

A Continuous-Time Principal-Agent Model with Privately Observable Cash Flows. **Preliminary and Incomplete**

Peter M. DeMarzo and Yuliy Sannikov

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Abstract

We consider a principal-agent model, in which the agent needs to borrow from the principal to finance a project. Our model is based on DeMarzo and Fishman (2003), except that the agent's cash flows are given by a Brownian motion with drift in continuous time. The difficulty in writing an appropriate financial contract in this setting lies in the fact that the agent can hide cash flows and not pay back the principal. To enforce payments, the principal has the ability to terminate the agent's project. We design a new way to analyze the problem of private information in continuous time. Using techniques from stochastic calculus, we characterize the optimal contract by a differential equation. We show that this contract is equivalent to the limiting case of a discrete time model with binomial cash flows. The optimal contract can be interpreted as a combination of equity, a credit line, and either long-term debt or a compensating balance requirement (i.e., a cash position). The project is terminated if the agent defaults on the debt or exceeds the limit on the credit line. Once the credit line is paid off, excess cash flows are used to pay dividends. The agent is compensated with equity alone. Unlike the discrete time setting, our differential equation for the continuous time model allows us to compute contracts easily, as well as compute comparative statics. Currently, we are working on a number of extensions to the basic model.

1 Introduction.

In this paper, we consider a dynamic contracting environment in which a risk-neutral agent/entrepreneur with limited resources manages an investment activity. While the investment is profitable, it is also risky, and in the short-run can generate large losses. The agent will need outside financial support to cover these losses and continue the project. The difficulty is that while the distribution of the cash flows is publicly known, only the

agent observes the realization of the cash flows. Therefore, from the perspective of the principal or investors funding the project, there is the concern that the agent may divert the cash flows and overstate losses or understate profits. The only means for investors to provide incentives to the agent is to withdraw their financial support for the project and force its early termination. We seek to characterize an optimal contract in this framework.

A discrete-time model of this sort is considered by DeMarzo and Fishman (2003), hereafter denoted DF. Here we extend their analysis to a continuous-time setting in which the cumulative cash flows generated by the investment follow a Brownian motion with positive drift. This has two advantages. First, in DF, the project cash flows are bounded below. With Brownian motion, however, the losses on the project over any interval of time can be arbitrarily large. An optimal contract must specify the level of losses that investors will tolerate before terminating their support. The second advantage of a continuous-time model is that the optimal contracts and payoffs can be characterized in terms of an ordinary differential equation, making the solution and comparative statics simpler to quantify. It also makes the model easier to calibrate and embed within other standard finance models.

Another important contribution of the paper is methodological. We solve the model in two ways. First, we represent the project cash flows as a discrete-time binomial tree. The agency problem is that the agent may report low cash flows when they are really high. Given the discrete-time setting, we can apply the results of DF to describe the solution. We show that the limit of this solution as the time increments vanish leads to an ordinary differential equation characterizing equilibrium payoffs.¹ Second, we formulate the model directly in continuous time. Here, we solve the model by first reformulating the agency problem. Rather than stealing the cash flows directly, we suppose that the agent can reduce the drift of the cash flows for a private gain (as in a standard "costly effort" principal-agent model). Then, using techniques similar to those introduced by Sannikov (2003), we again characterize the solution in terms of an ordinary differential equation.

In the discrete-time setting, DF demonstrate that the optimal contract can be implemented using a combination of standard securities: equity, long-term debt, and a credit line. Dividends are paid when cash flows exceed long-term debt payments and the credit line is paid off. If long-term debt payments are not made or the credit line is overdrawn, the project is terminated with a probability that depends on the size of the cash shortfall.

In the setting of this paper, a similar implementation results, with some distinctions. First, termination is no longer stochastic, but occurs the moment the credit line is overdrawn or there is a default on the long-term debt. Another distinction is that, due to the fact that the project can generate large short-term losses, risky projects will not use long-term debt but instead require a compensating balance with the credit line. (A compensating balance is a cash deposit that the firm must hold with the lender to maintain the credit line.) The compensating balance serves two roles. First, it allows for a larger credit line, which is valuable given the risk of the project. Second, it provides for an inflow

¹While using a binary-tree to approximate Brownian motion is a natural idea in finance, it need not work in the context of an agency problem. See, for example, Hellwig and Schmidt (2002) for a discussion of the difficulties of this approach in the Holmstrom-Milgrom (1987) continuous-time principal-agent model.

of interest payments to the project that can be used to somewhat offset operating losses. While investors could provide this cash inflow contractually, because the agent "owns" the compensating balance it provides the inflow in a way that maintains the investors' participation constraint without violating limited liability.

After characterizing the implementation of the optimal contract, we compute a number of comparative statics as well as determine the dynamics of security prices. In both cases, our differential equation characterization proves very useful for the analysis.

1.1 Related Literature.

Our paper is part of a growing literature on dynamic optimal contracting models using recursive techniques that began with Green (1987), Spear and Srivastava (1987), Phelan and Townsend (1991), and Atkeson (1991) among others. (See, for example, the text by Ljungqvist and Sargent (2000) for a description of many of these models.)

Surprisingly, few of these models have been formulated in continuous time. While discrete time models are adequate conceptually, in many cases a continuous-time setting may prove to be analytically more convenient. An important example of this is the principal-agent model of Holmstrom and Milgrom (1987), hereafter HM, in which the optimal continuous-time contract is shown to be linear. Schattler and Sung (1993) develop a more general mathematical framework for analyzing agency problems of this sort in continuous time, and Sung (1995) allows the agent to control volatility as well. Hellwig and Schmidt (2002) look at the conditions for a discrete-time principal-agent model to converge to the HM solution. See also Bolton and Harris (2001), Ou-yang (2003), Detemple, Govindaraj and Loewenstein (2001), Cadenillas, Cvitannic and Zapatero (2003) for further generalization and analysis of the HM setting.

Several features distinguish our model from the HM problem: the investor's ability to terminate the project, the agent's consumption while the project is running, and the nature of the agency problem. In HM, the agent runs the project until date T , and then receives compensation. In our model, the agent receives compensation many times while the project is running, until the contract calls for the agent's termination. Also, HM analyze a setting in which the agent takes hidden actions. In our setting the agent observes private payoff-relevant information. The termination decision is a key feature of the optimal contract in our setting. Here, as in DF, we demonstrate how this decision can be implemented through bankruptcy.²

Sannikov (2003) and Williams (2004) analyze principal-agent models, in which the principal and the agent interact dynamically. Their interaction is characterized by evolving state variables. In their models, the agent continuously chooses actions (e.g. hidden effort) that are not directly observable to the principal, and the principal takes actions (e.g. payments to the agent) that affect the agent's payoff. Besides having a dynamic nature in the spirit of Sannikov (2003) and Williams (2004), our paper develops a new method to

²Spear and Wang (2002) also analyze the decision of when to fire an agent. They do not consider the implementation of the decision through standard securities.

deal with the problem of private observations in continuous time. A key idea is to limit the number of ways in which the agent can respond to what he observes, and view the agent's response to his observations as a hidden action. After the principal creates incentives for a right action among the limited set of actions, we need to verify that these incentives are valid among all ways that the agent can respond to his observations.

This paper is organized as follows. Section 2 presents a discrete-time model with binary cash flows, summarizes the optimal contract that was found by DF, and derives the form of the contract in the limit as cash flows arrive more frequently. Section 3 presents a continuous-time model, in which cash flows arrive via a Brownian motion with a positive drift. Section 4 formally derives the optimal contract in the continuous-time setting, which coincides with the contract in the limit of the discrete-time settings. Section 5 discusses the implementation of the optimal contract in terms of familiar securities: credit line, debt and equity. Section 6 focuses on comparative statics results. The introduction to Section 6 intuitively discusses our results. The remainder of Section 6 is technical: it describes how comparative statics can be found, and derives many such results.

2 The Discrete Time Model.

There is an agent and investors. Investors are risk neutral, have unlimited capital, and value a cash flow stream $\{dC_t\}$ as $E \sum_t e^{-rt} dC_t$, where r is the riskless interest rate. The agent is also risk neutral, has limited capital, and values a cash flow stream $\{dC_t\}$ as $E \sum_t e^{-\gamma t} dC_t$, where $\gamma \geq r$ is the subjective discount rate.

The agent has a risky project that requires capital K . The agent has initial wealth $Y_0 \geq 0$. If $K > Y_0$, the agent must borrow to finance the project. Alternatively, even if $Y_0 \geq K$, if $\gamma > r$ the agent would like to borrow for consumption purposes. If the project is funded, it produces cash flows at interval dt . The cash flow at date t is given by the random variable dY_t . We assume that the cash flows $\{dY_t\}$ are i.i.d. with distribution

$$dY_t = \begin{cases} y_1 = \mu dt + \sigma \sqrt{\frac{1-q}{q}} dt & \text{with probability } q \\ y_0 = \mu dt - \sigma \sqrt{\frac{q}{1-q}} dt & \text{with probability } 1 - q \end{cases}$$

That is, each cash flow has a Bernoulli distribution with mean μdt and variance $\sigma^2 dt$. Note that y_0 may be negative. In this case the firm must have cash, or established credit, of at least $-y_0$ at the start of each period for the project to continue.

At the end of each period, the project may be terminated. If it is terminated, the agent receives a reservation payoff $R \geq 0$, and the assets of the firm can be liquidated for $L \leq K$. We also assume that the investment is efficient, so that $Kr + R\gamma < \mu$, and therefore that termination is inefficient. We assume that cash flows up to y_0 are observable and collectible by investors, but investors do not know whether y_0 or y_1 has occurred. Specifically, the agent privately observes the realization of dY_t . Investors must rely on the agent to report this realization. Of course, the agent may lie about the cash flow in order to cheat investors.

If the cash flow in period t is y_1 , the agent may conceal the excess $y_1 - y_0$. The excess may be stored within the firm or diverted for the agent's own consumption. Any cash concealed within the firm grows at rate $\rho \leq r$. If the agent diverts cash from the firm for his own consumption, he can consume a fraction $\lambda \in [0, 1]$ of the cash and the remainder is lost. When $\lambda = 1$, diversion is costless, which is the typical assumption in these models. When $\lambda = 0$, diversion is not possible and the agency problem disappears. For our analysis we will consider $\lambda > 0$.

In contrast to the operating cash flows, liquidation of the assets is observable and contractible. In particular, the division of the proceeds L can be contractually specified. This modeling reflects the idea that the agent can divert the profits but not the assets.

