuation value of Player 2 when she has misrepresented her cost is weakly less than her continuation value when her cost was truthfully revealed. Under $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ a truthful Player 2 is weakly indifferent between cooperating or not. Hence it is an equilibrium of the continuation game for a misrepresenting Player 2 not to cooperate at h^1 and hence trigger exit. Hence there need not be any exit on the equilibrium path after h^1 . Using $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ reduces the need for inefficient exit but also transfers utility from Player 2 to Player 1. This transfer can be undone by randomizing the initial revelation stage: with some constant probability $d > 0$ the revelation stage is delayed to the next period where the same ambiguous state occurs.

When $r_{max} = 1$, even when she has misrepresented, Player 2 might want to cooperate at state \underline{w} to keep the partnership going. In those cases we cannot yet rule out exit occurring on the equilibrium path after h^1 . However, some exit is unnecessary if Player 2 does not cooperate with probability one at every state $w \in \{\underline{w}, w_a^i\}$. Indeed, if Player 2 does not cooperate at such a history h , then one can demand Player 2 to cooperate at h . If cooperation is not IC for Player 2 under truthful revelation, one can reduce future exit so that cooperation becomes incentive compatible for the truthful type. This weakly reduces the utility of a misrepresenting Player 2 since, she benefits weakly less from the possible reduction in exit. As in the case where $r_{max} = 1$ this induces a transfer of utility from Player 2 to Player 1 that can be undone by randomizing the initial revelation stage. This concludes the proof. ■

Proof of Proposition 4: This proof builds on the proof of Lemma 4. We can restrict our attention to the class of Pareto efficient equilibria described in Lemma 4 where Player 2 cooperates at the maximum possible rate following inconclusive revelation. Let us first consider the case where $r_{max} < 1$. Denote by h_t the inconclusive revelation stage. We first show that having potential exit occur over h_{t+1} and h_{t+2} does not improve upon exiting only in period h_{t+1} .

Since $r_{max} < 1$, we know there is no need for exit after cooperation occurs. Denote by

$\mathbf{q} = (q_1, q_2^{\bar{w}}, q_2^{w_a^i})$ the staying rates at h_{t+1} and h_{t+2} , depending on what state is realized at h_{t+1} . Given that Player 2 cooperates as much as is incentive compatible, the players' values at h_t are,

$$V_{P_2}^{Liar}(\mathbf{q}) = q_1\pi + q_1(q_2^{\bar{w}}\bar{p} + q_2^{w_a^i}p_a)\frac{\beta(\bar{p} + p_a)}{1 - \beta(\bar{p} + p_a)}\pi \quad (19)$$

$$\begin{aligned} V_{P_2}^{Truth}(\mathbf{q}) &= q_1(\pi + (p_a + \underline{p})(-c_L + \beta V_{P_2}^{rmax})) \quad (20) \\ &+ q_1(q_2^{\bar{w}}\bar{p} + q_2^{w_a^i}p_a)\frac{\beta(\bar{p} + p_a)}{1 - \beta(\bar{p} + p_a)}(\pi + (p_a + \underline{p})(-c_L + \beta V_{P_2}^{rmax})) \end{aligned}$$

$$\begin{aligned} V_{P_1}(\mathbf{q}) &= q_1(-e + (p_a + \underline{p})(b + V_{P_1}^{rmax})) \quad (21) \\ &+ q_1(q_2^{\bar{w}}\bar{p} + q_2^{w_a^i}p_a)\frac{\beta(\bar{p} + p_a)}{1 - \beta(\bar{p} + p_a)}(-e + (p_a + \underline{p})(b + \beta V_{P_1}^{rmax})) \end{aligned}$$

Any change in \mathbf{q} that keeps $V_{P_2}^{Liar}(\mathbf{q})$ constant keeps $V_{P_1}(\mathbf{q})$ and $V_{P_2}^{Truth}(\mathbf{q})$ constant. Hence delaying exit by one period does not provide any efficiency gains.

This implies that there is also no efficiency gain from delaying exit over $T + 1$ periods rather than T and by induction no efficiency gain from delaying exit over $T + 1$ periods rather than 1. Given that delaying exit over an infinite number of periods can be arbitrarily approximated by strategies in which exit can happen over long but finite horizons, delaying exit over an infinite horizon does not provide efficiency gains either. A similar proof holds for the case where $r_{max} = 1$. ■

Proof of Lemma 5: We know that exit need only happen in the period following inconclusive revelation. Let us denote q the probability that Player 1 stays after an inconclusive revelation. When she misrepresents her cost, Player 2 gets a value $V_{P_2}^{Liar}(q) = qV_{P_2}^{Liar}$, where $V_{P_2}^{Liar}$ is defined in Proposition 3. Truthful revelation is incentive compatible if and only if

$$V_{P_2}^{Coop} - V_{P_2}^{Liar}(q) \geq \frac{c_L}{\beta}$$

which yields $q \leq \min \left\{ 1, \frac{\beta V_{P_2}^{Coop} - c_L}{\beta V_{P_2}^{Liar}} \right\}$. Pareto efficient equilibria will use the greatest such q , which yields the result. ■