Suppose investors fund the agent. Investors do not observe the actual cash flows or their diversion, and do not observe the agent's consumption or any savings. Investors only observe the agent's payments and reports. A contract therefore specifies payments made from investors to the agent as a function only of messages sent and past payments made by the agent to investors. The contract can also specify circumstances under which control of the project passes from the agent to investors, who may then terminate the project. It is this threat of termination of the project that induces the agent to pay investors some share of the cash flows. Finally, we assume the contract signed at date 0 remains in force for the life of the project. That is, the agent and investors can commit not to renegotiate.

2.1 The Optimal Contract.

DeMarzo and Fishman (2003), or DF, describe a recursive method for determining the optimal contract in a general discrete time setting that includes the model described above. The optimal contract depends only on the promised payoff to the agent at any date t , W_t . That is, W_t is a sufficient statistic for the history of the interaction.

There are three regions that govern the behavior of the optimal contract, determined by a liquidation boundary $W^L \geq R$ and a dividend boundary $W^1 \geq W^L$. For $W_t < W^L$, the project is (stochastically) terminated. For $W_t > W^1$, the agent receives compensation $W_t - W^1$ in the form of cash dividends. For $W \in [W^L, W^1]$, all cash flows are paid to investors, and the agent is rewarded through the promise of future payoffs only. Since the agent is only compensated through the future payoff, the expected future payoff must be higher than the agent's current payoff according to the agent's discount rate, γ . In addition, in order to maintain incentive compatibility, the agent's payoff must increase by λ for each dollar paid to investors. Thus, the agent's promised payoff evolves according to

$$W_{t+dt} = e^{\gamma dt} W_t + \lambda(dY_t - \mu dt) \tag{1}$$

The investor's future payoff can then be given in terms of the agent's through the continuation function,

$$b(W) = \text{maximal investor payoff given agent earns payoff } W \geq R.$$

DF demonstrate that this continuation function is concave and, in the region $W \in [W^L, W^1]$, satisfies

$$b(W_t) = e^{-rdt} (\mu dt + E [b(W_{t+dt})]) \quad (2)$$

The intuition for equation (2) is straightforward: the investors' current payoff is the present value (at discount rate r) of this period's expected cash flow plus their expected future payoff. The future payoff is also described by the continuation function, evaluated at the agent's future payoff.

DF show that the dividend boundary W^1 is determined by the lowest payoff for the agent such that $b(W^1) = -1$. That is, for $W < W^1$, $b(W) \geq -1$, so that it is cheaper to compensate the agent using future promises than with cash. On the other hand, to provide the agent with payoff $W > W^1$, it is optimal to give the agent an immediate cash transfer from the investors of $W - W^1$. That is, in the dividend region, $W \geq W^1$, the continuation function is linear with $b'(W) = -1$. Since immediate compensation is better than deferred compensation, this implies that in this region,

$$b(W_t) \geq e^{-rdt} (\mu dt + E [b(W_{t+dt})]) \quad (3)$$

Finally, in the liquidation region $W \leq W^L$, the project is terminated with probability $(W^L - W)/(W^L - R)$, or continued with the agent receiving payoff W^L . Thus, the continuation function is linear with slope $b'(W) = l = \frac{b(W^L) - L}{W^L - R}$, and again equation (3) holds.

2.2 The Continuous Limit.

Using the characterization of the optimal contract in discrete time, we now decrease the length of the period to determine the characterization for the continuous-time limit. As $dt \rightarrow 0$, the binomial-tree process for the cash flows converges to Brownian motion with mean μ and volatility σ . In this case, we will show that equation (2) becomes a second-order differential equation for the optimal continuation function (see (4)). The liquidation and dividend boundaries determine the boundary conditions for this differential equation. Liquidation occurs if and only if the agent's payoff equals the outside option R , so that the liquidation boundary becomes $b(R) = L$. The dividend boundary is determined by an implicit condition (see (5)).

Proposition 1. *Let b be the limit of the optimal continuation function as $dt \rightarrow 0$. Then b is concave and twice continuously differentiable. The liquidation boundary $W^L \rightarrow R$ and $b(R) = L$. In the region $W \in [R, W^1]$,*

$$rb(W) = \mu + \gamma W b'(W) + \frac{1}{2} \lambda^2 \sigma^2 b''(W) \quad (4)$$

and $dW = \gamma W dt + \lambda(dY_t - \mu dt)$. Finally, $b'(W^1) = -1$ and

$$rb(W^1) = \mu - \gamma W^1 \quad (5)$$

Proof. First, dW follows immediately from (1). For b , the limit of concave functions is concave. Since the agent's future continuation payoff is noisy ($\lambda^2\sigma^2 > 0$), b' must be continuous since otherwise there would be no way to achieve the payoff at a "kink." First we show that (4) holds in the region $W \in [W^L, W^1]$. Using Taylor expansions and ignoring terms that are $o(dt)$, we can rewrite (2) as follows:

$$b(W) = (1 - r dt) \left(\mu dt + E \left[b(W) + b'(W) dW + \frac{1}{2} b''(W) dW^2 \right] \right).$$

Since $E[dW] = \gamma W dt$ and $E[dW^2] = \lambda^2\sigma^2 dt + o(dt)$,

$$b(W) = (1 - r dt) \left(\mu dt + b(W) + b'(W)\gamma W dt + \frac{1}{2} b''(W)\lambda^2\sigma^2 dt \right),$$

which reduces to (4) on elimination of dt^2 terms and dividing by dt . Note that (4) also implies that b'' is continuous on (W^L, W^1) .

Next we verify (5). For $W > W^1$, we can use (3) and Taylor expansions to derive

$$\begin{aligned} b(W) &\geq (1 - r dt) \left(\mu dt + b(W) + b'(W)\gamma W dt + \frac{1}{2} b''(W)\lambda^2\sigma^2 dt \right) \\ &= (1 - r dt) (\mu dt + b(W) + b'(W)\gamma W dt), \end{aligned}$$

where we use the fact that b is linear in this region. Collecting terms and dividing by dt yields

$$rb(W) - \gamma W b'(W) \geq \mu \tag{6}$$

On the other hand, for $W \leq W^1$, from (4),

$$rb(W) - \gamma W b'(W) = \mu + \frac{1}{2} \lambda^2 \sigma^2 b''(W) \leq \mu \tag{7}$$

Since $b'(W^1) = -1$ by definition, together these imply $b''(W^1) = 0$, or equivalently (5).

Finally, we verify the liquidation boundary $W^L = R$. If $R \leq W < W^L$, (6) also holds. We show this leads to a contradiction, and therefore that $W^L = R$.

Let $l = b'(W^L)$. Then (6) becomes $l \leq \frac{rb(W) - \mu}{\gamma W}$. Letting $W = R$ and using the fact that liquidation is inefficient so that $rb(R) + \gamma R = rL + \gamma R < \mu$, this becomes

$$l \leq \frac{rL - \mu}{\gamma R} < \frac{-\gamma R}{\gamma R} = -1.$$

Since $l \geq -1$, this is a contradiction. \square

The intuition for (5) is as follows: To receive b , investors must earn total return rb . They earn this return by receiving the expected cash flow μ , less the cost of paying the agent his required return, $\gamma W b'$, less the incentive cost associated with the agent's risk, $\frac{1}{2} \lambda^2 \sigma^2 b''$. The boundary condition (6) follows from (5) plus the smoothness of b . Finally, note that unlike

in the discrete-time model of DF, in continuous-time termination is no longer stochastic. Stochastic termination is required in discrete-time since in order to maintain incentives, the agent's promised payoff can be too close to R . This is not the case in continuous time, since only infinitesimal incentives are required over each instant of time, and thus incentives can be maintained until the agent's payoff drops to R .

3 The Continuous Time Model.

In this section we develop a continuous-time formulation of the contracting problem. There are two important reasons for this. First, the discrete-time method of the previous section relies on the considerable machinery developed in DeMarzo and Fishman (2003). Here we propose a methodology that can be used directly to analyze the continuous-time model, which will prove useful in extensions of this analysis. Second, there is the possibility that the continuous-time setting may introduce superior contracting possibilities not available in discrete time. In fact, we will show that this is not the case, and that the limiting characterization of the optimal contract described in section 2 is also the solution to the continuous-time problem.

In the continuous-time model, the agent manages a project that generates a stochastic stream of cash flows, given by

$$dY_t = \mu dt + \sigma dZ_t,$$

where Z is a standard Brownian motion on a probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ with an augmented filtration \mathcal{F}_t , $0 \leq t \leq \infty$ generated by the Brownian motion. The cash flows Y from the project are observable only by the agent and not the principal. The agent makes a report $\{\hat{Y}_t; t \geq 0\}$ of the realized cash flows to the principal. The principal does not know whether the agent is lying or telling the truth. The principal extracts the reported cash flows from the agent and gives him back a transfer of D_t that is based on the agent's report. The transfer process D_t is a nondecreasing and \hat{Y} -measurable. If the agent underreports realized cash flows, he steals the difference. Stealing is costly: the agent is able to enjoy only a fraction $\lambda \in (0, 1]$ of what he steals. Also, the agent can overreport and put his own money back into the project. As a result, the agent receives a flow of income of

$$dI_t = [dY_t - d\hat{Y}_t]^\lambda + dD_t, \quad \text{where} \quad [dY_t - d\hat{Y}_t]^\lambda = \lambda(dY_t - d\hat{Y}_t)^+ - (dY_t - d\hat{Y}_t)^- \quad (8)$$

To make sure that the agent does not receive income of minus infinity, we assume that process $\hat{Y}_t - Y_t$ has to have bounded variation.³

³Note that (8) implies that the agent receives only λ for each dollar diverted from the firm, but must contribute one dollar to increase cash flows by a dollar. While this is natural, it is not necessary. At the cost of a realistic interpretation, we could allow the agent to contribute only λ to increase cash flows by one dollar; the analysis would not change.

The agent chooses a nondecreasing consumption process $C = \{C_t; t \leq \tau\}$. He also maintains a private savings account, from which he consumes and into which he deposits his income. The principal cannot observe the balance of the agent's savings account. The agent's balance S_t grows at interest rate ρ :

$$dS_t = \rho S_t dt + dI_t - dC_t. \quad (9)$$

The agent must maintain a nonnegative balance on his account.

The agent is risk-neutral and discounts his consumption at rate $\gamma > \rho$. This continues until a termination time τ that is contractually specified by the principal. Once the contract is terminated, the agent receives payoff R from an outside option available for him. Assume that $R \geq 0$. Therefore, the agent's total expected payoff is given by⁴

$$W_0 = E \left[\int_0^\tau e^{-\gamma t} dC_t + e^{-\gamma \tau} R \right]. \quad (10)$$

The principal discounts cash flows at rate r , such that $\gamma > r \geq \rho$. Once the contract is terminated, he receives expected liquidation payoff L . The principal's total profit is

$$b = E \left[\int_0^\tau e^{-rt} (d\hat{Y}_t - dD_t) + e^{-r\tau} L \right].$$

The project requires an investment of $K \geq L$ in order to be started, and the agent has initial wealth Y_0 . The principal specifies a contract before date 0. A contract (τ, D) specifies a termination time τ and a payments $\{D_t; 0 \leq t \leq \tau\}$ that are based on reports \hat{Y} . Formally, D is a \hat{Y} -measurable continuous process, and τ is a \hat{Y} -measurable stopping time.

In response to a contract (τ, D) , the agent will choose a strategy. A feasible strategy is a pair of processes (C, \hat{Y}_t) adapted to Y , such that

- (i) process C_t is nondecreasing and
- (ii) the savings process, defined by (9), stays nonnegative

The agent's strategy (C, \hat{Y}) is *incentive-compatible*, if it maximizes his total expected payoff W_0 given a contract (τ, D) . We wish to find a contract (τ, D) and an incentive-compatible response strategy (C, \hat{Y}) that maximize the principal's profit subject to delivering to the agent a required payoff \hat{W} . We can use a solution to this problem to analyze two settings: when the investor acts as a monopolist and when investors act competitively.

Remark. We do not explicitly model the agent's option to quit. Even if the agent had this option, he would choose not to exercise it. Indeed, rather than quit, it is strictly better for the agent to steal at a sufficiently high rate and wait for the principal to terminate the contract.

⁴At time τ , the agent can enjoy his remaining savings by consuming it immediately.

4 Solving the Continuous-Time Model.

We solve the problem of finding an optimal contract in several steps. First, we will show that it is sufficient to look for an optimal contract within a smaller class of contracts, namely contracts in which the agent chooses to report cash flows truthfully and maintain zero savings. Next, when we look for an optimal contract within this smaller class of contracts, we solve a relaxed optimization problem, in which we consider the constraint that the agent reports cash flows truthfully, but ignore the constraint that the agent must choose to maintain zero savings. We find an optimal contract subject to these relaxed incentive constraints. Finally, we show that under this contract happens to satisfy all incentive constraints.

Proposition 2. *There is an optimal contract in which the agent chooses to tell the truth, and maintain zero savings.*

Proof. See Appendix.

4.1 Formulation with Restrictions on the Agent's Strategies.

To form a relaxed problem, we will add several restrictions on the agent's strategies. First, assume that the report process \hat{Y} must have the form

$$d\hat{Y}_t = dY_t - A_t dt, \quad A_t \in \{0, 1\}.$$

Second, assume that the agent must maintain zero balance in his savings account at all times until his employment is terminated. Therefore, the agent's consumption process satisfies

$$dC_t = dI_t = dD_t + \lambda A_t dt.$$

The problem is to find an optimal contract which induces the agent to reveal cash flows truthfully and which maximizes the principal's total payoff. This can be interpreted as a principal-agent problem with a hidden action: the action of the agent reduces the mean of cash flows that the principal receives, and gives the agent a flow of utility of λ .

Define the value of transfers and termination utility that the agent receives if he tells the truth as

$$V_t = E \left[\int_0^\tau e^{-\gamma s} dD_s + e^{-\gamma \tau} R \mid \mathcal{F}_t \right].$$

Let us investigate how the agent's choice of A_t affects V_t . First, we need a representation

Lemma 1. *There is a Y -measurable process $\{\beta_t; 0 \leq t \leq \tau\}$ such that*

$$V_t = V_0 + \int_0^t e^{-rs} \beta_s \underbrace{\frac{dY_s - \mu ds}{\sigma}}_{dZ_s} \quad (11)$$

Proof. Process V_t is a martingale. By the Martingale Representation Theorem, there is a process σ such that (11) holds. \square

When the agent reports Y , then

$$V_\tau = \int_0^\tau e^{-\gamma t} dD_t + e^{-\gamma\tau} R = V_0 + \int_0^\tau e^{-rt} \beta_t \frac{dY_t - \mu dt}{\sigma}.$$

is the agent's value of transfers plus the termination value. Because D and τ depend exclusively on the agent's report, when the agent reports \hat{Y} then he gets the utility of

$$W_0 = E \left[\underbrace{V_0 + \int_0^\tau e^{-rt} \beta_t \frac{d\hat{Y}_t - \mu dt}{\sigma}}_{\text{termination and transfers}} + \underbrace{\int_0^\tau e^{-rt} \lambda A_t dt}_{\text{stealing}} \right].$$

We are ready to formulate our incentive compatibility condition.

Proposition 3. *Truthtelling ($A_t = 0$ for all t) is incentive-compatible if and only if $\beta_t \geq \lambda\sigma$.*

Proof. Suppose $\beta_t \geq \lambda\sigma$ for all t . Then for an arbitrary reporting strategy A the agent's value is

$$E \left[V_0 + \int_0^\tau e^{-rt} \frac{\beta_t}{\sigma} \underbrace{(dY_t - A_t dt - \mu dt)}_{d\hat{Y}_t} + \lambda A_t dt \right] = V_0 + E \left[\int_0^\tau e^{-rt} A_t \left(\lambda - \frac{\beta_t}{\sigma} \right) dt \right] \leq V_0,$$

where V_0 is the value that the agent receives if he tells the truth.

On the other hand, suppose $\beta_t < \lambda\sigma$ on a set of positive measure. Define

$$A_t = \begin{cases} 1 & \text{if } \beta_t < \lambda\sigma \\ 0 & \text{if } \beta_t \geq \lambda\sigma \end{cases}$$

Then for this reporting strategy the agent's value is

$$V_0 + E \left[\int_0^\tau e^{-rt} A_t \left(\lambda - \frac{\beta_t}{\sigma} \right) dt \right] \geq V_0,$$

so truthtelling is not optimal. \square

In order to construct an optimal contract under the restricted formulation and prove that there is no contract which does better, it will be convenient to work with the agent's continuation value

$$W_t = E \left[\int_t^\tau e^{-\gamma(s-t)} dD_s + e^{-\gamma(\tau-t)} R \mid \mathcal{F}_t \right].$$

Then W_t evolves according to equation

$$\int_0^t e^{-\gamma s} dD_s + e^{-\gamma t} W_t = V_t(A) \quad \Rightarrow \quad dW_t = \gamma W_t dt - dD_t + \beta_t dZ_t \quad (12)$$

Proof. First, let us show that $b''(W^1) = 0$. Assume that W^1 is the smallest value, for which $\hat{b}'(W^1) = -1$. Then $\hat{b}(W) > -1$ for $W < W^1$, so $\hat{b}''(W^1) \leq 0$. If $\hat{b}''(W^1) < 0$, then by increasing the slope $\hat{b}'(R)$ one would still obtain a solution that achieves slope -1 at a point near W^1 . This cannot happen, so $\hat{b}''(W^1) = 0$, and b is twice continuously differentiable.

From equation (13)

$$rb(W^1) = \mu - b'(W^1)$$

For $W > W^1$, (15) also holds because

$$rb(W) - \mu - b'(W)\gamma W = \mu - \gamma W^1 - r(W - W^1) - \mu + \gamma W = (\gamma - r)(W - W^1) \geq 0.$$

Lastly, \hat{b} is concave because for all $W < W^1$, $\hat{b}'(W) > -1$, so $\hat{b}(W) < (\mu - \gamma W)/r$ and

$$\hat{b}''(W) = \frac{2}{\lambda^2 \sigma^2} (r\hat{b}(W) - \mu - \hat{b}'(W)\gamma W) < \frac{2}{\lambda^2 \sigma^2} (\mu - \gamma W - \mu + \gamma W) = 0.$$

□

We are ready to construct an optimal contract and show that function b gives the principal's maximal profit. Before we go on, we need an upper bound on the principal's profit

Observation. Consider any contract. Since D is an increasing process and $\mu \geq 0$,

$$E \left[\int_t^\tau e^{-r(s-t)} (dY_s - dD_s) \mid \mathcal{F}_t \right] \leq \frac{\mu}{r} - E \left[\int_t^\tau e^{-\gamma(s-t)} dD_s + e^{-\gamma\tau} R \right] + R = \frac{\mu}{r} - W_t + R. \quad (16)$$

Proposition 4. *A contract that maximizes the principal's profit and delivers to the agent value $\hat{W} \in [R, W^1]$ takes the following form: W_t evolves as*

$$W_0 = \hat{W}, \quad dW_t = \gamma W_t dt - dD_t + \lambda (d\hat{Y}_t - \mu dt) \quad (17)$$

in the interval $[R, W^1]$. When $W_t \in [R, W^1)$, D_t stays constant. When $W_t = W^1$, variable D increases appropriately so that W_t is reflected at W^1 . The contract is terminated at the first time τ when W_t hits R .

Proof. Under an arbitrary incentive-compatible contract, W_t evolves as

$$dW_t = \gamma W_t dt - dD_t + \beta_t dZ_t \quad (18)$$

Define

$$G_t = \int_0^t e^{-rs} (dY_s - dD_s) + e^{-rt} b(W_t).$$

Then using (18) and Ito's lemma

$$dG_t = e^{-rt} \left(\underbrace{\left(\frac{\beta_t^2}{2} b''(W) - (rb(W_t) - \mu - b'(W_t)\gamma W_t) \right)}_{\leq 0} dt - \underbrace{(1 + b'(W_t)) dD_t}_{\leq 0} + (1 + \beta_t b'(W_t)) dZ_t \right).$$

From (13), (15) and the fact that $b'(W_t) \geq -1$, that G_t is a supermartingale. It is a martingale if and only if $\beta = \lambda\sigma$, $W_t \leq W^1$ and D_t is increasing only when $W_t = W^1$.

Let us evaluate the principal's profit for a arbitrary incentive-compatible contract. Note that $W_\tau = R$ and $b(W_\tau) = L$. For all $t < \infty$, using (16)

$$\begin{aligned} E \left[\int_0^\tau e^{-rs} (dY_s - dD_s) + e^{-r\tau} L \right] &= E \left[G_{t \wedge \tau} + 1_{t \leq \tau} \left(\int_t^\tau e^{-rs} (dY_s - dD_s) + e^{-r\tau} L - e^{-rt} b(W_t) \right) \right] \\ &\leq b(W_0) + E \left[1_{t \leq \tau} e^{-rt} \underbrace{\left(\frac{\mu}{r} + R - W_t + e^{-r(\tau-t)} L - b(W_t) \right)}_{\text{bounded}} \right] \rightarrow b(W_0) \end{aligned}$$

as $t \rightarrow \infty$.