Proof of Proposition 5: Consider a Pareto efficient equilibrium with revelation and denote $V_{P_1}^{Rev}$ the value expected by Player 1 at some revelation stage with ambiguous state w_a^{-i} . The maximum value Player 1 can get in an under-cooperating equilibrium is $V_{P_1}^{u, acc} = \frac{\beta}{1-\beta}(-e + \underline{p}b)$. Under-cooperation will be optimal whenever $V_{P_1}^{Rev} \leq \frac{\beta}{1-\beta}(-e + \underline{p}b)$, under-cooperation will be optimal. Using Proposition 4 and Lemma 5, we obtain that

$$V_{P_1}^{Rev} = \frac{1}{2}(b + \beta V_{P_1}^{Coop}) + \frac{1}{2}\beta q(V_{P_2}^{Coop})V_{P_1}^{1, rmax}$$

where $V_{P_1}^{Coop}$ is the value Player 1 would get should Player 2 cooperate, $q(V_{P_2}^{Coop})$ is equal to $\frac{\beta V_{P_2}^{Coop} - c_L}{\beta V_{P_2}^{Liar}}$, and $V_{P_1}^{1,rmax}$ is the greatest value Player 1 can obtain. Because $V_{P_1}^{Coop}$ is linear in $V_{P_2}^{Coop}$, $V_{P_1}^{Rev}$ is also linear in $V_{P_2}^{Coop}$. Hence, $V_{P_1}^{Rev}$ is maximized either for $V_{P_2}^{Coop} = V_{P_2}^{rmin}$ or $V_{P_2}^{rmax}$. We have

$$V_{P_1}^{Rev}(V_{P_2}^{rmax}) \leq \frac{1}{2} \left(b + \frac{\beta}{1-\beta} (-e + b(p_a + \underline{p})) \right) \quad (22)$$

$$V_{P_1}^{Rev}(V_{P_2}^{rmin}) \leq \frac{\beta}{2} \left(b + \frac{\beta}{1-\beta} (-e + b(p_a + \underline{p})) \right). \quad (23)$$

Inequalities (22), (23) and simple algebra yield that whenever $b + \frac{\beta}{1-\beta} p_a b \leq \frac{\beta}{1-\beta} (-e + \underline{p}b)$, under-cooperation is optimal. ■

Proof of Lemma 6: This is a direct application of Lemma 13. ■

Proof of Lemma 7: Because at \bar{w} , $c(\bar{w}) = +\infty$, Player 2 never cooperates at state \bar{w} on the Pareto frontier of Γ_{FI} . Hence if some revealing equilibrium of Γ_A is on the Pareto frontier of Γ_{FI} , then Player 2 must never cooperate at \bar{w} . Hence under such an equilibrium, Player 2 can guarantee a value greater than $\underline{V}_{P_2}^{Liar}$ even when she misrepresents her cost. The incentive compatibility of revelation implies that

$$\beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar}) \geq c_L.$$

Hence whenever $\beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar}) < c_L$, revelation must imply some inefficiency. ■

Proof of Proposition 6: Consider a possible revelation stage and fix Player 1's conditional value V_{P_1} . There exists some non-revealing strategy of Player 2 that delivers V_{P_1} and gives Player 2 a value of the form $V_{P_2}^{No Rev} = \frac{1}{1-\beta} (\pi - \underline{p} - p_a r (c_L + c_H))$, with $r \leq 1$. The question is whether there exists an equilibrium with revelation that gives Player 2 a greater value $V_{P_2}^{Rev}$ while keeping Player 1's value constant.

When $\beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar}) < c_L$ there must be inefficiency on the equilibrium path to induce revelation. Whether this takes the form of exit or cooperation at what Player 1 presumes is the high cost state. inefficient behavior is more costly to Player 2 when she has behaved truthfully than when she has misrepresented her cost. Hence, if $V_{P_2}^{FI}$ is the value that Player 2 could expect under full information, we have

$$V_{P_2}^{Rev} \leq V_{P_2}^{FI} - \left(\frac{c_L}{\beta} + \underline{V}_{P_2}^{Liar} - V_{P_2}^{max} \right). \quad (24)$$

Since

$$V_{P_2}^{No Rev} \geq V_{P_2}^{FI} - \frac{1}{1-\beta} p_a (c_H - c_L) \quad (25)$$

we get by combining inequalities (24) and (25) that whenever $\frac{\beta}{1-\beta}p_a(c_H - c_L) < c_L - \beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar})$, over-cooperation dominates revelation. ■

Proof of Lemma 8: The proof is identical to that of Lemma 13. ■

Proof of Lemma 9: Given that signals are unbiased, proof similar to that of Proposition 1 holds. Player 1 simply takes the information given by Player 2 seriously. From an ex ante perspective, Player 2 does not benefit from Player 1's confusion. ■

Proof of Lemma 10: Under fast learning when Player 2 does not cooperate at a state where her cost is c_L , Player 1 may learn it with certainty in finite time by sampling n consecutive signals. Hence, when β is large enough, there is an equilibrium in which Player 1 entirely trusts the declarations of Player 2 but exits if she learns that Player 2 has misbehaved. ■

Proof of Lemma 11: Under slow learning, it is possible that $\tilde{c} = c_H$ for arbitrarily long sequences independently of the signals that Player 1 observes. If there was no exit on the equilibrium path, then Player 1 would tolerate arbitrarily long sequences in which Player 2 does not cooperate. In that setting, it would not be incentive compatible for Player 2 to cooperate. ■

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