For a contract that satisfies the conditions of Proposition 3, G is a martingale until time τ because $b(W_t)$ stays bounded. Therefore,

$$E \left[\int_0^\tau e^{-rs} (dY_s - dD_s) + e^{-r\tau} L \right] = E [G_\tau] = b(W_0).$$

□

4.3 Removing the Restrictions on the Agent's Strategies.

Now it is our task to show that the contract proposed under Proposition 4 is fully incentive compatible under our original model, without any restrictions on the agent's strategies.

Proposition 5. *Consider any nondecreasing process D_t adapted to \hat{Y} , and suppose W_t solves*

$$W_0 = \hat{W}, \quad dW_t = \gamma W_t dt - dD_t + \lambda (d\hat{Y}_t - \mu dt) \quad (19)$$

until a stopping time $\tau = \min\{t | W_t = R\}$. Suppose that the process W_t is uniformly bounded for all paths of \hat{Y} . Then the agent can get a payoff at most W_0 from any feasible strategy in response to a contract (τ, D) . Moreover, if the agent reports truthfully and maintains zero savings then his payoff is exactly W_0 .

Proof. Let us show that under an arbitrary feasible strategy (C, \hat{Y}) of the agent,

$$V_t = \int_0^t e^{-\gamma s} dC_s + e^{-\gamma t} S_t + e^{-\gamma t} W_t$$

is a supermartingale for $t \leq \tau$. Using (9) and (19) we get

$$\begin{aligned} dV_t &= e^{-\gamma t} (dC_t - \gamma S_t dt + dS_t - \gamma W_t dt + dW_t) = \\ e^{-\gamma t} &\left(dC_t - \gamma S_t dt + \rho S_t dt + [dY_t - d\hat{Y}_t]^{\lambda} + dD_t - dC_t - \gamma W_t dt + \gamma W_t dt - dD_t + \lambda (d\hat{Y}_t - \mu dt) \right) = \\ &e^{-\gamma t} \left((\rho - \gamma) S_t dt + \lambda dZ_t - (1 - \lambda)(dY_t - d\hat{Y}_t)^{-} \right) \end{aligned} \quad (20)$$

Because $\gamma > \rho$, $S_t \geq 0$, and $(dY_t - d\hat{Y}_t)^{-}$ is nondecreasing, V must be a supermartingale. If $S_t = 0$ and $\hat{Y} = Y$ for all t , then V is a martingale. Thus,

$$W_0 = V_0 \geq E[V_{\tau}] = E \left[\int_0^{\tau} e^{-\gamma t} dC_t + e^{-\gamma \tau} S_{\tau} + e^{-\gamma \tau} R \right].$$

We conclude that the agent cannot get a payoff higher than W_0 from any feasible strategy. If the agent maintains zero savings and reports truthfully, then his payoff is exactly W_0 . \square

We conclude that the contract suggested in Proposition 4 is incentive compatible even when the agent can tell arbitrary lies and have hidden savings. From (20) we see that the agent is indifferent between following the truthtelling strategy with zero savings and underreporting cash flows, but he strictly prefers not to overreport cash flows and not to save.

Our continuous-time contract coincides with the optimal contract in the limit of discrete-time settings as cash flows arrive more frequently. The clean link between discrete and continuous time adds to our sense of security about this contract. The differential equation, which characterizes the optimal contract, is useful in many ways. First, it facilitates the computation of an optimal contract. Second, it allows us to derive many properties of an optimal contract analytically. Third, provides intuition about how the interaction between the principal and the agent evolves over time.

5 Implementation.

DF show that the optimal discrete-time contract can be implemented using standard securities: equity, long-term debt, and a credit line. Specifically, the firm has long-term debt with coupons $x dt$ due each period, a credit line with a fixed credit limit and interest rate γ , and the agent receives the fraction λ of the firm's equity. The agent uses cash flows to pay the debt coupons and credit line first. Once the credit line is fully repaid, the remaining cash flow is paid out as dividend. If the credit limit is exceeded, stochastic termination results.

For a stationary discrete-time setting, DF show that

$$Credit = \frac{W^1 - W^L}{\lambda} \quad \text{and} \quad x dt = \mu dt + \frac{(1 - e^{\gamma dt})W^1}{\lambda} \quad (21)$$

Letting $dt \rightarrow 0$, these expressions become

$$Credit = \frac{W^1 - R}{\lambda} \quad \text{and} \quad x = \mu - \frac{\gamma W^1}{\lambda} \quad (22)$$

We will establish that this implementation also holds in continuous-time, and offer some new interpretations.

First, we establish incentive compatibility of this class of contracts. Rather than work in terms of the debt coupon x , we normalize the coupon rate of the debt to be r and define the face value of the debt to be

$$Debt = \frac{x}{r}$$

Proposition 6. *Suppose the firm has debt and credit such that*

$$r Debt + \gamma Credit = \mu - \gamma R/\lambda \quad (23)$$

and the agent holds fraction λ of the equity. Termination occurs if the agent defaults on the debt or exhausts the credit line.

Then it is incentive compatible for the agent to use the project cash flows to pay the debt coupons and the credit line before issuing dividends, and to refrain from stealing. Moreover, the agent's continuation payoff at any point is determined by the current draw M_t on the credit line:

$$W_t = R + \lambda(Credit - M_t) \quad (24)$$

The proposition describes a class of incentive compatible contracts, parameterized by the level of debt versus the credit line that is used. Under this contract, the balance on the credit line evolves according to

$$dM_t = \gamma M_t dt + r Debt dt + dDividend_t - dY_t \quad (25)$$

and the agent consumes $dD_t = \lambda dDividend_t$.

Proposition 7. *If Debt and Credit are chosen as in (22), then we implement the optimal contract of Section 4.*

We now consider the market values of these securities. The debt is normally risky, since the debt holders may not obtain the entire face value of debt in the case of termination. Assuming that the debt is senior to the credit line, the debtholders get $\min(L, Debt)$ upon termination. As a function of the balance M on the credit line, the market value of debt is

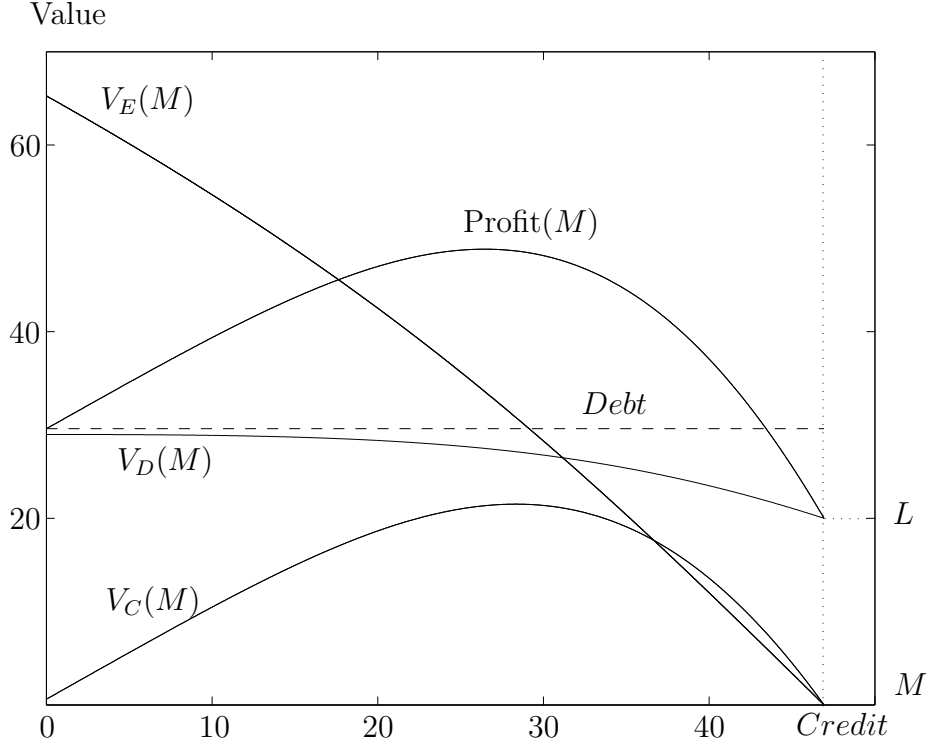


Figure 2: Values of securities for $r = .1$, $\gamma = .15$, $\mu = 10$, $\sigma = 10$, $L = 20$, $R = 0$ and $\lambda = 1$.

$$V_D(M) = E \left[\int_0^\tau e^{-rt} x dt + e^{-r\tau} \min(L, Debt) \mid M \right].$$

Until termination, the equity holders get dividends $\frac{dD_t}{\lambda}$, of which the agent owns fraction λ . At termination, the outside equity holders receive the remaining part of liquidation value $\max(0, L - Debt - Credit)$ after debt and credit line have been paid off. Lemma 10 in the Appendix shows that $Debt + Credit \geq L$ when there are no outside equity holders (i.e. $\lambda = 1$). The value of equity (per share) to outside equity holders is

$$V_E(M) = E \left[\int_0^\tau e^{-rt} \frac{dD_t}{\lambda} + e^{-r\tau} \frac{\max(0, L - Debt - Credit)}{1 - \lambda} \mid M \right].$$

The market value of credit line is

$$V_C(M) = E \left[\int_0^\tau e^{-rt} \left(d\hat{Y}_t - x dt - \frac{dD_t}{\lambda} \right) + e^{-r\tau} \min(Credit, L - \min(Debt, L)) \mid M \right].$$

Then the principal's profit is expressed as

$$\text{Profit}(M) = V_C(M) + V_D(M) + V_E(M)(1 - \lambda).$$

The Appendix shows how to compute these values as a function of the credit line M . Figure 2 shows a computed example, in which debt is risky, i.e. $Debt > L$. In this case, when a termination occurs, the liquidation payoff goes exclusively to the debtholders. When the balance on the credit line is 0, the principal's profit equals to the face value of debt. The market value of debt is decreasing towards L as the balance on the credit line increases towards the credit limit. The value of equity is concave and decreasing in M . This implies that the volatility of equity is increasing in M . It is interesting to note that the value of risky debt can be mimicked by a stationary portfolio of equity, credit and riskless debt:

$$V_D(M) = L + \frac{x - rL}{\mu - x}(V_C(M) + V_E(M)).$$

6 Comparative Statics.

The optimality equation

$$b''(W) = \frac{2}{\lambda^2 \sigma^2} (rb(W) - \mu - b'(W)\gamma W). \quad (26)$$

gives the principal's profit as a function of the agent's value W . The problem of finding an optimal contract is defined by the following parameters: liquidation payoff L , agent's outside value R , the mean and variance of cash flows μ and σ , the principal's and the agent's discount rates r and γ , and the fraction λ of diverted cash flows that the agent can consume. In our comparative statics analysis we assume that the agent has no initial capital. A solution of the optimality equation with boundary conditions $b(R) = L$, $b(W^1) = \frac{\mu - \gamma W}{r}$ and $b'(W^1) = -1$ gives the maximal profit as a function of the agent's value and defines an optimal contract.

We consider two main settings: a setting where the principal is a monopolist and chooses a contract that maximizes his profit, and a setting where the agent receives competitive offers from many investors, so the agent maximizes his value and the investors break even. In the second setting, assume that the initial investment required to start the project is $K = L$. A solution of the optimality equation with boundary conditions $b(R) = L$, $b(W^1) = \frac{\mu - \gamma W}{r}$ and $b'(W^1) = -1$ defines an optimal contract. The principal's profit and the agent's starting value W_0 , depending on whether the principal is a monopolist or acts competitively, are illustrated in Figure 3.

Let us discuss comparative statics results, which we prove later in this section. The defining features of an optimal contract are credit line $\frac{W^1 - R}{\lambda}$, and long-term debt $\frac{\mu}{r} - \frac{\gamma W^1}{\lambda r}$; their optimal choice does not depend on whether the principal acts as a monopolist or competitively. Both the credit line and debt are used to finance initial investment L . An optimal credit line results from the following trade-off: a long credit line delays the agent's consumption, but also gives more flexibility to delay termination. The debt complements the credit line to provide the agent with initial funding. Also, coupon payments on debt can be viewed as a way to extract cash flows from the agent.

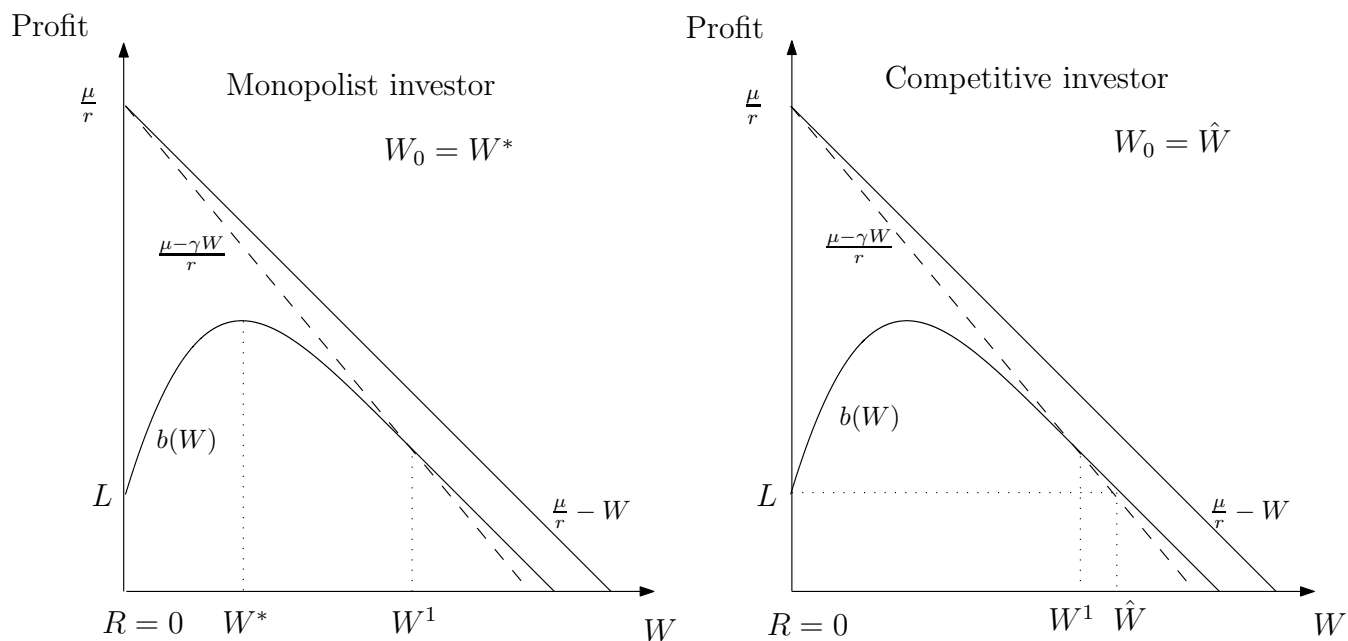


Figure 3: The Principal's Profit.

First, let us discuss our results about the credit line. If the agent's outside option R or the liquidation value L increases, the credit line shortens because termination becomes less costly. If the mean of cash flows μ increases, the credit line increases to delay termination. The risk of termination exists due to the variance of cash flows σ . The optimal credit line is increasing in σ . If the agent's discount rate γ increases, delayed consumption becomes more costly, so the credit line should decrease. If λ decreases, the credit line increases: it becomes easier to provide incentives and the project becomes more profitable; to delay termination, the credit line should be longer (measured in terms of standard deviations of the cash flows). Figures 4, 5 and 6 illustrate the effect of various parameters of the model on debt and credit line. In these Figures, we vary one of the parameters, while keeping the rest of the parameters at "base values:" $L = R = 0$, $\mu = 10$, $\sigma = 10$, $r = .1$, $\gamma = .11$ and $\lambda = 1$.

Let us discuss the face value of the long term debt $\frac{\mu}{r} - \frac{\gamma W^1}{\lambda r}$. Intuitively, payments on long-term debt allow the principal to extract profit from the agent, but higher payments necessitate termination sooner. If the agent's outside option increases, the payments on debt decrease to make the contract more attractive for the agent. If the liquidation value decreases, payments on debt decrease to avoid early termination. If the mean of cash flows μ increases, the effect on debt payments is ambiguous. Debt payments could increase to extract higher expected cash flows from the agent, or decrease if the credit line increases sufficiently. If σ increases, debt payments decrease and the credit line increases to give the agent more flexibility. If γ increases, the size of debt could increase due to the shortening

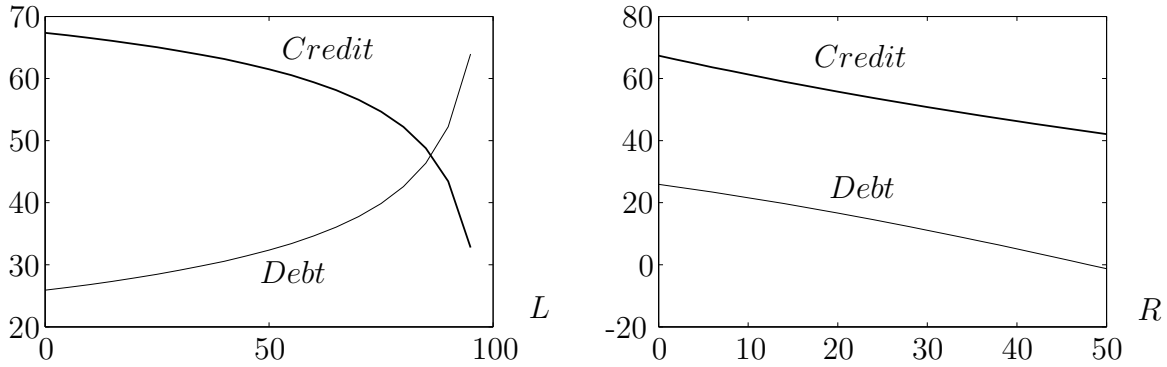


Figure 4: Credit line and debt for various values of L and R .

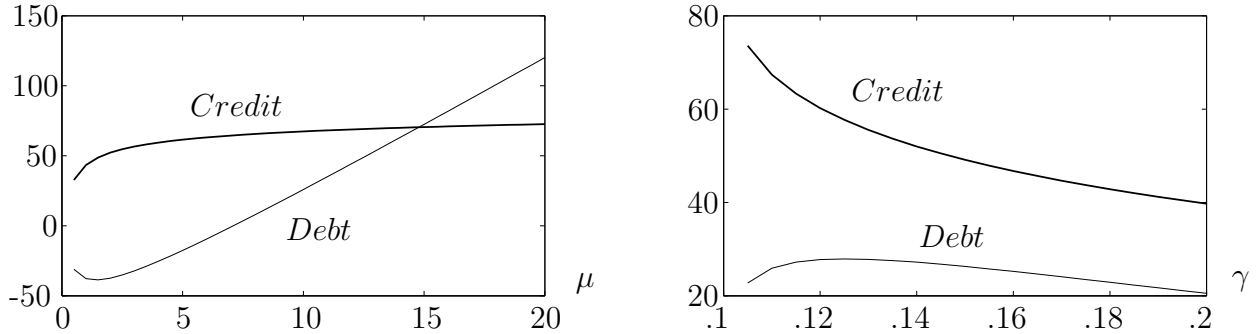


Figure 5: Credit line and debt for various values of μ and γ .

of the credit line, or decrease if the project becomes so unprofitable that it is more difficult to extract payments from the agent. If λ increases, the project becomes more valuable, so debt payments increase to extract more profit from the agent.

6.1 Techniques to derive comparative statics.

The optimality equation for our continuous-time model makes it possible to derive comparative statics results analytically. Our techniques involve comparing solution of the optimality equation drawn through the phase space for various values of parameters.

First, let us simplify the optimality equation. Note that $b(W) = F(W) + \mu/r$, where F solves equation

$$\frac{\lambda^2 \sigma^2}{2} F''(W) = rF(W) - \gamma W F'(W) \quad (27)$$

with boundary conditions $F(R) = L - \frac{\mu}{r}$, $F'(W^1) = -1$ and $rF(W^1) = -\gamma W^1$. We focus on equation (27) as if $\mu = 0$. To find an optimal contract when $\mu > 0$, we only need to

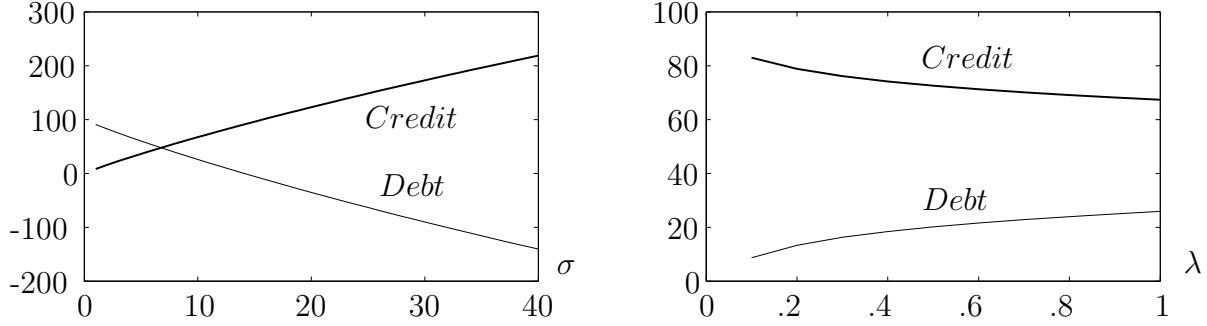


Figure 6: How credit line and debt change in σ and λ . (Credit line may increase in λ if $R > 0$.)

translate F up by μ/r . We are interested in solutions of (27) in region

$$\Theta = \left\{ (W, F) \mid W > 0, F < -\frac{\gamma W}{r} \right\}.$$

The following definition gives an important basis for comparing solutions of (27) drawn through phase space Θ .

Where do these properties of the phase diagram come from? They can be derived by comparing various solutions of (26) and their translates drawn through the phase space. The following definition gives an important basis for comparing solutions.

Definition. A twice continuously differentiable curve $\tilde{b} : [W_L, W_H] \rightarrow \Re$ in region Θ is not sufficiently concave if

$$\forall W \in (W_L, W_H), \quad \tilde{F}''(W) > \frac{2}{\lambda^2 \sigma^2} \left(r \tilde{F}(W) - \gamma W \tilde{F}'(W) \right).$$

If the inequality is reversed, it is too concave.

The following Lemma allows us to compare curves.

Lemma 3. (Comparison of Curves). Consider a twice continuously differentiable curve $\tilde{F} : [W_L, W_H] \rightarrow \Re$ in region Θ , which is not sufficiently concave. If $F : [W_L, W_H] \rightarrow \Re$ solves the optimality equation from initial conditions $F(W_H) \leq \tilde{F}(W_H)$ and $F'(W_H) \geq \tilde{F}'(W_H)$, then for $W \in [W_L, W_H]$

$$F'(W) - \tilde{F}'(W) \quad \text{and} \quad \tilde{F}(W) - F(W) \tag{28}$$

are strictly decreasing in W , as illustrated in Figure 7.

Proof. Since \tilde{F} is not sufficiently concave, property (28) holds locally for $W \in (W_H - \epsilon, W_H]$. If (28) does not hold for all $W \in [W_L, W_H]$, let V^* be the largest value for which it fails. Since property (28) holds for all $W \in (V^*, W_H]$ we must have

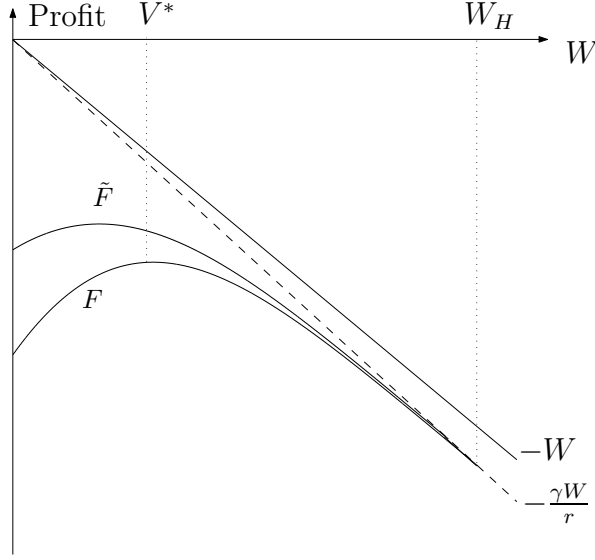


Figure 7: \tilde{F} is not sufficiently concave; F solves (27).

$$F'(V^*) - \tilde{F}'(V^*) > F'(W_H) - \tilde{F}'(W_H) \geq 0 \quad \text{and} \quad \tilde{F}(V^*) - F(V^*) > \tilde{F}(W_H) - F(W_H) \geq 0.$$

If (28) fails at V^* , it must be because $F''(V^*) = \tilde{F}''(V^*)$. However, note that

$$F''(V^*) = \frac{2}{\sigma^2} (rF(V^*) - \gamma V^* F'(V^*)) < \frac{2}{\sigma^2} (r\tilde{F}(V^*) - \gamma V^* \tilde{F}'(V^*)) \leq \tilde{F}''(V^*),$$

contradiction. □

6.2 Proving Comparative Statics Results.

In this subsection we prove results about the effects of the parameters of our model on the characteristics of an optimal contract.

Lemma 4. *For fixed values of σ , r and γ , the phase diagram of solutions of equation (27) that satisfy boundary conditions $rF(W^1) = -\gamma W^1$ and $F'(W^1) = -1$ looks as shown in Figure 8. If F_1 and F_2 are two solutions with right endpoints $W_1^1 < W_2^1$ respectively, then*

$$F_1(W) - F_2(W + (W_2^1 - W_1^1)) \quad \text{and} \quad F_2(W) - F_1(W + (W_2^1 - W_1^1)) \quad (29)$$

are decreasing in W . Also, $W^1 - W^$ decreases as W^1 increases.*

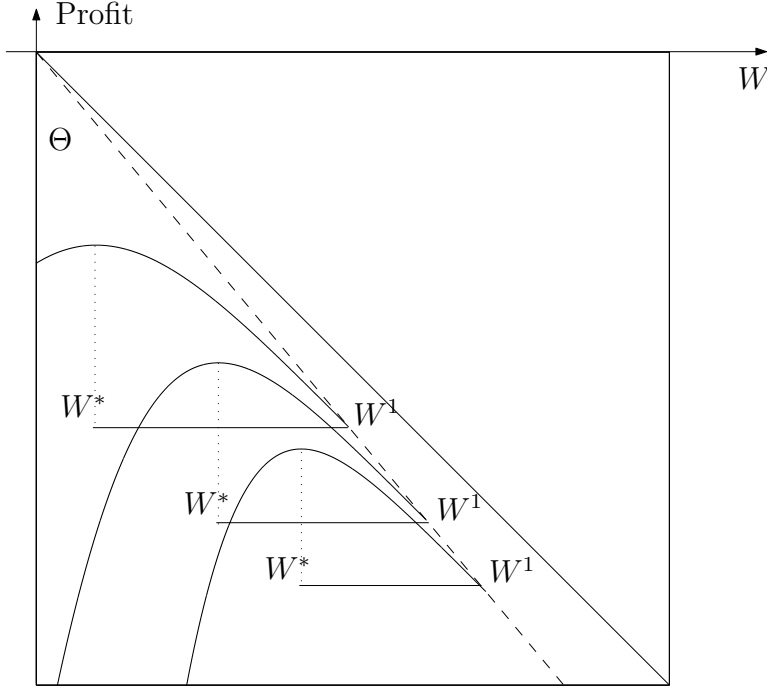


Figure 8: A Phase Diagram.

Proof. Consider two solutions F_1 and F_2 with right endpoints W_1^1 and W_2^1 respectively, as shown in Figure 9. Consider curve \tilde{F} , a translate of F_1 obtained by carrying point $(W_1^1, F_1(W_1^1))$ into point $(W_2^1, F_2(W_2^1))$. Formally,

$$\tilde{F}(W + (W_2^1 - W_1^1)) = F(W) - \frac{\gamma}{r}(W_2^1 - W_1^1).$$

Then for $W \in (R, W_1^1)$

$$\begin{aligned} r\tilde{F}(W + (W_2^1 - W_2^1)) - \gamma(W + (W_2^1 - W_2^1))\tilde{F}'(W + (W_2^1 - W_2^1)) = \\ F(W) - \gamma W F'(W) - \underbrace{\gamma(W_2^1 - W_2^1)(1 + F'(W))}_{>0} < F''(W) = \tilde{F}''(W). \end{aligned}$$

Curve \tilde{F} is not sufficiently concave. Therefore, Lemma 3 allows us to compare \tilde{F} and F_2 and derive (29). We have $0 = F_1'(W_1^*) < F_2'(W_1^* + (W_2^1 - W_1^1))$ at point W_1^* where F_1 is maximized. If W_2^* is the point where F_2 is maximized, then

$$W_2^* > W_1^* + (W_2^1 - W_1^1) \quad \Rightarrow \quad W_1^1 - W_1^* > W_2^1 - W_2^*.$$

□

Figure 8 shows a phase diagram for a fixed set of values of σ , r and γ . From this diagram, we can deduce how various features of an optimal contract depend on L , μ and R . The

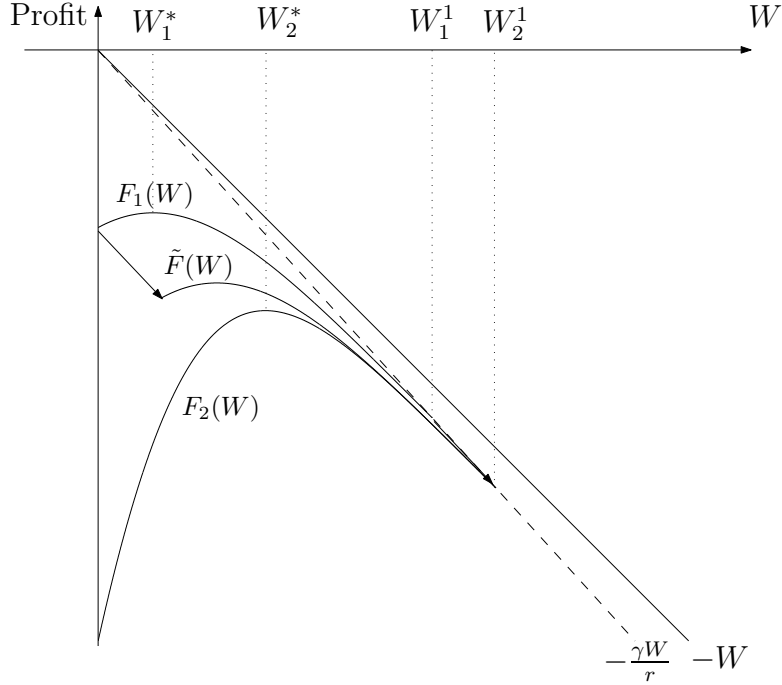


Figure 9: Comparison of solutions F_1 and F_2 , Lemma 4.

phase diagram is a family of curves. This family can be conveniently parameterized by point W^1 , at which the agent has zero balance in his credit line. Lower liquidation values L correspond to higher credit line $\frac{W^1 - R}{\lambda}$, lower principal's profit and higher agent's starting value W^* (in the case when the principal is a monopolist). When the principal's liquidation value is lower, the agent starts less deeply in the credit line ($W^1 - W^*$ is decreasing in W^1). This is intuitive, because for lower L longer credit line serves to avoid termination, not to delay the agent's consumption. Debt $\frac{\mu}{r} - \frac{\gamma W^1}{\lambda r}$ decreases when the principal's liquidation value L decreases.

From by comparing (27) and (13) we know that a decrease in L has the same effect on W^1 and W^* as an increase in μ . Therefore, if the mean of cash flows μ increases, then credit line $\frac{W^1 - R}{\lambda}$ increases, the agent's starting value W^* increases, and $W^1 - W^*$ decreases. Of course, when μ increases, the principal's profit also increases. The effect of an increase of μ on debt $\frac{\mu}{r} - \frac{\gamma W^1}{\lambda r}$ is ambiguous, since W^1 is increasing in μ .

Lemma 5. *Suppose $F'(W) > 0$. If the agent's outside option R increases, then the credit line $\frac{W^1 - R}{\lambda}$ decreases, the agent's starting value W^* increases, and the agent starts less deeply in the credit line.*

Proof. The conclusion follows from Lemma 4. When R increases, clearly W^1 also increases, so W^* increases and $W^1 - W^*$ decreases. Also, if F_1 and F_2 are solutions for two different outside options $R_1 < R_2$, then

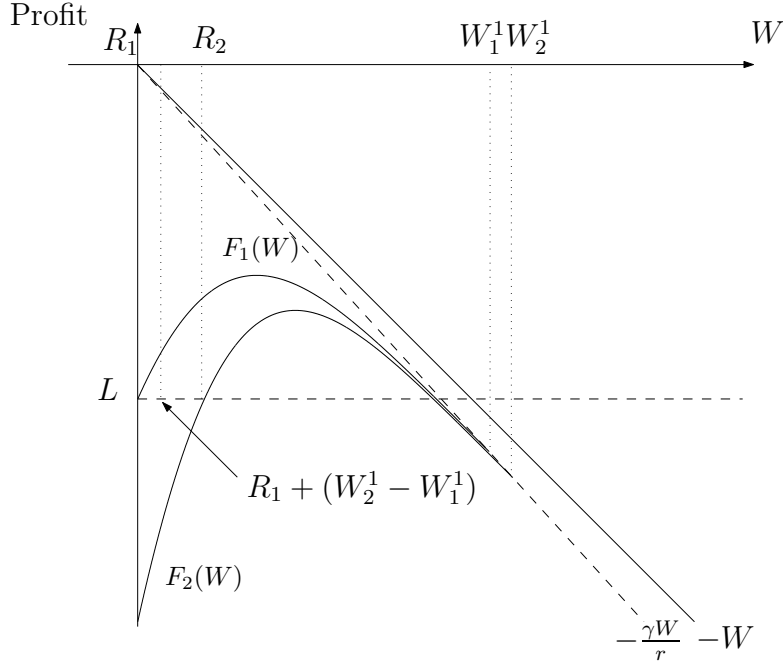


Figure 10: Comparison of solutions F_1 and F_2 , Lemma 5.

$$F_1(R_1) - F_2(R_1 + (W_2^1 - W_1^1)) > F_1(W_1^1) - F_2(W_2^1) = \frac{\gamma(W_2^1 - W_1^1)}{r} \Rightarrow$$

$$F_2(R_1 + (W_2^1 - W_1^1)) < L - \frac{\gamma(W_2^1 - W_1^1)}{r} < L,$$

as shown in Figure 10. Therefore, $R_2 > R_1 + (W_2^1 - W_1^1)$, so $W_1^1 - R_1 > W_2^1 - R_2$. \square

Lemma 6. *If λ or σ increases by a factor of $a > 1$, then W^1 increases by a factor of less than a and $W^1 - W^*$ increases by a factor of more than a .*

Proof. An increase of λ or σ has the same effect on equation (27), so let us discuss a change in σ . Consider profit functions F and H for liquidation value L and volatilities $\sigma' = a\sigma$ and σ respectively. Let W_F^1 and W_H^1 be the right endpoints of F and H . Because the principal's profit decrease in σ , $F(W) > H(W)$ for all $W > R$, so $W_F^1 < W_H^1$, as shown in Figure 11. To show that $W_H^1 < aW_F^1$, note that $G(W) = aF(W/a)$ solves equation

$$\frac{a^2 \lambda^2 \sigma^2}{2} G''(W) = rG(W) - \gamma W G'(W),$$

so it represents the principal's profit for volatility of cash flows $\sigma' = a\sigma$ and liquidation value L' lower than L (Figure 11 shows the case when $R = 0$.) Therefore, the right endpoint of H

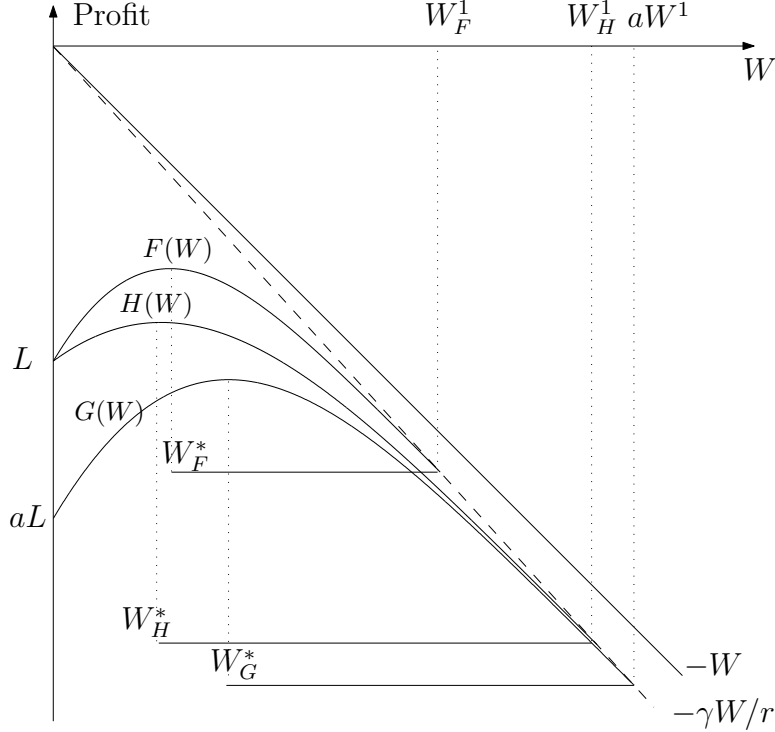


Figure 11: Proof of Lemma 6.

is smaller than the right endpoint of G , which is aW_F^1 . Finally, let us show that $W^* - W^1$ increases by a factor of more than a as σ increases to $a\sigma$. This follows because

$$W_H^* - W_H^1 > W_G^* - W_G^1 = a(W_F^* - W_F^1),$$

where the inequality follows from Lemma 4, and W_H^* , W_G^* and W_F^* maximize H , G and F respectively. \square

Using Lemma 6, we conclude that when σ increases, the credit line $\frac{W^1 - R}{\lambda}$ increases, and the agent starts significantly more deeply in the credit line. The long-term debt $\frac{\mu}{r} - \frac{\gamma W^1}{r\lambda}$ decreases. The effect of an increase in σ on the agent's starting value is ambiguous: W^* could increase to delay termination, or decrease if the profit drops significantly due to higher noise. From computation, W^* is U-shaped in σ .

The effect of changes in λ is more complicated. When λ increases by a factor of a , then W^1 increases by a factor of less than a , so long-term debt $\frac{\mu}{r} - \frac{\gamma W^1}{r\lambda}$ increases. The credit line decreases in λ if $R = 0$, but may increase if $R > 0$.

Lemma 7. *If γ increases, then the optimal credit line decreases.*

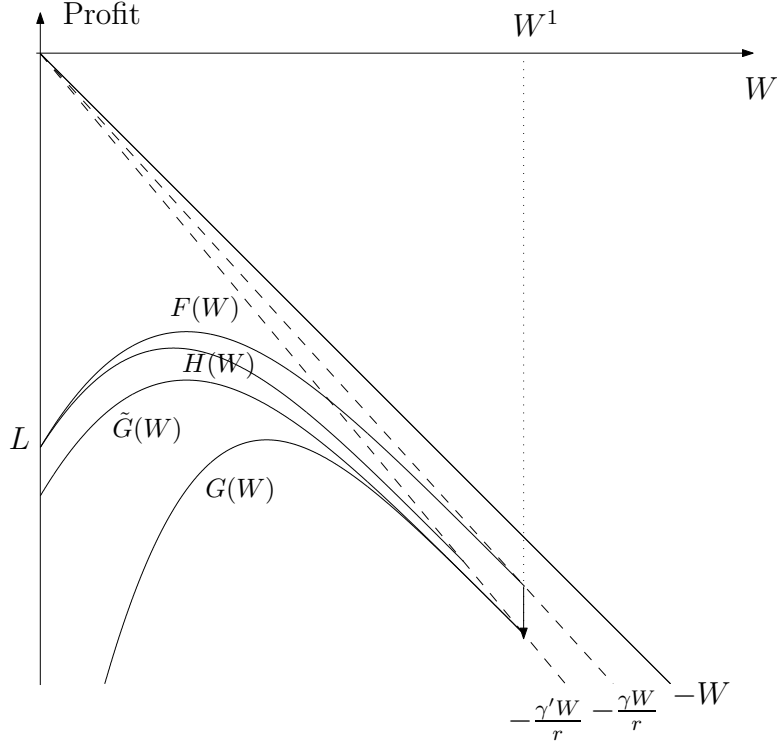


Figure 12: Proving that credit line is decreasing in γ .

Proof. Consider $\gamma' > \gamma > r$. Let F be the principal's profit for agent's discount rate γ . Denote $\tilde{G}(W) = F(W) - \frac{\gamma' - \gamma}{r}W^1$. Then

$$r\tilde{G}(W) - \gamma'W\tilde{G}'(W) = rF(W) - (\gamma' - \gamma)W^1 - \gamma WF'(W) - (\gamma' - \gamma)WF'(W) = rF(W) - \underbrace{(\gamma' - \gamma)(W^1 - W)}_{<0} - \gamma WF'(W) - (\gamma' - \gamma)W \underbrace{(F'(W) + 1)}_{>0} < F''(W) = \tilde{G}''(W).$$

Therefore, \tilde{G} is less concave than the optimality equation and by Lemma 3 function G that solves (27) from boundary conditions $G(W^1) = \tilde{G}(W^1)$ and $G'(W^1) = -1$ looks as shown in Figure 12 with $G(R) < \tilde{G}(R) < L$. Using the phase diagram in Figure 8, we conclude that the credit line for the agent's discount rate γ' and liquidation value L should be smaller than that for discount rate γ . In Figure 12 function H illustrates the principal's profit for parameters γ' and L . \square

As γ increases, debt can move either way. In the Figure 5, debt is U-shaped.

7 Appendix

Proof of Proposition 2. The Proposition follows from the following Lemmas 8 and 9:

Lemma 8. *Consider any incentive-compatible contract. Then there is another incentive-compatible contract, which gives the same profit to the agent and the same or greater payoff to the principal, under which the agent chooses to reveal cash flows truthfully.*

Proof. Our argument has a flavor of the revelation principle. However, the revelation principle does not apply directly, because the agent's payoff depends not only on the principal's action but also directly on his report (see (8)).

Consider an incentive-compatible contract with transfer process is $D(\hat{Y}(\cdot))$ and termination time $\tau(\hat{Y}(\cdot))$. We would like to define a new contract such that

- (i) the agent gets the same payoff as under the old contract
- (ii) the agent chooses to reveal cash flows truthfully
- (iii) the principal gets the same or greater profit as under the old contract

Given the agent's report Y' , define the transfer process D' under the new contract to be such that

$$dD'_t(Y') = [dY'_t - d\hat{Y}_t(Y')]^\lambda + dD_t(\hat{Y}(Y')),$$

where $\hat{Y}(Y')$ is the report generated by the agent under the old contract, when he observes Y' . Also, define the termination time under the new contract as $\tau(\hat{Y}(\cdot))$. It is easy to see that in the new contract, transfer process D' and termination time $\tau(\hat{Y}(\cdot))$ are \hat{Y} -measurable.

First, if the agent tells the truth, then he receives the same stream of income as if he reported \hat{Y} under the old contract. Second, if the agent lies and says Y' , he receives less income than he would by telling $\hat{Y}(Y')$ under the old contract, because

$$\begin{aligned} [dY_t - dY'_t]^\lambda + dD'_t(Y') &= [dY_t - dY'_t]^\lambda + [dY'_t - d\hat{Y}_t(Y')]^\lambda + dD_t(\hat{Y}(Y')) \\ &\leq [dY_t - d\hat{Y}_t(Y')]^\lambda + dD_t(\hat{Y}(Y')) \end{aligned}$$

Because the agent found it optimal to report \hat{Y} under the old contract, he prefers to tell the truth under the new contract. Third, because

$$dY_t - dD'_t(Y) = dY_t - [dY_t - d\hat{Y}_t(Y')]^\lambda - dD_t(\hat{Y}(Y')) \geq \hat{Y}_t - dD_t(\hat{Y}),$$

the principal's profit under the new contract is the same as or greater than under the old contract. Therefore, the new contract that we constructed satisfies conditions (i), (ii) and (iii), as required. \square

Lemma 9. *Consider any incentive-compatible contract (τ, D) under which the agent reports truthfully, and consumes C . Then there new contract (τ, D') with an alternative payment process D' , under which the agent chooses to maintain zero savings (since the principal does savings for the agent). This new contract gives the agent the same payoff as before; the principal receives the same or higher profit.*

Proof. Let

$$S_t(Y) = \int_0^t e^{-\rho(t-s)} (dD_s(Y) - dC_s(Y)) ds \geq 0$$

be the savings process under the old contract (τ, D) . For any report \hat{Y} define

$$D'_t(\hat{Y}) = C_t(\hat{Y}) \tag{30}$$

If the agent tells the truth and consumes C_t under the new contract (τ, D') , then he maintains zero savings. The agent's total expected payoff under the new contract is

$$W'_0 = E \left[\int_0^\tau e^{-\gamma t} dC_t + e^{-\gamma \tau} R \right],$$

which is the same as under the old contract.

The principal's profit under the new contract is greater or equal than his profit under the old contract. Indeed, when the principal does savings for the agent and the principal's interest rate r is greater than the agent's interest rate ρ , then the principal's expected profit improves by

$$E \left[\int_0^\tau e^{-rt} (r - \rho) S_t dt \right].$$

Lastly, we need to show that the new contract is incentive-compatible. Incentive compatibility follows if we show that the new contract does not allow any new feasible strategies for the agent: let us show that if any alternative strategy (\hat{C}, \hat{Y}) is feasible in response to (τ, D') , then it is also feasible in response to (τ, D) . A strategy is feasible if it generates a nonnegative savings process. Note that the income processes for strategy (\hat{C}, \hat{Y}) in response to (τ, D) and (τ, D') are related by

$$d\hat{I}_t = d\hat{I}'_t - dD'_t(\hat{Y}) + dD_t(\hat{Y}).$$

We have

$$\begin{aligned} \underbrace{\int_0^t e^{\rho(t-s)} (d\hat{I}_s - d\hat{C}_s) ds}_{\text{savings for } (\tau, D)} &= \underbrace{\int_0^t e^{\rho(t-s)} (d\hat{I}'_s - d\hat{C}_s) ds}_{\geq 0, \text{ savings for } (\tau, D')} \\ \int_0^t e^{\rho(t-s)} \underbrace{(dD'_s(\hat{Y}) - dC_s(\hat{Y})) ds}_{=0 \text{ by (30)}} + \underbrace{\int_0^t e^{\rho(t-s)} (dD_s(\hat{Y}_s) - dC_s(\hat{Y})) ds}_{=S_t(\hat{Y}) \geq 0} &\geq 0. \end{aligned}$$

This completes the proof that the new contract is incentive-compatible. \square

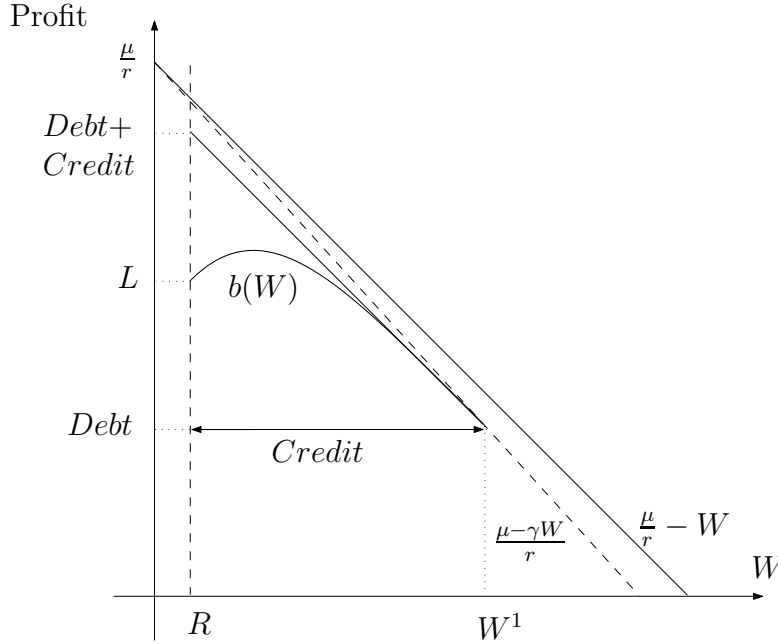


Figure 13: Proving that $Debt + Credit > L$ when $\lambda = 0$.

7.1 Market Value of Credit Line, Debt and Equity.

Lemma 10. *If $\lambda = 0$ then $Credit + Debt > L$.*

Proof. Lemma 10 follows from Figure 13. We know that $D = \frac{\mu - \gamma W^1}{r} = b(W^1)$. Draw a line through $(W^1, b(W^1))$ with slope -1 and find its intercept $(R, Debt + Credit)$ with a vertical line passing through R . From the concavity of b , and knowing that $b'(W^1) = -1$, we deduce that $Debt + Credit > L$. \square

The value of credit line, debt and equity can be conveniently expressed in terms of functions

$$G_\tau(W) = E[e^{-r\tau} | W_0 = W] \quad \text{and} \quad G_D(W) = E\left[\int_0^\tau e^{-rt} dD_t | W_0 = W\right].$$

Both of these functions solve differential equation

$$\frac{\lambda^2 \sigma^2}{2} G''(W) = rG(W) - G'(W)\gamma W \tag{31}$$

with boundary conditions

$$G_\tau(R) = 1, \quad G'_\tau(W^1) = 0 \quad \text{and} \quad G_D(R) = 0, \quad G'_D(W^1) = 1.$$

Then

$$\frac{1 - G_\tau(W)}{r} = E \left[\int_0^\tau e^{-rt} dt \mid W_0 = W \right].$$

We can express the values of the credit line, debt and equity equivalently as functions of the balance on the credit line, or the agent's value. The values of these securities can be conveniently expressed in terms of functions G_τ and G_D as

$$\begin{aligned} V_C(W) &= E \left[\int_0^\tau e^{-rt} \left(d\hat{Y}_t - x dt - \frac{dD_t}{\lambda} \right) + e^{-r\tau} \min(\text{Credit}, L - \min(\text{Debt}, L)) \mid W_0 = W \right] = \\ &\frac{\gamma W^1}{\lambda r} (1 - G_\tau(W)) - \frac{G_D(W)}{\lambda} + G_\tau(W) \min(\text{Credit}, L - \min(\text{Debt}, L)). \end{aligned}$$

$$\begin{aligned} V_D(W) &= E \left[\int_0^\tau e^{-rt} x dt + e^{-r\tau} \min(L, \text{Debt}) \mid W_0 = W \right] = \\ &\left(\frac{\mu}{r} - \frac{\gamma W^1}{\lambda r} \right) (1 - G_\tau(W)) + G_\tau(W) \min(L, \text{Debt}). \end{aligned}$$

$$\begin{aligned} V_E(W) &= E \left[\int_0^\tau e^{-rt} \frac{dD_t}{\lambda} + e^{-r\tau} \frac{\max(0, L - \text{Debt} - \text{Credit})}{1 - \lambda} \mid W_0 = W \right] \\ &= \frac{G_D(W)}{\lambda} + G_\tau(W) \frac{\max(0, L - \text{Debt} - \text{Credit})}{1 - \lambda}. \end{aligned}$$

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